Optimal Trading Strategies in Illiquid Markets

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Abstract

An institutional trader who wishes to trade a large position in the Paris stock market faces the choice between a block trade or several small trades. Actual data reveals that block trades increase subsequent to agitated markets. For an investor who wishes to trade through the electronic market, we extend the optimal trading algorithm of Almgren and Chriss (2001) to allow for volume impact. To do so, we extend Sadka’s (2006) microstructure model to allow for differential price impact of buys and sells. Estimations for a large set of companies reject restrictions that would discard volume impact and the buy/sell distinction. A comparative static exercise reveals the importance of the general order arrival rate for the optimal trading speed. Priors concerning the general order arrival rate are more important than priors concerning price variations or changes in the microstructure parameters to capture market crashes.

Keywords: Optimal trading strategy, liquidity risk, price impact, high frequency data, microstructure.

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1 Introduction

An institutional investor who wishes to trade a large position may either seek a counterpart in the upstairs market, to trade the shares in one block, or consider splitting the order into small parts that would then get executed by an electronic system. The splitting of the order has the advantage of minimizing the price impact, at the same time it exposes the investor to the risk of price deterioration. Out of the desire to minimize price impact and to minimize price risk arises a trade-off that can be optimized, as demonstrated in the seminal work by Almgren and Chriss (2001).

In this paper, we reconsider the trading of a large position. To do so, we extend the microstructure model on which the trading model is built and validate this extension for a large set of companies. We wish to state upfront that we use data for the Paris stock exchange, an electronic market to which block trades need to be reported. One reason for this choice is the quality of that data. Data of this market has been used in previous research by Biais, Hillion, and Spatt (1995).

The first level of decision, that is block-trading versus splitting, requires information that is generally difficult to measure such as a measure of the search-cost of finding a counterpart. To obtain, nonetheless, some guidance on when to perform a block trade, in the absence of data on when and how an investor actually decides to trade, we measure in a descriptive manner when block trades actually take place. Clearly, given the lack of data for an actual calibration, we do not attempt to create a theoretical model for the decision to trade a block.\footnote{To calibrate such a model, we would need information on the decision to trade blocks by all the institutional investors. Typically, proprietary data sets only contain information on a subset of institutional trades.} For the market under consideration, we show that subsequent to increases in liquidity measured by the volume of shares traded, the number of block trades increases. We also reveal an asymmetry in block trading where the number of traded blocks increases more subsequent to drops in markets than after rises of the same magnitude. This phenomenon exacerbates when the fall of returns is particularly large. It appears therefore that institutional investors have a preference for block trading subsequent to large variations in the market.

Given the convenience of electronic trading systems, the main thrust of research of this work, is however, the determination of the optimal splitting strategy of a large order that is to be channeled through the electronic system. We contribute to this literature from a theoretical point of view, by introducing into the model of Almgren and Chriss (2001) an extension of the more general microstructure model of Sadka (2006). This model allows, similar to many other microstructure models (Glosten and Harris, 1988, Huang and Stoll, 1997, Madhavan, Richardson, and Roomans, 1997), for temporary and permanent price impacts of a trade. Moreover, it considers the additional impact due to
the size of a trade. Indeed, since one would intuitively expect that a large trade impacts prices differently than a small trade, such an extension appears theoretically relevant. This important feature has, however, been neglected in the initial version of Almgren and Chriss (2001). More recently, Almgren and Chriss (2006) consider a model where the size of a trade affects the uncertainty of the price. We model, however, the impact of volume in a direct manner. Another theoretical contribution is that we extend the microstructure model of Sadka (2006) by allowing for a differential impact of buy orders from sell orders.

As expected, we confirm the earlier findings of Hung and Stoll (1997), Madhavan, Richardson, and Roomans (1997) as well as Sadka (2006) who found positive price impacts of the permanent (fixed and variable) components as well as of transitory ones. Moreover, we confirm that also for the French market, there exist volume discounts translating into negative transitory variable costs. Thus, the signs of the parameters appear stable. Next, we tested our extension which allows an asymmetric price impact depending on a trade being a buy or a sell order. Out of 40 stocks, we can reject the differential price impact for only 6 stocks. Taking this asymmetric price reaction into account appears, therefore, of relevance for the calibration of microstructure models, since it will imply different strategies depending on an order being a buy or a sell.

Having extended the theoretical trading model and demonstrated the relevance of our microstructure model, we perform a comparative static exercise. We demonstrate that volume effects are relevant for optimal trading. Changes in the general order arrival rate are found to be particularly relevant. Overall, our model indicates that orders should be executed very quickly with most of the shares traded within one hour. The confrontation of our model with the prior that markets will crash shows that changes in the permanent or transitory price components alone do not affect significantly the trading strategy. Again, modification of the general level of order arrivals matters most.

The rest of the paper is organized as follows. In Section 2, we discuss the decision to trade blocks versus channeling an order through an electronic system. In Section 3, we present the optimization problem and its solution. We also introduce the microstructure model describing the price evolution and we present the estimation methodology. In Section 4, we estimate the model for various companies. We consider two cases: when the parameters describing the price dynamics are constant throughout the day and when they change hourly. For both cases, we then determine the optimal liquidation strategy and study the sensitivity of the solution to changes in the levels of the parameters. We also study how the optimal liquidation strategy behaves when the value of the parameters evolves according to several pre-established patterns. Section 5 provides concluding comments.
2 Trading in Blocks or Not?

When evaluating the decision to trade, a trader faces the question of whether to contract with counterparts in the so-called upstairs market or to trade directly through an electronic trading system. A formal model of this decision requires specification of the costs and benefits of one market with respect to the other. From a modeling point of view, both markets represent different challenges. It turns out that modeling the electronic market is less challenging than modeling the workings of the upstairs market. The reason for this is that the upstairs market is more opaque and based on the search of a counterpart. Even though there is an emergence of theoretical models which describe such markets (see Duffie, Gărleanu, and Pedersen, 2005), their actual calibration requires information on the holdings of investors. Such data is not available, to our knowledge, for the French market.\(^2\) As a consequence, to glean at least an insight on how an investor may proceed with the decision between trading in the upstairs market versus splitting the orders, we perform a statistical analysis of when upstairs trades take place.

Before discussing the way the Paris upstairs market works, we want to relate our work on block trades with some of the additional existing literature. There is a large body of literature that considers the impact of block trades on prices. See Chan and Lakonishok (1995, 1997), or Kavajecz and Keim (2005). This literature considers the impact of a block trade on prices, rather than the decision of how to trade a large order. Keim and Madhavan (1995) represents one exception by providing descriptive statistics of the type of order chosen by institutional traders in the US market.

We now discuss block-trading in the Paris market. The possibility of block trading has been introduced on the Paris Stock Exchange in 1994. Only the largest and most liquid companies are eligible for block trades. In addition, there is a lower bound on what constitutes a block, which depends on the average amount of a given stock that is traded. Once a transaction has been decided upon, it must be channeled through an electronic system, which verifies that the trade is sufficiently large and that the price of the transaction is based on the volume-weighted average price. Trades that do not satisfy the size and price criteria are rejected by the system. The accepted trades are called applications and they must be declared to the market authorities immediately.

To gain insight when to trade applications, we gathered a continuous record of prices and trading volumes for 37 stocks traded between 1996 and 2003 from the Paris Stock Exchange. We first computed several statistics related to applications for each company.\(^3\)

\(^2\)Several papers such as Chiyachantana et al. (2006a and b), or Keim and Madhavan (1995, 1996, 1997) use proprietary data. Unfortunately, such data does not appear to cover all institutional trades.

\(^3\)Unlike Keim and Madhavan (1995), for this data one does not know if a trade was buy or sell initiated.
We report summary statistics of those averages in Table 1. Focusing on the column ‘Days with applic,’ we see that the company with the fewest block trades had block trades on 76% of all days, whereas there was a block trade for each and every day for other companies. For a company selected at random, there are block trades in 95% of the days on average. A subdivision of the sample shows that the estimates are comparable over time. The next columns of the table show that applications represent only a small number of all trades (about 1.35%). However, the number of blocks traded on a given day can be as large as 30. The number of shares traded via blocks represents about 18% of all shares traded, on average. An average block trade is about 640,000 euros. The average block size can be as large as 2 million euros for some companies. Comparing these basic figures across the subsamples reveals that the proportion of block trades has decreased from 2000 to 2003. However, the size of the average block trade has increased over this period. The figures between 2000 and 2003 are typically twice as large as those observed between 1996 and 1999. All in all, the proportion of shares traded through applications is broadly constant through time.

The last two columns of Table 1 display the average and maximal amounts of euros traded for the 37 companies. As one would expect, the average amount of euros for an application is much larger than the amount traded in a regular trade. Comparing the average block value and the largest regular trades reveals an important feature: the amounts traded in blocks are smaller than the largest regular trades. This observation suggests that block trades are not the only way by which large orders can be executed. Many large trades appear to be channeled through the electronic system.

To further investigate the choice of the upstairs market versus the electronic market, we run a panel regression over 37 companies, with data sampled at daily frequency, covering the period 1996-2003. The dependent variable is the number of applications per day. The retained regression is:

\[
\text{nbApplic}_{it} = \mu_i + a_0 \text{nbTr}_{it-1} + a_1 V_{it-1} + a_2 \text{vol}_{it-1} \\
+ a_3 R_{it-1} + a_4 R^-_{it-1} + a_5 R^{-\infty}_{it-1} + u_{it},
\]  

\(1\)

where the regressors are lags of the number of trades (\(\text{nbTr}\)), the daily realized variance (\(V\)),\(^4\) the volume traded in Euros (\(\text{vol}\)), the return (\(R\)), negative returns (\(R^-\)), and extremely negative returns (\(R^{-\infty}\)). The extremely negative returns variable is obtained by interacting returns with a dummy that takes the value 1 if a return is below \(-4\%\). The crash variable is aimed at capturing the behavior of extreme market conditions such as arose for instance during 2008. We also experimented with a variable capturing positive extreme returns but the estimates did not turn out significant. We also experimented

\(^4\) Our measure of variance is the realized variance computed by cumulating squared five-minute returns.
with interactions between volume or trade variables and extreme realizations but did not capture significant results. Last, we investigated if there is an end-of-the-month seasonality. Again, we could not exhibit such a phenomenon.

Table 2 presents the preferred estimation results. This table demonstrates that, on heuristic grounds, an investor may wish to perform a block trade subsequent to days with more trades, greater trading activity measured by the value of traded shares, or after days with greater realized volatility. Chiyachantana et al. (2006b, Table 3) also report for the French market an increase of volatility two days before institutional trades. Moreover, the parameters on $R^{-}$ and $R^{e-}$ indicate that, after negative market developments (which presumably represent bad news), the number of block trades increases significantly. These findings suggest that after a particularly large drop in the market, large investors become aware of others wishing to also trade significant blocks. In this case, instead of trading through the electronic system, those investors may wish to trade in the upstairs market.

One possible explanation of this feature is that institutional investors follow positive feedback trading as also shown by Dornbusch and Park (1995). This does, however, not explain the difference in asymmetry between the buy and sell side. This finding is also compatible with the notion that a buy order may be motivated by a different reason than a sell order, as suggested by Keim and Madhavan (1996) or Chan and Lakonishok (1995). Typically, among others, a buy order is motivated by the information that the stock is undervalued whereas a sell is triggered by liquidity needs. After falling markets, as the needs for liquidity by institutional investors increases, they may need to sell even more.

To conclude, we note that block trades are a common and important way to liquidate large positions. However, as the descriptive statistics of the trade sizes reveal, investors also appear to channel large trades through the electronic market directly. Also Keim and Madhavan (1995) notice for US data, that institutional investors channel about 90% of their trades through market orders. Such investors may therefore have a particular incentive to use an optimal trading strategy.

3 Optimal Trading Strategy

The optimal trading strategy for a large position through the electronic market requires a description of the price dynamic, which needs to take into account the price impact of the trade. It also requires a theoretical model, which allows an optimization of the tradeoff between speed which leads to a price impact and execution risk. In this section, we first set the frame of the trading model, then we present a realistic microstructure model and its estimation, last we present the formulae corresponding to the optimal
trading strategy for the retained microstructure model.

3.1 The Trading Strategy

The definition of trading strategy and the optimization problem that we use are closely related to those adopted by Almgren and Chriss (2001) and Huberman and Stanzl (2000). Huberman and Stanzl (2000), assuming slightly different price dynamics than Almgren and Chriss (2001), find a closed-form solution to the optimal static selling strategy when the price impact of the trade size is constant, and a recursive solution when the price impact changes over time. In recent years, papers dealing with optimal trading strategies have flourished. One may mention for instance Bertsimas and Lo (1998) who, however, unlike Almgren and Chriss (2001), do not incorporate the volatility of the total cost of trading into the optimization problem. Dubil (2002) and Hisata and Yamai (2000) use a setting similar to Almgren and Chriss (2001) and obtain the optimal total time of liquidation assuming a constant liquidation speed. Mönch (2004) incorporates two microstructure features in the price function: the U-shape in the intraday market liquidity and the resiliency of the order book. Then, assuming that price dynamics only incorporate permanent impacts, he determines the optimal liquidation strategy numerically. Contrary to other papers, he does not force the time between trades to be constant.

Our paper differs from Almgren and Chriss (2001), Dubil (2002), and Hisata and Yamai (2000) in various aspects. First, it does not restrict the price impact of the trade direction and the trade size to be constant over the time. This difference is achieved by allowing the parameters representing these impacts to be time-varying. Second, we allow for a more general microstructure model. Third, using tick-by-tick data from the Paris Stock Exchange, we actually estimate the parameters that characterize the price process. It is worth emphasizing that, in this context, only Mönch (2004) has estimated the parameters of his proposed model using high-frequency data from the Helsinki Stock Exchange. He assumes two particular functional forms for the price and traded size functions.

Presently, we describe the mechanism of the optimal selling strategy, deferring the discussion of the optimal purchase strategy to a later section. Assume that an investor has a large position $X$ in a security and that she wants to liquidate it completely by time $T$. A liquidation strategy is defined by a sequence of positive numbers $(x_{t_0}, \ldots, x_{t_N})$, where $x_{t_i} \geq 0$ is the number of units the investor intends to hold at time $t_i$, for $i = 0, \ldots, N$. The strategy is implemented at time $t_0$, where $t_0 = 0 < t_1 < \cdots < t_N = T$, and $x_{t_i} \geq x_{t_j}$ if $i < j$. Times $t_i$ are assumed to be discretely and equally spaced.\footnote{Since the liquidation strategies may find that a liquidation of 0 unit in some periods is optimal, and since we determine the optimal number of liquidation periods, our assumption that the time between}
that \( x_{t_0} = X \) and \( x_{t_N} = 0 \). The liquidation trajectory \((x_{t_0}, \ldots, x_{t_N})\) can be expressed as a trade list \((n_{t_1}, \ldots, n_{t_N})\), where \( n_{t_k} \) is the number of units sold immediately before time \( t_k \). Hence, the variables \( x_{t_k} \) and \( n_{t_k} \) are related by

\[
x_{t_k} = X - \sum_{j=1}^{k} n_{t_j}, \quad \text{for} \ k = 0, \ldots, N.
\]

This equation states that the investor’s holding at time \( t_k \) is the initial holding, \( X \), minus the sum of the quantities sold up to time \( t_k \), \( \sum_{j=1}^{k} n_{t_j} \). The liquidation strategy is based on the information available at time \( t_0 \).\(^6\) The investor’s optimization problem consists of finding the liquidation strategy \((x_{t_0}, \ldots, x_{t_{N-1}})\), or the equivalent list \((n_{t_1}, \ldots, n_{t_N})\), that maximizes the expected total income of liquidation, penalized by its variance, with the penalization factor denoted by \( \eta \). We also assume affine transactions costs \( K + n_t \kappa \).

The parameters \( K \) and \( \kappa \) represent a fixed and a variable cost for each transaction, respectively. This leads us to the following optimization program:\(^7\)

\[
\max_{\{n_{t_1}, \ldots, n_{t_N}\}} E \left[ \sum_{i=1}^{N} n_{t_i} (p_{t_i} - \kappa) \right] - \eta V \left[ \sum_{i=1}^{N} n_{t_i} p_{t_i} \right] - NK, \tag{2}
\]

\[\text{s.t.} \quad \sum_{i=1}^{N} n_{t_i} = X, \quad \text{and} \quad 0 \leq n_{t_i} \leq X, \quad \text{for} \ i = 1, \ldots, N.\]

The optimization problem (2) assumes a specific number of trading periods \( N \). To find the optimal number of trading periods, we solve the optimization problem for \( N = 1, \ldots, N_{\text{max}} \), where \( N_{\text{max}} \) is large. We choose the optimal \( N \) to be the value that maximizes the objective function in problem (2). Note that the introduction of an affine transaction cost has two consequences. First, the variable cost \( n_t \kappa \) corresponds to a change in the price. Second, the fixed cost \( NK \) only intervenes in the selection of the optimal number of trades \( N \) and not in the determination of \( n_t \).

It is clear from problem (2) that the solution to the optimization problem depends on the dynamics of the trading price, \( p_{t_k} \). We will therefore incorporate several microstructure features into the optimization, such as the time-varying impact of the direction and magnitude of the order flow. For instance, larger sales require larger discounts. Further, slow execution of the liquidation strategy implies more risk.

Regarding the purchase strategy, the problem is symmetric to the one discussed.

\(^6\)At time \( t_0 \), we define deterministic strategies, which determine the quantities to sell over the next \( N \) periods of time. Ideally, one would like to use adaptive strategies whereby the arrival of news affects the trading strategy.

\(^7\)The decision criterion is mean-variance. Experiments with other utility functions such as aversion towards downside risk reveal greater numerical complexity.
above. The objective function becomes:

\[
\min_{\{n_{t_1}, \ldots, n_{t_N}\}} \mathbb{E} \left[ \sum_{i=1}^{N} n_{t_i} (p_{t_i} + \kappa) \right] + \eta V \left[ \sum_{i=1}^{N} n_{t_i} p_{t_i} \right] + NK,
\]  

(3)

\[s.t. \quad \sum_{i=1}^{N} n_{t_i} = X, \quad \text{and} \quad 0 \leq n_{t_i} \leq X, \quad \text{for} \quad i = 1, \ldots, N,
\]

where \(x_{t_0} = 0\) and \(x_{t_N} = X\). In this case, \(n_{t_i}\) denotes the number of units purchased immediately before time \(t\).\(^8\) The interpretation and impact of \(K\) and \(\kappa\) are as before.

Having understood the theoretical impact of transaction costs, we set \(\kappa = K = 0\) from now on. It would be easy to modify the numerical implementation to account for these components.

We discuss the dynamics of the trading price in the following section, which incorporates separate dynamics for the buy and the sell sides of the order book.

### 3.2 The Price Dynamics

From now on, we drop the index \(k\) from the time \(t_k\), denoting it simply by \(t\). Therefore, time \(t\) represents the event time of the trade, and \(t - 1\) represents the time of the trade occurring immediately before. In other words, we account for the time between trades, \(\tau\), in the estimation. The transaction price, \(p_{t_i}\), is assumed to follow an extension of the dynamics proposed by Glosten and Harris (1988), Brennan and Subrahmanyam (1996), Madhavan, Richardson, and Roomans (1997), Huang and Stoll (1997), and Sadka (2006). The dynamics of price movements are related to the dynamics of the fundamental value of the stock, denoted \(m_t\). We define \(m_t\) as:

\[
m_t = \mu + m_{t-1} + \psi^+ \left(1_{\{D_t=1\}} - E_{t-1} \left[1_{\{D_t=1\}}\right]\right) - \psi^- \left(1_{\{D_t=-1\}} - E_{t-1} \left[1_{\{D_t=-1\}}\right]\right) \\
+ \lambda^+ \left(n_t - E_{t-1} \left[n_t\right]\right) 1_{\{D_t=1\}} - \lambda^- \left(n_t - E_{t-1} \left[n_t\right]\right) 1_{\{D_t=-1\}} + y_t,
\]

(4)

where \(y_t \sim iid(0, \sigma^2_{y,t})\).

The variable \(m_t\) represents the expected value of the security, conditional on the information available at time \(t\). For a given transaction at time \(t\), the indicator variable \(D_t\) represents the direction of the order flow, which is defined as +1 if the trade is buy-initiated and -1 if it is sell-initiated. The variable \(n_t\) represents the order flow (i.e., the traded volume). The variable \(y_t\) is an exogenous shock representing the arrival of

\(^8\)For institutional investors who seek to minimize tracking error with respect to some index, an alternative strategy may be to trade as many shares as possible right before the close of the market. Another interesting strategy would be to allow limit orders for the trading strategy. Such strategies would demand significant changes in the model. Given the complexity of currently available dynamic order books (see Goettler, Parlour, and Rajan, 2005), we leave this analysis to further research.
new information. Our model of price dynamics is innovative as it assumes differentiated responses for the buy and sell sides. Furthermore, we introduce a parameter $\mu$, which describes market sentiment. This parameter represents a mean rate of drift and may be difficult to estimate; however, it may be useful in describing investors’ expectations of bullish or bearish markets.

Equation (4) describes the permanent effect of both private information, $D_t$ and $n_t$, and public information, $y_t$, on price. The intuition behind this representation is that some traders have some private information about the price of the security, which is reflected in the decisions that they make. These decisions include the decision to buy or sell (i.e., the value of $D_t$), and the quantity to buy or sell (i.e., the value of $n_t$). Hasbrouck (1991a and b) and Foster and Viswanathan (1993), among others, have documented the presence of predictability in the order flow.\footnote{The predictability of prices is related to several empirical issues. For instance, the decisions of some investors to break large trades into small orders to reduce the price impact can create serial correlation in the order flow.}

Equation (4) does not assume that the direction and magnitude of the order flow have an impact on the change in the fundamental value of the stock. Instead, only their innovation processes, represented respectively by, $1\{D_t=1\} - E_t - 1\{D_t=1\}$, $1\{D_t=-1\} - E_t - 1\{-1\{D_t=-1\}\}$, and $(n_t - E_t - 1\{n_t\})$, respectively, have an impact. In equation (4), the parameters $\psi_t$ and $\lambda_t$ measure the permanent (fixed) and variable impacts on price. The superscripts $+$ and $-$ of $\psi_t$ and $\lambda_t$ indicate whether a trade is a buy or a sell order.

Following Glosten and Harris (1988), the dynamics of the transaction price depend on the transitory impact of the order flow and the direction of the order flow:

$$p_t = m_t + \psi_t T 1\{D_t=1\} - \psi_t T 1\{D_t=-1\} + \lambda_t T n_t 1\{D_t=1\} - \lambda_t T n_t 1\{D_t=-1\} + \xi_t,$$

where $\xi_t \sim iid(0, \sigma_{\xi_t}^2)$. This equation incorporates transient microstructure phenomena into the price process, generating a wedge between the fundamental value of the security, $m_t$, and the actually traded price. The parameter $\psi_t$ can be interpreted as a fixed cost per order and the parameter $\lambda_t$ as a variable (unitary) cost. Parameters $\psi_t$ and $\lambda_t$ are compatible with inventory cost models. Intuitively, these costs should affect the price only at the current time and in a transitory manner. The error $\xi_t$ captures features that are not explicitly modeled, such as stochastic rounding induced by price discreteness. It is worth mentioning that in equations (4) and (5) the parameters, $\psi_t$, $\psi_t$, $\lambda_t$, and $\lambda_t$, are time-varying functions.\footnote{This parametrization relaxes the assumption that the permanent and transitory impacts of the trade characteristics on the price are constant throughout the day, as assumed by Almgren and Chriss (2001), Dubil (2002), Hisata and Yamai (2000), and Sadka (2004).}
related to the order book.\textsuperscript{11}

The estimation of the parameters is based on the set of trades (i.e., buy-initiated as well as sell-initiated trades), whereas we focus only on buying or selling strategies in the optimal trading strategy. Consequently, when determining the optimal strategy, we will assume that the predictable part of the series (captured by the terms $E_{t-1}[D_t]$ and $E_{t-1}[n_t]$) is negligible. In other words, the predictable component captures the effect of all other trades. Therefore, for a pure selling or buying strategy, the fundamental value of the stock evolves according to:

$$m_t = \begin{cases} 
\mu + m_{t-1} - \psi^-_t - \lambda^-_t n_t + \sigma_t \tau^{1/2} \omega_t, & \text{for a sale,} \\
\mu + m_{t-1} + \psi^+_t + \lambda^+_t n_t + \sigma_t \tau^{1/2} \omega_t, & \text{for a purchase,}
\end{cases}$$  

(6)

where $\omega_t = y_t / \sigma_{y,t}$ ~ iid$(0, 1)$, and the decision variable $n_t$ denotes the quantity the investor decides to sell or buy at time $\tau t = \frac{T}{N} t$ for $t = 1, \ldots, N$. It is convenient to define $\sigma_{y,t} = \sigma_t \tau^{1/2}$, where $\sigma_{t}$ is the instantaneous volatility.

The trading price is given by:

$$p_t = \begin{cases} 
m_t - \psi^-_t - \lambda^-_t n_t + \xi_t, & \text{for a sale,} \\
m_t + \psi^+_t + \lambda^+_t n_t + \xi_t, & \text{for a purchase,}
\end{cases}$$  

(7)

which corresponds to equation (5) for a pure selling or buying strategy (i.e., $D_t = -1$ or $D_t = +1$).

We use the price dynamics defined in equations (6) and (7) to determine the optimal liquidation strategy, which corresponds to the solution to the optimization problem described by problem (2).

Before determining the optimal strategy, we describe the estimation procedure in the next section. In Section 4, we will discuss the optimal trading strategy which results from our calibration.

### 3.3 Estimation of the Microstructure Model

Some filtering of the data is required before estimation: (i) the first trade of each day is eliminated, since the process generating the opening price (an auction) is different from the process generating the prices during the day and (ii) original volumes are seasonally adjusted. The rationale for this filter is that traded volume can be seen as the sum of a stochastic part and a deterministic part. The latter usually includes a time-of-the-day effect. This effect reflects systematic variation in traded volume during a normal day. Empirical evidence suggests that traded volume tends to be greatest just after opening and just before closing. We verified these features in our dataset. The seasonal

\textsuperscript{11}Empirical evidence of this fact is found in Hasbrouck (1991b) and Mönch (2004) and the references mentioned therein.
adjustment procedure filters out this effect in order to keep only the stochastic effect. To correct for this effect, we adjust the raw traded volume series by the method proposed by Engle and Russell (1998).\textsuperscript{12}

### 3.3.1 Estimation - Step 1

The estimation is done in two steps. In the first step, we adjust and estimate parametric processes for the direction of the order book, \(D_t\), and for the volume of the order flow, \(n_t\). The estimated processes are used to infer the series \(E_{t-1}[D_t]\) and \(E_{t-1}[n_t]\). These series are then used in the second step for the estimation of equations (4) and (5).

There is no consensus as to which model best describes the directions of the order book or the order flow. For instance, Hasbrouck (1991a and b) expresses quote-midpoints and order flows through a vector autoregressive process of order five. Based on this early model, Brennan and Subrahmanyam (1996) use five lags of price and order flow to estimate the unexpected order flow. Huang and Stoll (1997) model the direction of the order book as an autoregressive process of order one. Sadka (2006) estimates an \(AR(5)\) process for the (signed) order flow.

We assume that the order flow, \(n_t\), follows an autoregressive process and that the direction of the order flow, \(D_t\), follows a Markov Chain process with state-space \(S = \{-1, 0, 1\}\).

Therefore, the order flow is described by: \(n_t = c + \varphi_1 n_{t-1} + \cdots + \varphi_p n_{t-p} + \varepsilon_t\), with \(\varepsilon_t \sim iid(0, \sigma^2)\). Once the parameters of the processes \(n_t\) (i.e., \(c, \varphi_1, \cdots, \varphi_p\)) and \(D_t\) (i.e., the transition probabilities \(\pi_{i,j} = Pr\{D_t = j \mid D_{t-1} = i\}\)) for \(i, j = -1, 0, 1\) are estimated, we calculate

\[
\hat{E}_{t-1}[n_t] = \hat{c} + \hat{\varphi}_1 n_{t-1} + \cdots + \hat{\varphi}_p n_{t-p},
\]

and

\[
\hat{E}_{t-1}[D_t|D_{t-1} = i] = -\pi_{i,-1} + \pi_{i,1}, \quad \text{for} \quad i = -1, 0, 1.
\]

Estimation of an AR model for \(n_t\) reveals different patterns, depending upon the stock under consideration. For instance, using a Bayesian information criterion, we found that Orange’s order flow is best described by an \(AR(6)\) model.\textsuperscript{13}

\textsuperscript{12}We regress the logarithm of the raw volume (traded quantity) on time-of-the-day dummies. More specifically, the day is divided into \(K\) sub-periods and we consider the regression \(\log(n_t) = \sum_{k=1}^{K} a_k x_{kt} + \varepsilon_t\), where \(x_{kt} = 1\) if the time of the trade \(t\) belongs to the intraday sub-period \(k\) for \(k = 1, \cdots, K\), and 0 otherwise. The seasonally adjusted series is defined by \(\hat{n}_t = n_t \exp(-\hat{a}' x_t)\), where \(\hat{a}\) denotes the OLS estimate of \(a\).

\textsuperscript{13}Autocorrelation was not detected in other series, such as Suez or Sodexho-Alliance. Others, such as Alcatel, required as many as 10 lags in the AR process. This autocorrelation pattern is not related to the size of the company nor is it related to liquidity, since all these stocks have large daily trading
3.3.2 Estimation - Step 2

We use Generalized Method of Moments (GMM) to estimate the remaining set of parameters \((\psi^+, \psi^-, \bar{\psi}^+, \bar{\psi}^-, \lambda^+, \lambda^-, \bar{\lambda}^+, \bar{\lambda}^-, \sigma_{\gamma,t}^2, \sigma_{\xi,t}^2)\) describing the dynamics of the expected and traded prices in equations (4) and (5). Following Madhavan, Richardson, and Roomans (1998), we take the first difference of \(p_t\) in equation (5), yielding:

\[
p_t - p_{t-1} = m_t - m_{t-1} + \bar{\psi}_t^+ \mathbf{1}_{\{D_t=1\}} - \bar{\psi}_t^- \mathbf{1}_{\{D_t=-1\}} - \psi_t^+ \mathbf{1}_{\{D_t=1\}} + \psi_t^- \mathbf{1}_{\{D_t=-1\}} + \lambda_t^+ n_t \mathbf{1}_{\{D_t=1\}} - \lambda_t^- n_t \mathbf{1}_{\{D_t=-1\}} + \bar{\lambda}_t^+ n_{t-1} \mathbf{1}_{\{D_{t-1}=1\}} - \bar{\lambda}_t^- n_{t-1} \mathbf{1}_{\{D_{t-1}=-1\}} + \xi_t - \xi_{t-1}.
\]

By plugging in \(m_t - m_{t-1}\) from equation (4) and grouping some terms, we find that the dynamics of the actual trade price are given by

\[
p_t - p_{t-1} = \mu + \psi_t^+ (1 - \mathbb{E}_{t-1}[1]) - \psi_t^- (1 - \mathbb{E}_{t-1}[1]) + \lambda_t^+ (n_t - \mathbb{E}_{t-1}[n]) - \lambda_t^- (n_t - \mathbb{E}_{t-1}[n]) + \bar{\lambda}_t^+ n_t \mathbf{1}_{\{D_t=1\}} - \bar{\lambda}_t^- n_t \mathbf{1}_{\{D_t=-1\}} + \bar{\lambda}_t^+ n_{t-1} \mathbf{1}_{\{D_{t-1}=1\}} - \bar{\lambda}_t^- n_{t-1} \mathbf{1}_{\{D_{t-1}=-1\}} + y_t + \xi_t - \xi_{t-1}.
\]

If the parameters \((\psi^+, \psi^-, \bar{\psi}^+, \bar{\psi}^-, \lambda^+, \lambda^-, \bar{\lambda}^+, \bar{\lambda}^-, \sigma_{\gamma,t}^2, \sigma_{\xi,t}^2)\) are constant, we use equation (10) and define \(u_t = y_t + \xi_t - \xi_{t-1}\) to find that

\[
u = p_t - p_{t-1} - \mu - \left(\psi^+ + \bar{\psi}^+\right) \mathbf{1}_{\{D_t=1\}} + \left(\psi^- + \bar{\psi}^-\right) \mathbf{1}_{\{D_t=-1\}} + \lambda^+ n_t \mathbf{1}_{\{D_t=1\}} - \lambda^- n_t \mathbf{1}_{\{D_t=-1\}} + \bar{\lambda}^+ n_{t-1} \mathbf{1}_{\{D_{t-1}=1\}} - \bar{\lambda}^- n_{t-1} \mathbf{1}_{\{D_{t-1}=-1\}}.
\]

From this equation, we notice that innovations \(u_t\) are linear functions of the parameters. We have \(V[u_t] = V[y_t] + V[\xi_t] + V[\xi_{t-1}] = \sigma_{\gamma,t}^2 + 2\sigma_{\xi,t}^2\), and \(Cov[u_t, u_{t-1}] = Cov[y_t + \xi_t - \xi_{t-1}, y_{t-1} + \xi_{t-1} - \xi_{t-2}] = -Cov[\xi_{t-1}, \xi_{t-1}] = -\sigma_{\xi,t}^2\). Hence, the parameters \(\sigma_{\xi,t}^2\) and \(\sigma_{\gamma,t}^2\) can also be identified from orthogonality conditions. To obtain the orthogonality conditions, let

\[
\begin{align*}
U_{t,1} &= [u_t u_t \mathbf{1}_{\{D_t=1\}} u_t \mathbf{1}_{\{D_t=-1\}} u_t \mathbb{E}_{t-1}[1] u_t \mathbb{E}_{t-1}[1]]', \\
U_{t,2} &= [u_t n_t \mathbf{1}_{\{D_t=1\}} u_t n_t \mathbf{1}_{\{D_t=-1\}} u_t \mathbb{E}_{t-1}[n_t] u_t \mathbb{E}_{t-1}[n_t]]', \\
U_{t,3} &= [u_t \mathbf{1}_{\{D_{t-1}=1\}} u_t \mathbf{1}_{\{D_{t-1}=-1\}} u_t n_{t-1} \mathbf{1}_{\{D_{t-1}=1\}} u_t n_{t-1} \mathbf{1}_{\{D_{t-1}=-1\}}]', \\
U_{t,4} &= [u_t^2 - (\sigma_{\gamma,t}^2 + 2\sigma_{\xi,t}^2) u_t u_{t-1} + \sigma_{\xi,t}^2].
\end{align*}
\]

volume.
be column vectors. By staking these conditions, we obtain the orthogonality conditions

$$E \begin{bmatrix} U_{t,1} \\ U_{t,2} \\ U_{t,3} \\ U_{t,4} \end{bmatrix} = 0,$$

which can be used for GMM estimation.14

### 3.4 The Optimal Selling Strategy

In this section, we solve the optimization problem described by equation (2) to derive the optimal liquidation strategy. These propositions follow Almgren and Chriss (2001).

**Proposition 1** When the price function is given by equation (7), the solution to the optimization problem (2) corresponds to the solution to the linear system

$$c_k + \sum_{t=1}^{k-1} (-\lambda_t^* - b_t) n_t + (d_k - b_k)n_k - \sum_{t=k+1}^{N} b_k n_t = 0, \quad \text{for } k = 1, \ldots, N,$$

where $c_t = - \left( \sum_{j=1}^{t} \psi_j^* + \psi_t^* - t \mu \right), b_t = 2 \eta \tau \left( \sum_{j=1}^{t} \sigma_j^2 \right), d_t = -2 \left( \lambda_t^* + \bar{\lambda}_t^* + \eta \sigma_{\xi,t}^2 \right)$, and $(n_1, \ldots, n_N)$ satisfies the constraints of the original problem.

Rather than expressing the solution in terms of $n_t$, it is possible to express it as a function of $x_t$. This function is defined in the following proposition.

**Proposition 2** When the price function is given by (7), the solution to the optimization problem (2) corresponds to the solution to the linear system

$$a_t + b_t x_{t-1} + 2 c_t x_t + b_{t+1} x_{t+1} = 0, \quad \text{for } t = 1, \ldots, N - 1,$$

where $a_t = (\mu + \bar{\psi}_t^* - \bar{\psi}_{t+1}^* - \psi_{t+1}^*), b_t = \lambda_t^* + 2 \bar{\lambda}_t^* + 2 \eta \sigma_{\xi,t}^2, c_t = -\lambda_t^* - \bar{\lambda}_{t+1} - \bar{\lambda}_t^* - \eta \sigma_{\xi,t+1}^2 - \eta \sigma_{\xi,t}^2, n_t = x_{t-1} - x_t$, and $(x_0, \ldots, x_N)$ satisfies the constraints of the original problem.

The proof of Proposition 1 is relegated to the Appendix. The proof of Proposition 2 is analogous and not presented here. We use Proposition 1 to determine the optimal

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14 An estimation takes less than 3 minutes in MATLAB even for samples containing more than 50,000 observations.
liquidation strategy. This strategy is easy to implement in a software such as MATLAB by using the associated optimization toolbox.\footnote{It is also possible to use different utility functions such as those that emphasize downside risk. For such choices of preference structures, a direct optimization of the utility may be considered by using numerical optimization. Determination of a trading strategy (optimal number of trades and actual trade sizes) typically takes less than a minute.}

It is worth noting that problem (2) assumes a specific number of trades. To determine the optimal number of trading periods, we first obtain the optimal liquidation strategy, \((n_1^*, \cdots, n_N^*)\) for values of \(N\) between 1 and \(N^{\text{max}}\). Then, in a second step, we choose \(N_{\text{opt}}\) as the value of \(N\) that maximizes the objective function and retain \((n_1^*, \cdots, n_{N_{\text{opt}}}^*)\) as the optimal liquidation strategy.

### 3.5 The Optimal Buying Strategy

Our result is similar to Proposition 1, but for a buying strategy.

**Proposition 3** When the price function is given by equation (7), the solution to the optimization problem (3) corresponds to the solution to the linear system

\[
c_k + \sum_{t=1}^{k-1} (\lambda_t^+ + b_t)n_t + (d_k + b_k)n_k + \sum_{t=k+1}^{N} b_k n_t = 0, \quad \text{for} \ k = 1, \cdots, N,
\]

where \(c_t = \sum_{j=1}^{t} \psi_j^+ + \overline{\psi}_t^+ + t\mu\), \(b_t = 2\eta \tau \left(\sum_{j=1}^{t} \sigma_{j,t}^2\right)\), \(d_t = 2 \left(\lambda_t^+ + \overline{\lambda}_t^+ \eta \sigma_{\xi,t}^2\right)\), and \((n_1, \cdots, n_N)\) satisfies the constraints of the original problem.

The following proposition gives the solution in terms of \(x_t\).

**Proposition 4** When the price function is given by equation (7), the solution to the optimization problem (3) corresponds to the solution to the linear system

\[
a_t + b_t x_{t-1} + 2c_t x_t + b_{t+1} x_{t+1} = 0, \quad \text{for} \ t = 1, \cdots, N - 1,
\]

where \(a_t = (\mu + \psi_{t+1}^+ + \overline{\psi}_{t+1}^+ - \overline{\psi}_t^+), \ b_t = -(\lambda_t^+ + 2\overline{\lambda}_t^+ + 2\eta \sigma_{\xi,t}^2), \ c_t = \lambda_{t+1}^+ + \overline{\lambda}_{t+1}^+ + \overline{\lambda}_t^+ + \eta \sigma_{\xi,t+1}^2 + \eta \sigma_{\xi,t}^2, \ n_t = x_{t-1} - x_t\), and \((x_0, \cdots, x_N)\) satisfies the constraints of the original problem.

The proofs of Propositions 3 and 4 are similar to those of Propositions 1 and 2 and are not presented here.
4 Empirical Results

In this section, we use high frequency data to estimate the parameters $\psi_t$, $\psi_t$, $\lambda_t$, $\lambda_t$, $\sigma_{y,t}$, and $\sigma_{\xi,t}$, distinguishing the buy from the sell side.\textsuperscript{16} We use trades and quotes to infer the direction of the order flow, $D_t$, in addition to the following rule: If the trade price is larger than or equal to the last best ask price, then the trade is classified as buy-initiated (i.e., $D_t = +1$), if the trade price is smaller than or equal to the last best bid price, then the trade is classified as sell-initiated (i.e., $D_t = -1$), and if the trade is between the last best bid and the last best ask price, then the trade direction is assigned to $0$.\textsuperscript{17}

The estimation is performed using many stocks, although we present the results for only five randomly selected stocks: Alcatel, France Telecom, Orange, Sodexho-Alliance, and Suez. The optimal liquidation strategy is established in three cases: (1) when the value of the parameters is constant; (2) when the parameter $\lambda_t$ follows a hypothetical increasing function of time; and (3) when each parameter evolves continuously over time, the function governing the evolution being a spline approximation of the hourly estimates. In the first case, we also implement a sensitivity analysis to study how the optimal number of trading periods and the speed of liquidation are affected by changes in the value of the parameters.

4.1 Optimal Trading Strategy for Constant Parameters

We first assume that the parameters governing the price evolution in equations (4) and (5), $\psi_t$, $\psi_t$, $\lambda_t$, $\lambda_t$, $\sigma_{y,t}^2$, and $\sigma_{\xi,t}^2$, are constant throughout the day. For $\sigma_{y,t}^2$, this implies $\sigma_{y,t}^2 = \sigma^2 \tau$, where $\tau$ is the time between two consecutive trades. We describe the parameter estimates we obtain from the two-step procedure described in Section 3.2 and then determine the optimal liquidation strategy following the procedure described in Section 3.3.

\textsuperscript{16}The high frequency data is taken from the Paris Stock Exchange database. The original data contains information about all the buy and sell contracts, including date, price, and traded quantity, with a precision of 1 second. It also contains information about quotes, including date, best bid, best ask, depth at the best bid (maximum quantity to be bought at the best bid), and depth at the best ask (maximum quantity offered at the best ask).

\textsuperscript{17}To evaluate the quality of this inference methodology, we compared the direction of the order flow obtained from the inference method with the actual direction of the order flow content in the order book. For all the stocks studied, the performance of the inference methodology was very good. The aforementioned methodology consistently classifies more than 96 percent of the trades as sell-initiated or buy-initiated, remaining undefined for less than 4 percent of the trades. It is worth clarifying that the classification methodology of Lee and Ready (1991) was developed for quote-driven markets and therefore does not apply to our case, since the Paris Stock Exchange is an order-driven market.
4.1.1 Estimation

In the first step of the estimation procedure, we adjust and estimate an autoregressive process for the order flow, $n_t$, and assume that the direction of the order flow, $D_t$, evolves according to a Markov Chain. In the second step, the other parameters (i.e., $\psi$, $\bar{\psi}$, $\lambda$, $\bar{\lambda}$, $\sigma_y^2$, and $\sigma^2_\xi$) are estimated using GMM, where the set of orthogonality conditions is given by equation (12). We also performed tests of the restrictions where the buy and sell parameters were the same (that is, if $\vartheta^+ = \vartheta^-$ for $\vartheta \in \{\psi, \bar{\psi}, \lambda, \bar{\lambda}\}$). For the cases where we could not reject the null of equality of the parameters, we present the same parameters for the buy and the sell sides.

Table 3 reports the parameter estimates for various companies as well as summary statistics for all 40 stocks considered. Intuitively, the value of the parameters $\psi$ and $\lambda$ should be positive. Indeed, a sell order (i.e., $D_t = -1$) typically reflects the belief that the price is high (the security is overvalued) and should decrease. Conversely, a buy order (i.e., $D_t = +1$) reflects the belief that the price is low (the security is undervalued) and should increase. The same interpretation applies to the parameter $\lambda$, which measures the impact of the direction of innovation of the order flow on prices.

In relation to the sign of the transient price impact parameters, the parameter $\bar{\psi}$ represents a fixed cost and the parameter $\bar{\lambda}$ represents a unitary cost. Intuitively, both parameters are expected to be positive. However, Sadka (2006) documents that this parameter may be negative, in which case there is a price discount.

Overall, we find that the parameters have the expected sign, even though the variable transitory impact parameter, $\bar{\lambda}$, is insignificant for Sodexho-Alliance and Suez. More specifically, we observe that the permanent impact of the innovation of the direction of the order book, $\psi$, and the transitory impact of the direction of the order book, $\bar{\psi}$, are positive and always significant. The parameter $\psi$ can be interpreted as the expected change in the price after an unexpected change in the direction of the order flow (i.e., in $D_t - E_{t-1}[D_t]$) and $\bar{\psi}$ is a fixed cost by order whose effect on price is only transitory. We observe that the signs of the estimates agree with the microstructure interpretation claiming that trades incorporate traders’ private information. More specifically, traders decide to buy a security because they think it is undervalued and decide to sell because they think it is overvalued.

The permanent effect of the direction of innovation of the order flow and the transitory effect of the direction of the order flow are captured respectively by $\lambda$ and $\bar{\lambda}$, respectively. We observe that the former is positive and significant, meaning that a sell order has a negative impact on the price, while a buy order has a positive impact. The parameter $\bar{\lambda}$ is interpreted as the (transitory) marginal cost per unit. Its estimated value

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18 All prices were scaled as to yield comparable parameter estimates.
19 The discussion holds for parameters with + or − superscripts.
is negative, meaning that the seller of a certain amount of shares receives a discount for this sale. This finding is remarkable and confirms the previous findings of Huang and Stoll (1997) and Sadka (2006) on a completely different dataset.

4.1.2 Optimization

We determine the characteristics of the optimal trading strategy for the selected companies, assuming that the investor wishes to liquidate an order corresponding to 10% of the average daily traded volume. The available time horizon is $T = 8.5$ hours. We take $\eta = 4$ for the benchmark risk aversion. The parameters are those provided in Table 3.

A first illustration of the optimal trading strategy is given in Figure 1, where we focus on Orange. We observe that a large percentage of shares is traded immediately. The selling strategy is however more aggressive than the buying strategy (fewer trades are made and more shares are traded each time). This difference between sell and by orders is a direct consequence of the differential impact of trades on prices as documented in Table 3. It is interesting to note that the optimal trading strategy consists of a large initial trade, suggesting that the electronic system of this type is able to provide liquidity for large deals. One may wonder whether Orange is the only company for which such a trading strategy would hold. To answer this question, we present statistics for the companies under consideration in Table 4, focusing on an optimal liquidation. Concerning the first trade, we observe similar patterns for all companies. The first trade involves between 72% and 83% of shares. Next, following Mönch (2004), we define duration as the average weighted time needed to trade a position

$$D = \sum_{i=1}^{N} (t_i - 9) \frac{n_i}{X},$$  

(13)

where $t_i$ is the time passed since the trade was decided (measured in hours), $n_i$ represents the size of a trade, and $X$ is the initial position. The 9 corresponds to the opening hour. Thus, duration is an indicator of the speed at which the trade is executed. It also represents the time at which the number of shares that has been traded equals the number of shares that remain to be traded. As Table 4 documents, the duration of the various trades is, in all cases, under an hour, ranging from about 20 minutes to about 32 minutes. The total number of trading periods ranges between 5 and 7.

The next question that we address is which component of the microstructure model most strongly influences the expected revenue from the trading strategy. To do so, we consider the expected revenue relative to the revenue if one had sold all of the shares at one time and if one had no price impact at all. This theoretical single trade revenue
would be \( p_0 X \). Without loss of generality, we assume now that \( p_0 = 1 \). We have

\[
\frac{1}{X} E \left[ \sum_{t=1}^{N} n_t p_t \right] = 1 + \sum_{t=1}^{N} (\mu - \psi^-) \sum_{j=t}^{N} \frac{n_j}{X} - \sum_{t=1}^{N} \lambda^- n_t \sum_{j=t}^{N} \frac{n_j}{X} - \sum_{t=1}^{N} \psi^- n_t - \sum_{t=1}^{N} \bar{\lambda}^- n_t^2 \frac{1}{X}.
\]

To gain further insights into this formula, we consider now the constant-amounts trading strategy defined as a strategy where the same amount \( X/N \) gets traded in each transaction. We have, as shown in Appendix 2, that

\[
\frac{1}{X} E \left[ \sum_{t=1}^{N} n_t p_t \right] = 1 + (\mu - \psi^-) \frac{N + 1}{2} - \lambda X \frac{N + 1}{2N} - \psi^- - \frac{\bar{\lambda}^-}{N}.
\]

We will refer to the four terms on the right hand side as the \( \psi, \lambda, \bar{\psi}, \) and \( \bar{\lambda} \) effects. All of those terms have a negative sign (for the first one, this is the case if \( \mu = 0 \)) meaning that, if a parameter is found to be positive, it will lead to a reduction of the revenue from the theoretical benchmark revenue. As could be expected, we also establish that the overall impact of the unitary components \( \lambda^- \) and \( \bar{\lambda}^- \) will depend on the total volume \( X \) traded.

The various components of revenue are presented in Table 4.\(^{20}\) We note that the gains due to the direct price impact, be it transitory (\( \psi^- \)) or permanent (\( \psi^- \)), are relatively small. We note that the unitary permanent impact (\( \lambda^- \)) is important and ranges from 1.3% to 20%. We also note that the transitory discount (\( \bar{\lambda}^- \)) ranges from about 0 for Suez to about 10.7% for Alcatel. This suggests that the optimal selling strategy involves a fraction of the trade, not because of an important direct impact, but because the optimizer tries to keep the cost of the permanent impact of a large sale under control. This effect is partially offset by the transitory impact of the sale.\(^{21}\) In the following subsection, we perform a comparative static exercise and study how the optimal selling strategy changes when the risk aversion of the price-impact parameters move away from the benchmark.

### 4.1.3 Sensitivity Analysis

We now investigate the consequences of changing either the risk aversion parameter, \( \eta \), or one of the microstructure parameters. Unfortunately, the impact of changing one of those parameters on either the size of the trades or the optimal number of trades is in general very difficult to predict due to the high non-linearity of the problem. To demonstrate this analytical complexity, we consider in Appendix 2 the expression of the

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\(^{20}\)Those components are obtained using optimal trades \( n_t \) and not the constant trades.

\(^{21}\)We found very similar results for the buying strategy and for all measures reported in Table 4. The strong similarity is the reason why we do not report the results. We found also very similar results for the strategy implemented in the case of continuously varying parameters.
utility in the case of a trading strategy in which each trade involves the same amount and the model parameters do not change over time. Because of the complex interaction between the amounts traded, the optimal number of trades to perform, and the total time required before a trade is executed, we perform now a comparative static exercise where we demonstrate the various impacts in an actual numerical exercise. We will use as stock Orange and report the ensuing selling strategies in Figure 2.

**Sensitivity to the risk aversion** $\eta$. We change the value of the parameter $\eta$ with respect to the benchmark from $\eta = 4$ to $\eta = 8$. The investor is expected to liquidate her position more quickly as she becomes more impatient, in order to avoid the risk of a potential price change. In Figure 2, we compare the cumulative number of shares sold obtained with the GMM estimates ($\eta = 4$) and with $\eta = 8$. This selling strategy permits the investor to avoid part of the exposure she faces due to price changes.

**Sensitivity to the parameter** $\psi$. We expect that the larger the value of the permanent impact parameter $\psi$, the smaller the optimal number of trading periods. The intuition is that each time the investor places an order, the price is expected to decrease permanently by $\psi$ units. Thus, if the value of the parameter increases, the investor has more incentive to avoid its negative price impact by placing fewer orders. To evaluate this hypothesis, the parameter $\psi$ is halved with respect to the benchmark. From Figure 2, we observe that the decrease in the value of the parameter $\psi$ implies that the liquidation speed slightly increases at the beginning and decreases towards the end (see the slopes of the curves). As a consequence, the optimal number of trading periods increases from 5 to 6. Therefore, on average, the liquidation speed increases as larger lots are sold.

**Figure 3** displays the optimal number of trading periods as a function of $\psi$. We observe that the optimal number of trading periods is decreasing in $\psi$ and that its sensitivity to changes in the parameter value grows with smaller $\psi$. The value of the objective function is lower at the new parameterization. This change is a consequence of the increase in the fixed costs that is incurred each time an order is placed.

**Sensitivity to the parameter** $\overline{\psi}$. The parameter $\overline{\psi}$ represents a transitory fixed cost. When this parameter is constant over time, as is shown in Appendix 2, its effect on the objective function is represented by the term $-\sum_{t=1}^{m} \overline{\psi}_t n_t = -\overline{\psi} X$. Thus, $\overline{\psi}$ affects the objective function through a constant term, implying that changes in its value do not alter the optimal liquidation strategy. Rather, it only changes the level of the objective function.
Sensitivity to the parameter $\lambda$. An increase in the value of the parameter $\lambda$ indicates that the permanent effect of each unit sold has a larger (negative) impact on price. If the unitary impact is larger, the investor might prefer to liquidate smaller quantities, packaging the total initial position in more trades. This observation implies that the investor should increase the number of trading periods. We double the parameter $\lambda$ in order to evaluate this hypothesis. As a consequence, the optimal number of trades rises from 5 to 7 and, in each of the first periods, the investor liquidates approximately 5% less shares than in the benchmark case.

Figure 2 displays the percentage that has been sold under the optimal liquidation strategy over time. At the beginning and at the end of the liquidation period, we observe that the investor liquidates more slowly than in the benchmark case, while in the middle she liquidates slightly faster. The duration decreases by minutes with respect to the benchmark, implying that the liquidation speed slightly increases given the parameterization.

Figure 3 displays the optimal number of daily trading periods as a function of $\lambda$. We observe that it is increasing in $\lambda$. We also observe that the number of trading periods is much more sensitive to the parameter $\lambda$ than to the parameter $\psi$.

Sensitivity to the parameter $\bar{\lambda}$. An increase in the value of the parameter $\bar{\lambda}$ causes a reduction in its positive impact on price. This relationship exists because $\bar{\lambda}$ is negative and has a negative effect in equation (6). If the unitary (positive) impact on price is smaller, the investor might prefer to liquidate smaller quantities and to increase the number of trading periods. To evaluate this hypothesis, we increase the value of the parameter $\bar{\lambda}$ with respect to the benchmark from $\bar{\lambda} = -0.032 \cdot 10^{-3}$ to $\bar{\lambda} = -0.016 \cdot 10^{-3}$. As a consequence, the optimal number of trades increases from 5 to 6.

As discussed in the context of Table 4, the effect of $\bar{\lambda}$ on the total expected revenue is important. Changes in $\bar{\lambda}$ are therefore expected to have important consequences for the optimal selling strategy. Indeed, Figure 2 shows a much shorter duration of the overall sale as well as a larger number of trades. This finding corroborates the importance of the parameter $\bar{\lambda}$. From Figure 3, we find that the optimal number of trading periods is increasing in $\bar{\lambda}$.

Sensitivity to $\sigma_y$. We first note that $\sigma_y$ is composed of $\tau$, the waiting time at time $t$, and $\sigma$, the fundamental uncertainty of an asset’s fundamental value. In discussing the impact of $\sigma_y$, we refer to the change of the average weighting time or equivalently, the impact of a change in $\sigma$. 

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As shown in Appendix 2, the contribution of $\sigma_y$ to the objective function is of the order of magnitude $-\sigma_y^2 N$. An increase in the parameter $\sigma_y$ will decrease the objective function because the price uncertainty of the shares that remain to sell increases. To compensate for this effect, the optimizer will seek an optimal number of trades that minimizes this negative impact by choosing a smaller $N$.

To further illustrate this point, we decrease the value of the parameter $\sigma_y$ from $50.36 \cdot 10^{-3}$ (benchmark) to $25.18 \cdot 10^{-3}$. From Figure 2, we see that the liquidation speed decreases strongly during the first half of the day, where approximately 90 percent of the initial position is liquidated. The duration increases significantly. As a comparison of the benchmark curve and the new curve shows, the impact of a change in $\sigma_y$ is quite dramatic.

Figure 3 shows that the optimal number of trading periods are strongly decreasing in $\sigma_y$. The consequence of a decrease in the variance is therefore significant. The optimal number of trading periods increases from 5 to 8.

**Sensitivity to $\sigma_\xi$.** Appendix 2 shows that the contribution to the objective function is of the order of magnitude $-\sigma_\xi^2 / N$. An increase in the parameter $\sigma_\xi^2$ will decrease the objective function. Contrary to the previous discussion concerning $\sigma_y$, the optimizer will propose an increase in $N$ in order to mitigate this negative impact.

In the numerical exercise, we increase the parameter $\sigma_\xi$ from $\sigma_\xi = 4.046 \cdot 10^{-3}$ (benchmark) to $\sigma_\xi = 6.069 \cdot 10^{-3}$, keeping the other parameters at their benchmark levels. From Figure 2, we observe that the number of trading periods increases from 5 to 6, and that the investor liquidates smaller quantities in the first periods. We conclude that the liquidation speed is about as sensitive to variation in the parameter $\sigma_\xi$ as to variation in the parameter $\sigma_y$. Figure 3 displays the evolution of the optimal number of trading periods as a function of $\sigma_\xi$, revealing that the function is increasing in $\sigma_\xi$.

To summarize, the higher the impact of a given trade on the fundamental value, $\psi$, the lower the optimal number of trades, $N^{opt}$. Inversely, if the impact of volume on the fundamental value, $\lambda$, or on the temporary price impact, $\bar{\lambda}$, increases, the higher the optimal number of trades. The impact of a change in the temporary impact of a trade on the price, $\bar{\psi}$, has relatively little impact on the optimal number of trades. If there is more fundamental uncertainty, that is a higher $\sigma_y$, then the optimal number of trades increases. Inversely, more temporary uncertainty, $\sigma_\xi$, will yield a more precautious sequence of trades. It should be emphasized that changes of the last two parameters, $\sigma_y$ and $\sigma_\xi$, will lead to a particularly strong modification of the optimal trading strategy.
4.2 Optimal Trading Strategy for Time-Varying Parameters

To empirically explore the time-varying behavior of the parameters in our model, we estimate them for different hours of the day. Figure 4 displays the hourly estimated values of the parameters for Orange, a telecommunications company. We observe that, in general, the parameters are not constant throughout the day, suggesting that this fact needs to be captured by the model. There are significant differences in the parameters over a trading day. In this section, we discuss the consequences of a time-varying price impact function $\lambda_t$. We then investigate the consequences of allowing the parameters to vary over the day. The study of such a dynamic setting is particularly relevant in light of evidence suggesting that different dynamics apply at the close of the market. Our focus is on the parameter $\lambda_t$, which is particularly important because it captures the permanent price impact of the order flow. We determine the optimal liquidation when the time-varying parameter $\lambda_t$ follows the increasing hypothetical pattern displayed in the top panel of Figure 5, while the other parameters remain at benchmark levels. This pattern corresponds to the specification $\lambda_t = a + bt + ct^2$, where $a = 0.257 \cdot 10^{-3}$, $b = -0.051 \cdot 10^{-5}$, $c = 10^{-6}$, and $t$ is measured in hours with respect to 9 a.m. Intuitively, we expect that the investor liquidates larger quantities at the beginning of the day, when the price impact of the order flow is smaller, rather than late in the afternoon when it is larger.

BALOU : Comment expliquer la grande différence entre la valeur des paramètres pour lambda dans les 2 graphiques 4 et 5 ?

The bottom panel in Figure 5 displays the corresponding optimal liquidation strategy. We observe that the optimization recommends a liquidation of the position in five trades as in the benchmark case. Importantly, the main effect of introducing an increasing function for the parameter $\lambda_t$ is an increase in the liquidation speed. The total position is liquidated faster relative to the situation without time variation in the parameter. In particular, the position is liquidated during the early moments of trading, when the permanent variable impact on price of the order flow is small.

We conclude that taking even relatively shallow variation in the parameters into account may have important consequences for the market. We also performed an exercise where we allowed the parameters to change over time according to the dynamics in Figure 4. From the hourly estimates, we obtained continuous functions using a spline interpolation. We found that the position was liquidated in four trades and in less time.
4.3 Market Sentiment and Trading

An investor who plans to trade a significant position may have priors on the evolution of the market. For instance, she may wish to sell a portfolio in a falling market. In this section, we analyze the responses of our model to particular market conditions, assuming that the investor uses prior information in the calibration of her model.\textsuperscript{22} Table 5 summarizes the results of the various cases we consider. Case 1 corresponds to our benchmark, as reported in Table 4. A first belief could be that markets will sharply fall. We parameterize this belief by assuming a decrease of $\mu = -5\%$ in the fundamental value during the day. As Case 2 in Table 5 indicates, this assumption will yield a larger first trade with respect to the benchmark in Case 1, but a greater liquidation duration. This result is not exactly what one may have in mind as a fire-sale.

Next, we consider the scenarios in which the investor accounts for her prior about the future evolution of the market and also uses realistic order flow parameters. Indeed, we re-estimated the parameters of the microstructure model ($\psi, \overline{\psi}, \lambda, \overline{\lambda}$) as well as the duration between trades $\tau$ corresponding to the five days with the largest price drop.\textsuperscript{23} In Table 5, under Case 3, we represent those figures corresponding to the situation where only $\mu$ and $\tau$ adjust to the new market conditions. Because of the increased speed of the order flow, we find that the position should be liquidated much faster. For instance, in Case 2, we find that the initial duration of 0.53 hour for Alcatel is reduced to 0.25 hour. The speed of the order flow seems, therefore, to be an important variable for the optimal trading strategy.

Next, we assume that our investor increments her information set and also uses the $\psi$ and $\overline{\psi}$ parameters corresponding to the worst possible day (Case 4). Unambiguously, the initial number of shares traded increases and the duration diminishes across all companies. Lastly, in Case 5, our investor uses the full information set: she believes that markets will fall sharply, that markets experience a strong increase in trading activity, resulting in a decrease in the time between trades $\tau$, and that the parameters of the price dynamics switched to a bearish market sentiment. As Case 5 documents, the size of the first trade ranges between 73\% and 90\% of the amount to be traded. This order of magnitude was also found in Case 2. However, the duration is significantly shorter than for the benchmark (Case 1) and the other situations. This appears to correspond to what one intends by the expression fire-sale.

As this section documents, market sentiment may be added to our model by incorporating either beliefs about the price behavior or by adjusting the calibration to a

\textsuperscript{22}In practice, trading floors are often oriented by an economist who will give indications if a market should be expected to rise or fall.

\textsuperscript{23}These days are not necessarily consecutive. We also estimated the parameters for the five days with the largest price increases. The results are qualitatively comparable with those reported here.
worst case scenario. Such market sentiment may significantly affect the optimal trading strategy.

5 Conclusion

In this paper, we consider an investor who desires to trade a large number of shares. Facing the choice between executing a block trade versus optimal liquidation through an electronic system, we show that an investor would choose block trading subsequent to high trading volumes, which presumably characterize liquid markets. Inversely, an investor would choose to trade through an electronic system on mostly normal days.

The thrust of our work is the extension and estimation of a realistic microstructure model whose price dynamic serves as an input to an optimal trading algorithm. The extension of the microstructure model accounts for the possibility of differentiated parameters for the buy and sell sides. This microstructure model contains permanent and transitory components. In addition, we allow for quantity-related price impacts.

The analysis of the recommended optimal trading strategies reveals that sell orders will be traded more aggressively than buy orders. A comparative static exercise demonstrates the importance of the various parameters and shows that changes in volatility have a particularly important impact. The observation that volatility plays an important role is corroborated in an exercise where we allow for market sentiment by assuming that the investor has bullish or bearish beliefs. Indeed, we show that the mere belief that markets will fall (which translates into a negative drift term, keeping all other parameters constant) will not significantly affect the trading strategy. Adjustments of the microstructure-model parameters and the volatility are also required to yield significant changes in the pattern of trades.
6 Appendices

6.1 Appendix 1: Proof of Proposition 1.

In this appendix, we obtain the solution to the optimization problem described in problem (2) when the price dynamics are given by equations (6) and (7). Replacing \( p_t \) from equation (7) in the total income of liquidation, \( \sum_{t=1}^{N} n_t p_t \), we have

\[
\sum_{t=1}^{N} n_t p_t = \sum_{t=1}^{N} n_t m_t + \sum_{t=1}^{N} n_t (\overline{\psi}_t - \overline{\lambda}_t n_t + \xi_t). \tag{14}
\]

Since equation (6) can be written as \( m_t = m_{t-1} + h_t - \lambda_t n_t \), where \( h_t = \mu - \psi_t - \sigma_t \tau^{1/2} \omega_t \), we obtain

\[
\sum_{t=1}^{N} n_t m_t = m_0 X + \sum_{t=1}^{N} (h_t - \lambda_t n_t) \left( \sum_{j=t}^{N} n_j \right).
\]

Replacing this result in equation (14), it follows that

\[
\sum_{t=1}^{N} n_t p_t = m_0 X + \sum_{t=1}^{N} \left[ (h_t - \lambda_t n_t) \left( \sum_{j=t}^{N} n_j \right) \right] - \sum_{i=1}^{N} \overline{\psi}_i n_t + \sum_{t=1}^{N} n_t (\overline{\lambda}_t n_t + \xi_t).
\]

Taking the expectation and calculating the variance of \( \sum_{t=1}^{N} n_t p_t \), it holds that

\[
E \left[ \sum_{t=1}^{N} n_t p_t \right] = p_0 X + \sum_{t=1}^{N} E [h_t] \left( \sum_{j=t}^{N} n_j \right) - \sum_{t=1}^{N} \lambda_t n_t \left( \sum_{j=t}^{N} n_j \right) - \sum_{t=1}^{N} \overline{\psi}_t n_t - \sum_{t=1}^{N} \overline{\lambda}_t n_t^2,
\]

and

\[
V \left[ \sum_{t=1}^{N} n_t p_t \right] = \sum_{t=1}^{N} V [h_t] \left( \sum_{j=t}^{N} n_j \right)^2 + \sum_{t=1}^{N} n_t^2 \sigma_{\xi,t}^2.
\]

Replacing \( E \left[ \sum_{t=1}^{N} n_t p_t \right] \) and \( V \left[ \sum_{t=1}^{N} n_t p_t \right] \) in the objective function (2), grouping common terms, and using the results: \( E [h_t] = \mu - \psi_t \) and \( V [h_t] = \sigma_t^2 \tau \), we have

\[
E \left[ \sum_{t=1}^{N} n_t p_t \right] - \eta V \left[ \sum_{t=1}^{N} n_t p_t \right] = p_0 X + \sum_{t=1}^{N} \left[ (\mu - \psi_t) \left( \sum_{j=t}^{N} n_j \right) \right] - \sum_{t=1}^{N} \lambda_t n_t \left( \sum_{j=t}^{N} n_j \right) - \sum_{t=1}^{N} \overline{\psi}_t n_t - \sum_{t=1}^{N} \overline{\lambda}_t n_t^2 - \eta \sum_{t=1}^{N} \sigma_{\xi,t}^2 \left( \sum_{j=t}^{N} n_j \right)^2 - \eta \sum_{t=1}^{N} \sigma_{\xi,t}^2 n_t^2. \tag{15}
\]
Taking the derivative with respect to \( n_k \), we have

\[
\frac{\partial}{\partial n_k} \sum_{t=1}^{N} \left[ (\mu - \psi_t^-) \left( \sum_{j=t}^{N} n_j \right) \right] = \sum_{t=1}^{k} (\mu - \psi_t^-) = k\mu - \sum_{t=1}^{k} \psi_t^-,
\]

\[
\frac{\partial}{\partial n_k} \sum_{t=1}^{N} \left[ -\sum_{t=1}^{N} \lambda_t^- n_t \left( \sum_{j=t}^{N} n_j \right) \right] = -2\lambda_k^- n_k - \sum_{t=1}^{k-1} \lambda_t^- n_t,
\]

\[
\frac{\partial}{\partial n_k} \sum_{t=1}^{N} \left[ -\sum_{t=1}^{N} \psi_t^- n_t \right] = -\psi_k^-,
\]

\[
\frac{\partial}{\partial n_k} \sum_{t=1}^{N} \left[ -\sum_{t=1}^{N} \lambda_t^- n_t^2 \right] = -2\lambda_k^- n_k,
\]

\[
\frac{\partial}{\partial n_k} \sum_{t=1}^{N} \left[ -\eta \sum_{t=1}^{N} \sigma_t^2 \left( \sum_{j=t}^{N} n_j \right)^2 \right] = -\sum_{t=1}^{N} \left( \sum_{j=1}^{\min(t,k)} 2\eta \sigma_j^2 \right) n_t,
\]

\[
\frac{\partial}{\partial n_k} \sum_{t=1}^{N} \left[ -\eta \sum_{t=1}^{N} \sigma_{\xi,t}^2 n_t^2 \right] = -2\eta \sigma_{\xi,k}^2 n_k.
\]

Thus, the total derivative of \( L = E \left[ \sum_{t=1}^{N} n_t p_t \right] - \eta V \left[ \sum_{t=1}^{N} n_t p_t \right] \) is given by

\[
\frac{\partial L}{\partial n_k} = - \sum_{t=1}^{k} \psi_t^- + k\mu - 2\lambda_k^- n_k - \sum_{t=1}^{k-1} \lambda_t^- n_t - \psi_k^- - 2\lambda_k^- n_k - \sum_{t=1}^{N} \left( \sum_{j=1}^{\min(t,k)} 2\eta \sigma_j^2 \right) n_t - 2\eta \sigma_{\xi,k}^2 n_k,
\]

for \( k = 1, \cdots, N \). Therefore, the solution to the optimization problem (2) is given by the solution to a system of linear equations given by imposing \( \frac{\partial L}{\partial n_k} = 0 \), subject to the restrictions: \( \sum_{t=1}^{N} n_t = X \), and \( 0 \leq n_k \leq X \) for \( k = 1, \cdots, N \). It follows that the optimization problem (2) for the price dynamics boils down to a solution of the Phase-I of the Simplex algorithm.

### 6.2 Appendix 2: Comparative Statics.

In order to better understand the patterns of the optimal number of trades as the parameters change, we consider a simplified selling strategy with equally trades and constant parameters. For notational ease, we have dropped the superscript “−” from the parameters. Decomposing \( E \left[ \sum_{t=1}^{N} n_t p_t \right] - \eta V \left[ \sum_{t=1}^{N} n_t p_t \right] \) from equation (15) and
replacing \( n_t \) by \( X/N \), we obtain the influence of each parameter:

\[
\sum_{t=1}^{N} \left( \mu - \psi_t \right) \left( \sum_{j=t}^{N} n_j \right) = \sum_{t=1}^{N} \left( \mu - \psi \right) \left( \sum_{j=t}^{N} \frac{X}{N} \right) \\
= \left( \mu - \psi \right) \frac{X}{N} \sum_{t=1}^{N} (N - t + 1) \\
= \frac{1}{2} X (\mu - \psi)(N + 1),
\]

\[
\sum_{t=1}^{N} \lambda_t n_t \left( \sum_{j=t}^{N} n_j \right) = \sum_{t=1}^{N} \lambda \frac{X}{N} \left( \sum_{j=t}^{N} \frac{X}{N} \right) = -\lambda \left( \frac{X}{N} \right)^2 \sum_{t=1}^{N} (N - t + 1) \\
= -\frac{1}{2} \lambda X^2 \frac{N + 1}{N},
\]

\[
\sum_{t=1}^{N} \psi_t n_t = \sum_{t=1}^{N} \psi \frac{X}{N} = -X \psi,
\]

\[
\sum_{t=1}^{N} \bar{\lambda}_t n_t^2 = \sum_{t=1}^{N} \bar{\lambda} \left( \frac{X}{N} \right)^2 = -X^2 \bar{\lambda} \frac{1}{N},
\]

\[
-\eta \sum_{t=1}^{N} \sigma_t^2 \tau \left( \sum_{j=t}^{N} n_j \right)^2 = -\eta \sum_{t=1}^{N} \sigma^2 \tau \left( \sum_{j=t}^{N} \frac{X}{N} \right)^2 = -\eta \sum_{t=1}^{N} \sigma^2 \tau \left[ \frac{X}{N} (N - t + 1) \right]^2 \\
= -\eta \sigma^2 \tau \left( \frac{X}{N} \right)^2 \sum_{t=1}^{N} [N^2 - 2N(t - 1) + (t - 1)^2] \\
= -\frac{1}{6} \eta \sigma^2 \tau X^2 \frac{(N + 1)(2N + 1)}{N},
\]

\[
-\eta \sum_{t=1}^{N} \sigma_t^2 \xi_t n_t^2 = -\eta \sum_{t=1}^{N} \sigma^2 \xi \left( \frac{X}{N} \right)^2 = -\eta \sigma^2 \xi \frac{X^2}{N}.
\]

These computations demonstrate that, for the strategy involving trades with constant size, the utility is a complex function of the parameters:

\[
E \left[ \sum_{t=1}^{N} n_t \psi_t \right] - \eta V \left[ \sum_{t=1}^{N} n_t \psi_t \right] = 1 + X(\mu - \psi) \frac{(N + 1)}{2} - \lambda X^2 \frac{N + 1}{2N} - X \bar{\psi} - X^2 \bar{\lambda} \frac{1}{N} \\
- \eta \left[ \frac{1}{6} \sigma^2 \tau X^2 \frac{(N + 1)(2N + 1)}{N} + \sigma^2 \xi \frac{X^2}{N} \right].
\]

In particular, we notice that the number of trades appears in a highly non-linear manner.
References


Captions

Table 1: This table presents descriptive statistics of block trades (called ‘applications’) for 37 companies. Mean, min, max, std represent the average, minimum, maximum, and standard deviation, respectively, over the average of a given statistic computed for 37 companies. The first block of figures refers to the entire sample (1996-2003) and the lower two blocks to subsamples.

Table 2: This table reports the estimates from a panel regression relating the number of applications on a given day to several explanatory variables. The variables are as follows: \( nbApplic \), the daily number of block trades; \( nbTr \), the total number of trades; \( V \), a measure of daily realized volatility measured as the sum of squared five minute returns; \( Vol \), the daily traded volume (in Euros); \( R \), the daily return for a given company; \( R^- \), negative returns, and \( R^{e-} \) denoting extremely negative returns (worse than \(-4\%\)). The last two explanatory variables are included to account for leverage effects and extreme market corrections.

Table 3: This table reports GMM estimates of the model, keeping parameters in equations (4) and (5) constant. All parameters have been multiplied by \(10^3\). The last 3 columns correspond to averages, min and max of the estimates obtained from a cross section of 40 stocks. The numbers in parentheses represent t-statistics for the null hypothesis that a parameter is equal to 0. The \( J \) statistic is a broad measure of goodness of fit. Parameters labeled with “+” correspond to the sell side and those labeled with “−” to the buy side. \( \psi \) and \( \lambda \) are permanent fixed costs respectively variable costs. \( \bar{\psi} \) and \( \bar{\lambda} \) represent transitory fixed costs and variable costs, respectively. \( \sigma_{\xi} \) is the standard deviation of the rounding error. \( \sigma_{y} \) is the standard deviation of the permanent component of the prices. The total standard deviation would by \( \tau^{1/2}\sigma_{y} \), where \( \tau \) is the time that passes between two given trades. The model is estimated by considering the continuous flow of transactions. The samples contain at least 50,000 observations.

Table 4: This table reports characteristics of the optimal liquidation strategy with constant parameters. \( n_{t_1} \) is the number of shares sold at \( t_1 \) as a percentage of the total number of shares to be liquidated. Duration is defined in equation (13). \( N^{opt} \) is the optimal number of trading periods in the sales strategy. We have the following definitions: \( \psi^- \) effect = \( 100 \left[ \sum_{t=1}^{N} \psi^- t \left( \sum_{j=t}^{N} n_j \right) \right] / p_0 X \), \( \bar{\psi}^- \) effect = \( 100 \left[ \sum_{t=1}^{N} \bar{\psi}^- t n_t \right] / p_0 X \), \( \lambda^- \) effect = \( 100 \left[ \sum_{t=1}^{N} \lambda^- t n_t \left( \sum_{j=t}^{N} n_j \right) \right] / p_0 X \), \( \bar{\lambda}^- \) effect = \( 100 \left[ \sum_{t=1}^{N} \bar{\lambda}^- t^2 n_t^2 \right] / p_0 X \).

Table 5: This table reports characteristics of the optimal liquidation strategy under alternative scenarios. \( n_{t_1} \), duration, and \( N^{opt} \) are defined as in Table 4. Case 1 is the benchmark case that was considered in Table 4. In Case 2, the investor incorporates
beliefs about the evolution of the price over the next day given a market crash of $-5\%$. In Case 3, the investor accounts for an increased trading activity ($\tau$, the average duration between trades is increased). In Case 4, the investor believes that the impacts from $\psi$ and $\overline{\psi}$ correspond to bad days, and that the speed of the order flow $\tau$ remains increased. In Case 5, the investor believes a crash will take place $\mu = -5\%$, all other parameters remain adjusted to reflect worst case scenarios.

Figure 1: We present the optimal amounts of shares to be traded over the day for Orange. The investor must trade 6966 shares of stock (10% of the average daily volume). Stars represent the number of shares sold and triangles represent the number of shares bought.

Figure 2: We trace the cumulative amount of shares sold for various parameter settings. The company considered is Orange. The benchmark case is defined by parameters $X = 6966$, $T = 8.5$ hours, $\eta = 4$, $\psi^- = 2.199 \cdot 10^{-3}$, $\overline{\psi}^- = 5.448 \cdot 10^{-3}$, $\lambda^- = 0.042 \cdot 10^{-3}$, $\overline{\lambda}^- = -0.032 \cdot 10^{-3}$, $\sigma_y = 50.357 \cdot 10^{-3}$, and $\sigma_\xi = 4.046 \cdot 10^{-3}$. The benchmark curve is labeled GMM. Deviations are obtained by changing one parameter after another. $\eta$ increased to 8, $\psi$, $\overline{\lambda}$, $\sigma_y$ are halved, $\lambda$ is doubled and $\sigma_\xi$ increased by a factor of 1.5. We use the average time that passed between trades for $\tau$.

Figure 3: The various panels represent the effect of changing one of the microstructure parameters on the number of daily sell orders. The company considered is Orange. We use the numbers presented in Figure 2 as benchmark parameters. All parameters have been multiplied by $10^3$.

Figure 4: The evolution of the various parameters over the day is presented here. The stars represent the point estimates and the dashed lines correspond to the 90% confidence interval. All parameters have been multiplied by $10^3$.

Figure 5: The upper plot presents the function $\lambda(t)$ as estimated over the day and the lower plot presents the optimal liquidation over a given day.
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<td>traded per day non applic. (%)</td>
<td>traded per day non applic. (%)</td>
</tr>
<tr>
<td></td>
<td>min 268876.2</td>
<td>min 126144.2</td>
<td>min 649778.4</td>
</tr>
<tr>
<td></td>
<td>max 2940872.6</td>
<td>max 543762.1</td>
<td>max 5357.14</td>
</tr>
<tr>
<td></td>
<td>std 64523.1</td>
<td>std 75719.9</td>
<td>std 5357.14</td>
</tr>
</tbody>
</table>

**Table 1**: Descriptive statistics on block trades
<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>p-value</th>
<th>Parameter</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nbTr_{t-1}$</td>
<td>0.0020</td>
<td>0.00</td>
<td>0.0034</td>
<td>0.00</td>
</tr>
<tr>
<td>$nbTr_{t-2}$</td>
<td>0.0011</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$V_{t-1}$</td>
<td>0.0009</td>
<td>0.74</td>
<td>0.0051</td>
<td>0.04</td>
</tr>
<tr>
<td>$V_{t-2}$</td>
<td>0.0080</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$vol_{t-1}$</td>
<td>0.0159</td>
<td>0.00</td>
<td>0.0226</td>
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<td>$vol_{t-2}$</td>
<td>0.0159</td>
<td>0.00</td>
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<tr>
<td>$R_{t-1}$</td>
<td>0.2656</td>
<td>0.00</td>
<td>0.2188</td>
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<td>$R_{t-2}$</td>
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<td>0.89</td>
<td>-</td>
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<tr>
<td>$R_{t-1}^{-}$</td>
<td>-0.5942</td>
<td>0.00</td>
<td>-0.5315</td>
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</tr>
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<td>$R_{t-2}^{-}$</td>
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<tr>
<td>$R_{t-1}^{e}$</td>
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<tr>
<td>$R_{t-2}^{e}$</td>
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<td>0.19</td>
<td>-</td>
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<tr>
<td>$R^2$</td>
<td>0.259</td>
<td></td>
<td>0.255</td>
<td></td>
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<tr>
<td>(Nobs, Nvars)</td>
<td>(73257, 49)</td>
<td></td>
<td>(73294, 43)</td>
<td></td>
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Table 2: Panel regressions explaining the number of applications
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<th></th>
<th>Individual estimates</th>
<th>All firms</th>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Min)</td>
<td>(Max)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\psi^+$</td>
<td>1.246</td>
<td>11.033</td>
<td>5.107</td>
</tr>
<tr>
<td></td>
<td>(11.00)</td>
<td>(33.39)</td>
<td>(76.55)</td>
</tr>
<tr>
<td>$\psi^-$</td>
<td>1.461</td>
<td>11.033</td>
<td>5.107</td>
</tr>
<tr>
<td></td>
<td>(12.72)</td>
<td>(33.39)</td>
<td>(76.55)</td>
</tr>
<tr>
<td>$\overline{\psi}^+$</td>
<td>4.151</td>
<td>8.992</td>
<td>5.432</td>
</tr>
<tr>
<td></td>
<td>(54.1)</td>
<td>(25.20)</td>
<td>(88.23)</td>
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<tr>
<td>$\overline{\psi}^-$</td>
<td>4.373</td>
<td>8.992</td>
<td>5.432</td>
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<tr>
<td></td>
<td>(55.86)</td>
<td>(25.20)</td>
<td>(88.23)</td>
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<tr>
<td>$\lambda^+$</td>
<td>0.054</td>
<td>0.052</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>(7.53)</td>
<td>(5.87)</td>
<td>(2.06)</td>
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<td>$\lambda^-$</td>
<td>0.048</td>
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<tr>
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<td>(5.38)</td>
<td>(3.95)</td>
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<tr>
<td>$\overline{\lambda}^+$</td>
<td>-0.035</td>
<td>-0.026</td>
<td>-0.046</td>
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<tr>
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<td>(5.49)</td>
<td>(3.56)</td>
<td>(0.58)</td>
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<td>$\overline{\lambda}^-$</td>
<td>-0.031</td>
<td>-0.026</td>
<td>-0.046</td>
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<tr>
<td></td>
<td>(3.91)</td>
<td>(3.14)</td>
<td>(0.589)</td>
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<td>$\sigma_y$</td>
<td>22.446</td>
<td>50.357</td>
<td>76.232</td>
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<tr>
<td></td>
<td>(13.71)</td>
<td>(13.30)</td>
<td>(14.47)</td>
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<tr>
<td>$\sigma_\xi$</td>
<td>2.543</td>
<td>4.046</td>
<td>7.461</td>
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<tr>
<td></td>
<td>(93.17)</td>
<td>(24.57)</td>
<td>(67.19)</td>
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<td>$J\text{ stat}$</td>
<td>5.832</td>
<td>1.088</td>
<td>2.107</td>
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<tr>
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<td>3.173</td>
<td>2.107</td>
<td>1.200</td>
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<td>$p\text{ value}$</td>
<td>0.323</td>
<td>0.885</td>
<td>0.955</td>
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<td>0.702</td>
<td>0.885</td>
<td>0.955</td>
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</table>

Table 3: GMM estimates of the model with constant parameters
<table>
<thead>
<tr>
<th></th>
<th>Alcatel</th>
<th>France Tel.</th>
<th>Orange</th>
<th>Sodexho</th>
<th>Suez</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_t$ (%)</td>
<td>83.2412</td>
<td>72.9076</td>
<td>76.8118</td>
<td>78.4821</td>
<td>72.5812</td>
</tr>
<tr>
<td>Duration $D$</td>
<td>0.3398</td>
<td>0.4497</td>
<td>0.5072</td>
<td>0.4618</td>
<td>0.5316</td>
</tr>
<tr>
<td>$N_{opt}$</td>
<td>5</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>6</td>
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<tr>
<td>$\psi_-$ effect</td>
<td>0.0397</td>
<td>0.0395</td>
<td>0.0453</td>
<td>0.0644</td>
<td>0.0425</td>
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<tr>
<td>$\bar{\psi}_-$ effect</td>
<td>0.1020</td>
<td>0.0449</td>
<td>0.0726</td>
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<tr>
<td>$\lambda_-$ effect</td>
<td>19.5409</td>
<td>4.2284</td>
<td>3.8517</td>
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<tr>
<td>$\bar{\lambda}_-$ effect</td>
<td>-10.7213</td>
<td>-1.6088</td>
<td>-1.9920</td>
<td>-0.2709</td>
<td>-0.0523</td>
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</table>

Table 4: Characteristics of the optimal liquidation strategy with constant parameters
<table>
<thead>
<tr>
<th>Case</th>
<th>$n_t$</th>
<th>Alcatel</th>
<th>France Tel.</th>
<th>Orange</th>
<th>Sodexho</th>
<th>Suez</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$n_t$</td>
<td>83.241</td>
<td>72.908</td>
<td>76.812</td>
<td>78.482</td>
<td>72.581</td>
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<tr>
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<td>Duration $D$</td>
<td>0.340</td>
<td>0.450</td>
<td>0.507</td>
<td>0.462</td>
<td>0.532</td>
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<tr>
<td></td>
<td>$N^{opt}$</td>
<td>5</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Case 2</td>
<td>$n_t$</td>
<td>87.424</td>
<td>73.612</td>
<td>90.677</td>
<td>83.837</td>
<td>74.704</td>
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<tr>
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<td>0.534</td>
<td>0.722</td>
<td>0.396</td>
<td>0.687</td>
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<td>$N^{opt}$</td>
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<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Case 3</td>
<td>$n_t$</td>
<td>68.716</td>
<td>47.896</td>
<td>60.171</td>
<td>55.741</td>
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<td>5</td>
<td>3</td>
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<tr>
<td>Case 4</td>
<td>$n_t$</td>
<td>76.576</td>
<td>71.768</td>
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<td>69.113</td>
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<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>Case 5</td>
<td>$n_t$</td>
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<td>90.688</td>
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<td>Duration $D$</td>
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<td>0.157</td>
<td>0.078</td>
<td>0.135</td>
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<tr>
<td></td>
<td>$N^{opt}$</td>
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<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5: Characteristics of the optimal liquidation strategy under alternative scenarios
Figure 1: Optimal liquidation/purchase strategy
Figure 2: Sold percentage of the initial position
Figure 3: Number of daily trades depending on microstructure parameters
Figure 4: Evolution of the parameters over the day
Figure 5: Function $\lambda(t)$ over the day and optimal liquidation strategy