Lorentz's model with dissipative collisions

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Abstract

Propagation of a particle accelerated by an external field through a scattering medium is studied within the generalized Lorentz model allowing inelastic collisions. Energy losses at collisions are proportional to $(1-\alpha^2)$, where $0 \leq \alpha \leq 1$ is the restitution coefficient. For $\alpha = 1$ (elastic collisions) there is no stationary state. It is proved in one dimension that when $\alpha < 1$ the stationary state exists . The corresponding velocity distribution changes from a highly asymmetric half-gaussian ($\alpha = 0$) to an asymptotically symmetric distribution $\sim \exp[-(1-\alpha)v^4/2]$, for $\alpha \to 1$. The identical scaling behavior in the limit of weak inelasticity is derived in three dimensions by a self-consistent perturbation analysis, in accordance with the behavior of rigorously evaluated moments. The dependence on the external field scales out in any dimension, predicting in particular the stationary current to be proportional to the square root of the external acceleration.

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I. INTRODUCTION

The object of the present paper is to study the propagation of a particle through a medium composed of immobile spherically symmetric scatterers (of infinite mass), randomly distributed in space with some number density n. Between collisions the particle moves with acceleration $\mathbf{a}$, acted upon by a constant and uniform external field. At binary collisions its velocity $\mathbf{v}$ is instantaneously transformed according to the law

$$\mathbf{v} \quad \rightarrow \quad \mathbf{v}' = \mathbf{v} - (1+\alpha)(\mathbf{v}\cdot\hat{\sigma})\hat{\sigma} \tag{1}$$

where $0 \leq \alpha \leq 1$ is the so called restitution coefficient, and $\hat{\sigma}$ denotes the unit vector along the line passing through the centers of colliding particles at the moment of impact. Whereas the projection of velocity $\mathbf{v}$ on the direction tangent to the surface of the scatterer at the point of contact is not changed, the component $(\mathbf{v}\cdot\hat{\sigma})$ along the normal gets multiplied by $(-\alpha)$. In the extreme case of $\alpha = 0$, the postcollisional velocity reduces to the tangential component.

When $\alpha = 1$, collisions (1) are perfectly elastic, and we recover the situation of the classical Lorentz model of electric conductivity in metals [1]. It can be proved that there is no stationary state in this case [3], [4]. This is because the moving particle does not loose energy at collisions, and gets asymptotically heated by the field beyond any bounds. In particular, its mean kinetic energy diverges as $\sim t^{2/3}$ when the time $t \rightarrow \infty$.

The aim of this work is to show, that in the case of inelastic collisions (1) the stationary state becomes possible in the whole range of $0 \le \alpha < 1$. Once $\alpha < 1$, there occurs dissipation of the kinetic energy $E = mv^2/2$ at encounters, as in accordance with (1) E suffers the transformation

$$E \quad \rightarrow \quad E' = E - m(1-\alpha^2)(v \cdot \hat{\sigma})^2/2 \tag{2}$$

The inelastic dissipation (2) turns out to suffice to balance the energy flow from the external field.

The basis for the subsequent analysis is the linear Boltzmann equation satisfied by the probability density $f(\mathbf{v};t)$ for finding the propagating particle with velocity $\mathbf{v}$ at time t. In writing this equation one must take into account that the jacobian of transformation (1) equals α (dissipative collisions are contracting the volume in the velocity space), and that the inverse transformation is obtained by replacing α in (1) by α^{-1}. The kinetic equation reads

$$\left(\frac{\partial}{\partial t} + \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{v};t) = \frac{|\mathbf{v}|}{\lambda \pi} \int d\hat{\sigma} (\hat{\mathbf{v}} \cdot \hat{\sigma}) \theta(\hat{\mathbf{v}} \cdot \hat{\sigma}) \left\{ \alpha^{-2} f[\mathbf{v} - (1+\alpha^{-1}(\mathbf{v} \cdot \hat{\sigma})\hat{\sigma}); t] - f(\mathbf{v};t) \right\} \tag{3}$$

Here $\lambda = (\pi n R^2)^{-1}$ denotes the mean free path (R is the sum of the particle and the scatterer radii), $\hat{\mathbf{v}}$ is the unit velocity vector, and the integration with respect to $d\hat{\sigma}$ spreads over a unit sphere (solid angle). In the gain term (due to collisions) the factor α^{-2} compensates for the contraction of the velocity space and gives the proper value to the precollisional velocity of approach $(\mathbf{v} \cdot \hat{\sigma})/\alpha$. Owing to this factor the velocity integral of the collision term vanishes, which permits to deduce from (3) the continuity equation.

A convenient equivalent form of equation (3) reads

$$\left(\frac{\partial}{\partial t} + \mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{v};t) = \frac{|\mathbf{v}|}{\lambda}\left\{\alpha^{-2}\int \frac{d\hat{\mathbf{n}}}{4\pi} f\left[\frac{1+\alpha}{2\alpha}|\mathbf{v}|\hat{\mathbf{n}} - \frac{1-\alpha}{2\alpha}\mathbf{v};t\right] - f(\mathbf{v};t)\right\} \tag{4}$$

It can be obtained from (3) by changing the angular integration variables from $d\hat{\sigma} = sin\psi d\psi d\phi$ (where $cos\psi = (\hat{\mathbf{v}}\cdot\hat{\sigma})$) to $d\hat{\mathbf{n}} = sin\chi d\chi d\phi$, with $\chi = 2\psi$. When $\alpha = 1$, we recover the kinetic equation derived originally by Lorentz [1],

Our work is closely related to the study of spontaneous percolation in three dimensions of Wilkinson and Edwards [2]. The dynamics of particles falling under gravity through a fixed scattering medium was analyzed therein on the basis of a Boltzmann-like equation with a phenomenological scattering function. The perturbative methods developed in [2] permitted to analyze the low inelasticity limit leading to the same qualitative predictions as those derived here. The novelty of our contribution consists mainly in proving by an explicit construction the existence of a stationary state in one dimension, and in developing an original self-consistent perturbative expansion in three dimensions, inspired by the scaling structure appearing in one dimension. It is to be stressed that the rigorous one-dimensional results firmly support the predictions of the perturbative analysis in three dimensions, as the properties of the system in both cases are qualitatively the same. We could also calculate rigorously some of the moments of the stationary distribution in three dimensions. Their asymptotic behavior in the elastic limit coincides with that predicted by the proposed perturbation scheme. In Section II the dimensional analysis of the stationary solution to the kinetic equation (4) is performed. Then, in Section III the one-dimensional version of the model is considered. We prove therein the existence of a stationary solution and we analyze its dependence on parameter α. Section IV is devoted to the extension of our results to three dimensions. This is obtained in the low inelasticity limit $\alpha \to 1$ by a self-consistent construction of a stationary state with appropriate scaling behavior. The paper ends with concluding comments.

II. DIMENSIONAL ANALYSIS

It is as usual instructive to formulate the theory in terms of dimensionless variables. We thus put

$$\mathbf{v} = \sqrt{a\lambda}\mathbf{u} \tag{5}$$

where $a = |\mathbf{a}|$. Denoting by $F(\mathbf{u})$ the stationary dimensionless distribution we find from (4) the equation

$$\hat{\mathbf{a}} \cdot \frac{\partial}{\partial \mathbf{u}} F(\mathbf{u}) = |\mathbf{u}| \left\{ \alpha^{-2} \int \frac{d\hat{\mathbf{n}}}{4\pi} F\left[\frac{1+\alpha}{2\alpha} |\mathbf{u}| \hat{\mathbf{n}} - \frac{1-\alpha}{2\alpha} \mathbf{u} \right] - F(\mathbf{u}) \right\} \tag{6}$$

The amplitude a of the external field disappeared from equation (6). Only the unit vector $\hat{\mathbf{a}}$ shows there, indicating the direction of acceleration. Hence, the field dependence of the moments of distribution F can be obtained by the velocity scaling (5). In particular

$$< \mathbf{v} >= \sqrt{a\lambda} < \mathbf{u} >\sim \sqrt{a}, \quad < v^2 >\sim a \tag{7}$$

The fact that the stationary particle current is proportional to the square root of the field is of no surprise. The immobile scatterers represent a zero-temperature medium, and the only energy scale comes from the acceleration a. The relations (7) could be thus predicted on purely dimensional grounds. At fixed restitution coefficient α, one cannot expect the linear response even for very weak fields.

The above conclusions apply in any dimension. In the study of the stationary state we shall use in the coming sections the dimensionless form (6) of the kinetic equation.

III. ONE-DIMENSIONAL STATIONARY VELOCITY DISTRIBUTION

In one dimension equation (6) takes the form

$$\frac{d}{du}F(u) = |u| \left[\alpha^{-2}F(-\alpha^{-1}u) - F(u)\right] \tag{8}$$

Putting

$$F(u) \equiv G(u|u|/2) \tag{9}$$

we find

$$G'(s) + G(s) = \alpha^{-2}G(-\alpha^{-1}s), \tag{10}$$

where G' denotes the derivative of G. Applying to (10) the Fourier transformation one finds

$$\hat{G}(k) = (1+ik)^{-1}\hat{G}(-\alpha^2 k) = \hat{G}(0)\prod_{r=0}^{\infty}[1+i(-\alpha^2)^r k]^{-1} \tag{11}$$

where

$$\hat{G}(k) = \int ds e^{-iks} G(s)$$

For $0 \leq \alpha < 1$ the infinite product converges which proves the existence of the stationary state. This is an important conclusion indeed, showing the fundamental difference with respect to the Lorentz model with elastic collisions. Clearly, an arbitrary degree of inelasticity suffices to balance the energy flow from the external field.

A particularly simple result is found for perfectly inelastic collisions. Indeed, when $\alpha = 0$ equation (11) reduces to

$$\hat{G}(k) = \frac{\hat{G}(0)}{(1+ik)} \tag{12}$$

Inverting the Fourier transformation and using (9) we find

$$F(u) = \theta(u)\sqrt{\frac{2}{\pi}}\exp(-u^2/2) \tag{13}$$

In one dimension each collision with $\alpha = 0$ completely stops the particle, dissipating the whole energy absorbed from the field. So, in a stationary flow the velocities oriented against the field are not possible (the θ factor in (13)). It is quite remarkable that the half space of possible velocities gets a gaussian weight.

Let us turn now to equation (10). It is equivalent to the system of two coupled equations

$$G'_+(s) + G_-(s) = -\alpha^{-2} G_-(s/\alpha^2) \tag{14}$$

$$G'_-(s) + G_+(s) = \alpha^{-2} G_+(s/\alpha^2)$$

where $G_+(s) = [G(s) + G(-s)]/2$ and $G_-(s) = [G(s) - G(-s)]/2$ are the symmetric and antisymmetric parts of G, respectively. The second relation in (14) can be solved for $G_-(s)$ yielding

$$G_-(s) = \int_s^{s\alpha^{-2}} dz\, G_+(z) \tag{15}$$

We then get from (14) a closed equation for the symmetric part

$$G'_+(s) = -\int_s^{s\alpha^{-2}} dz\, G_+(z) - \alpha^{-2} \int_{s\alpha^{-2}}^{s\alpha^{-4}} dz\, G_+(z) \tag{16}$$

In order to study the low inelasticity limit we define a small parameter $\epsilon = 1 - \alpha$. When $\epsilon \ll 1$, equation (16) takes the simple asymptotic form $G'_+(s) = -4\epsilon s G_+(s)$, and we find

$$G_+(s) = C\epsilon^{1/4}\exp(-2\epsilon s^2) \tag{17}$$

Using then the relations (15) and (9) we can determine the asymptotic form of the velocity distribution for weakly inelastic scattering. It reads

$$F(u) = \frac{C\epsilon^{1/4}}{\sqrt{2}}(1 + \epsilon u|u|)\exp(-\epsilon u^4/2), \quad \epsilon \ll 1 \tag{18}$$

The value of constant C is independent of ϵ and follows form the normalization condition

$$C^{-1} = \int dw \exp(-2w^4)$$

The dominant term in (18) is symmetric and depends on the scaled variable $w = \epsilon u^4$. It implies the divergence $\sim \epsilon^{-1/2}$ of the kinetic energy $< u^2 >$ for $\epsilon \to 0$. This divergence reflects the disappearance of the stationary state in the case of elastic collisions. In fact, the kinetic equation (8) permits to calculate directly some of the moments of distribution F, and to predict divergences in the elastic limit. An interesting example is provided by the recurrence relation

$$< u^j|u|^j >= \frac{2j-1}{1-(-1)^j\alpha^{2j-1}} < u^{j-1}|u|^{j-1} >, \quad \text{with} \quad < u|u| >= (1+\alpha)^{-1} \tag{19}$$

which follows directly from (8) when mutiplied by $u^j|u|^{j-1}$ and integrated over the velocity space. For $j = 2$, we get from (19)

$$< u^4 >= 3[(1+\alpha)(1-\alpha^3)]^{-1},$$

implying the divergence $\sim \epsilon^{-1}$ of the fourth moment. It should be stressed that the asymptotic state (18) is consistent with the exact relations (19). This result can also be obtained directly from the explicit solution by studying the asymptotics of the infinite product in (11) as $\alpha \to 1$. The high kinetic energy for $\epsilon \to 0$ implies a large collision frequency, which makes the asymptotic distribution (18) almost symmetric. In this situation the stationary current is very weak. Indeed, we find

$$< u >= C\,\epsilon^{1/4}/\sqrt{2} \tag{20}$$

Before closing let us note that the above results can be generalized to the case where the collision frequency is proportional to $|u|^\gamma$, $\;0 \le \gamma < 1$, rather than to $|u|$. The corresponding generalization of the kinetic equation (8) reads

$$F'(u) = |u|^\gamma \left[\alpha^{-(1+\gamma)} F(-\alpha^{-1}u) - F(u)\right] \tag{21}$$

and the substitution

$$F(u) \equiv G[u|u|^{\gamma}/(\gamma+1)]$$

transforms (21) into

$$G'(s) = \alpha^{-(1+\gamma)} G(-s\alpha^{-(1+\gamma)}) - G(s) \tag{22}$$

Performing then the same analysis as for $\gamma = 1$, we find the generalization of the asymptotic formula (18)

$$F(u) = C\epsilon^{1/2(\gamma+1)}(\gamma+1)^{-1/\gamma+1}(1+\epsilon u|u|^{\gamma})\exp[-\epsilon u^{2(\gamma+1)}/(\gamma+1)], \quad \epsilon \ll 1 \tag{23}$$

Again the dominant term in (23) is symmetric, and the first two moments follow the asymptotic law

$$< u >\sim \epsilon^{\gamma/2(\gamma+1)}, \quad < u^2 >\sim \epsilon^{-1/\gamma+1}$$

An interesting case is that of the so called Maxwell gas, where $\gamma = 0$, so that the collision frequency becomes independent of the velocity. The stationary current stays then finite even in the limit $\epsilon \to 0$. However, the $\sim \epsilon^{-1}$ divergence in the kinetic energy indicates the infinite absorption of energy from the field for $\epsilon = 0$.

Our analysis for $\gamma = 1$ (free motion between collisions) shows that the stationary velocity distribution changes from a highly asymmetric half-gaussian (13) at the strongest dissipation, to an almost symmetric scaled distribution $\sim \exp(-\epsilon u^4)$ in the weak dissipation limit. The experience from the studies of the classical Lorentz model showed the independence from the spatial dimension in the qualitative behavior. We can thus expect the appearance of the same scaling structure in three dimensions.

IV. STATIONARY VELOCITY DISTRIBUTION IN THREE DIMENSIONS

In three dimensions the stationary velocity distribution $F(\mathbf{u})$ satisfies the equation (see (4))

$$\hat{\mathbf{a}} \cdot \frac{\partial}{\partial \mathbf{u}} F(\mathbf{u}) = |\mathbf{u}| \left\{ \alpha^{-2} \int \frac{d\hat{\mathbf{n}}}{4\pi} F \left[\frac{1+\alpha}{2\alpha} |\mathbf{u}| \hat{\mathbf{n}} - \frac{1-\alpha}{2\alpha} \mathbf{u} \right] - F(\mathbf{u}) \right\} \tag{24}$$

Multiplying (24) by a function $\psi(\mathbf{u})$ and integrating over the velocity space one obtains the dual equation of the form

$$< \left\{ \hat{\mathbf{a}} \cdot \frac{\partial}{\partial \mathbf{u}} \psi(\mathbf{u}) + |\mathbf{u}| \int \frac{d\hat{\mathbf{n}}}{4\pi} \psi \left[\frac{1-\alpha}{2} \mathbf{u} - \frac{1+\alpha}{2} |\mathbf{u}| \hat{\mathbf{n}} \right] - |\mathbf{u}| \psi(\mathbf{u}) \right\} > \; = \; 0 \tag{25}$$

Here $< ... >$ denotes the average with respect to the probability density F. One can obtain from (25) useful information by an appropriate choice of ψ. In fact, some moments of F can be evaluated exactly in this way for any value of α. So, for instance putting $\psi(\mathbf{u}) = \mathbf{u} \cdot \hat{\mathbf{a}}$ yields the formula

$$(1+\alpha) < |\mathbf{u}|(\mathbf{u} \cdot \hat{\mathbf{a}}) >= 2 \tag{26}$$

And when $\psi = |\mathbf{u}|^3$, we find

$$3 < |\mathbf{u}|(\mathbf{u} \cdot \hat{\mathbf{a}}) >= \left[1 - \frac{2(1-\alpha^5)}{5(1-\alpha^2)} \right] < |\mathbf{u}|^4 > \tag{27}$$

Combining (26) and (27) permits to determine the fourth moment

$$(1-\alpha)(3+6\alpha+4\alpha^2+2\alpha^3) < |\mathbf{u}|^4 >= 30 \tag{28}$$

In accordance with the remark made at the end of section III, we find here the same divergence $< |\mathbf{u}|^4 >\sim \epsilon^{-1}$ for $\epsilon = (1-\alpha) \to 0$ as in one dimension. Let us finally note the relation

$$4 < (\mathbf{u}\cdot\hat{\mathbf{a}}) >=< |\mathbf{u}|^3 > (1-\alpha^2) \tag{29}$$

following from (25) for $\psi(\mathbf{u}) = |\mathbf{u}|^2$.

In order to determine the distribution F in the limit of weak inelasticity we put $\alpha = (1-\epsilon)$ in equation (24), and expand the inelastic collision law up to terms linear in ϵ. Clearly, the stationary state can depend on two variables only, $u = |\mathbf{u}|$ and $\mu = \hat{\mathbf{a}}\cdot\hat{\mathbf{u}}$. Expressing the differential operator in (24) in terms of u and μ one finds eventually the following asymptotic equation

$$\left(\mu\frac{\partial}{\partial u} + \frac{1-\mu^2}{u}\frac{\partial}{\partial \mu} + u\right) F_A(u,\mu) = -\mu\frac{d}{du}F_S(u) + \epsilon\left(\frac{1}{2u^2}\frac{d}{du}[u^4F_S(u)] - \frac{\mu}{4}\frac{d}{du}[u^2\int_{-1}^{1} d\sigma\, \sigma F_A(u,\sigma)]\right) \tag{30}$$

By analogy with the approach developed in the one-dimensional case, we introduced here the spherically symmetric projection of F

$$F_S(u) = \int \frac{d\hat{\mathbf{u}}}{4\pi} F(\mathbf{u})$$

and the deviation form spherical symmetry $F_A = F - F_S$.

An important relation between F_S and F_A is obtained by integration of (30) over the whole range of the angular variable $-1 \leq \mu \leq 1$. One finds

$$\int_{-1}^{1} d\mu\, \mu F_A(u,\mu) = \epsilon u^2 F_S(u) \tag{31}$$

In order to proceed, we shall assume that for $\epsilon \ll 1$ the state F depends on velocity via a scaled variable $\epsilon^x u$. The exact relations (26)-(29) imply then the value $x = 1/4$. Indeed, we know from (28) that the fourth moment diverges like ϵ^{-1}, whereas the assumed scaling predicts the $\sim \epsilon^{-4x}$ behavior. So, taking also into account relation (31), we write the perturbative expansions of F_S and F_A as

$$F_S^{\epsilon}(w) = F_S^0(w) + \sqrt{\epsilon} F_S^1(w) + \epsilon F_S^2(w) + ... \tag{32}$$

$$F_A^{\epsilon}(w,\mu) = \sqrt{\epsilon} F_A^1(w,\mu) + \epsilon F_A^2(w,\mu) + ...,$$

where $w \equiv \epsilon^{1/4} u$.

The expansion of the non-spherical part F_A does not contain the zero order term in accordance with the relation (31) between F_S and F_A. Inserting expansions (32) into equations (30) and (31) to lowest order in ϵ we find

$$w F_A^1(w,\mu) = -\mu \frac{d}{dw} F_S^0(w), \qquad \int_{-1}^{1} d\mu \mu F_A^1(w,\mu) = w^2 F_S^0(w) \tag{33}$$

From (33) there follows a closed equation for F_S^0

$$3w^3 F_S^0(w) = -2 \frac{d}{dw} F_S^0(w)$$

The lowest order terms in expansion (32) are then readily determined. The asymptotic formula for the stationary distribution reads

$$F(w,\mu) = F_S^0(w) + \sqrt{\epsilon}F_A^1(w,\mu) = C\epsilon^{3/4}(1+3\sqrt{\epsilon}\mu w^2/2)\exp(-3w^4/8), \tag{34}$$

where the constant C assures the normalization. Inserting into (34) $w = \epsilon^{1/4}u$ and $\mu = \hat{\mathbf{a}}\cdot\hat{\mathbf{u}}$, we obtain the same structure of the state F as in one dimension (compare with (18)). The same ϵ-dependence of the moments of F thus also holds.

This perturbative analysis can be continued. We checked that at the next step the following relations are found

$$F_S^1(w) \equiv 0,$$

$$F_A^2(w,\mu) = P_2(\mu)[1 + w\frac{d}{dw}]F_S^0(w),$$

where $P_2(\mu) = (1-3\mu^2)/2$ is the second Legendre polynomial. This confirms self-consistency of expansion (32) in powers of $\sqrt{\epsilon}$.

V. CONCLUDING COMMENTS

We proved the existence of a stationary state in the case of a one-dimensional propagation of a uniformly accelerated particle suffering inelastic collisions with randomly distributed scatterers. The velocity distribution, strongly asymmetric at high dissipation, showed the scaling behavior in the elastic limit $\epsilon = 1-\alpha \to 0$, recovering asymptotically the spherical symmetry $F \sim \exp(-\epsilon \mathrm{u}^4)$.

The same scaling properties could be derived in three dimensions by a suitable perturbation method. Of course, it would be desirable to provide a rigorous existence proof also in three dimensions.

In the one-dimensional case we could determine the form of the velocity distribution for perfectly inelastic collisions (see (13)). In three dimensions, passing in the dual equation (25) to limit $\alpha \to 0$, one can deduce in a straightforward way the form of the equation satisfied by the state F at $\alpha = 0$

$$\pi \hat{\mathbf{a}} \cdot \frac{\partial}{\partial \mathbf{u}} F(\mathbf{u}) = \int d\mathbf{w} \, \delta(\mathbf{w} \cdot \mathbf{u}) F(\mathbf{w} + \mathbf{u}) - \pi |\mathbf{u}| F(\mathbf{u}) \tag{35}$$

It should be realized that in contradistiction to the one-dimensional case the particle is not stopped at encounters with $\alpha = 0$, but its postcollisional velocity reduces then to the component tangent to the surface of the scatterer. It is thus clear that the solution to (35) can give non-zero probability weight only when $\hat{\mathbf{a}} \cdot \mathbf{u} > 0$. Solving equation (35) is another open problem related to our study.

Finally, an interesting question would be to explore the dynamics of approach to the stationary distribution. This problem has been solved only in one dimension for $\alpha = 0$ [5].

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REFERENCES

[1] H.A. Lorentz, Arch.Néerl. **10**, 336 (1905).

[2] D.R.Wilkinson, S.F. Edwards, Proc.R.Soc.Lond. **A 381**, 33 (1982).

[3] J. Piasecki, E. Wajnryb, J.Stat.Phys. **21**, 549 (1979); see also J. Piasecki, Am.J.Phys. **61**, 718 (1993).

[4] K. Olaussen, P. Hemmer, J.Phys.A: Math.Gen. **15**, 3255 (1982).

[5] J.Piasecki, J.Stat.Phys. **30**, 185 (1983).