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ON THE CATEGORY OF PSEUDO-BCI-ALGEBRAS

Abstract. The category \textit{psBCI} of pseudo-BCI-algebras and homomorphisms between them is investigated. It is also shown that the category \textit{psBCI}_p of p-semisimple pseudo-BCI-algebras and homomorphisms between them is a reflective subcategory of \textit{psBCI}.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are (pseudo-)MV-algebras, (pseudo-)BL-algebras, (pseudo-)BCK-algebras, (pseudo-)BCI-algebras and others. They are strongly connected with logic. For example, BCI-algebras introduced by K. Iséki in 1966 ([7]) have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming.

The notion of pseudo-BCI-algebras has been introduced by W. A. Dudek and Y. B. Jun in [3] as an extension of BCI-algebras and it was investigated by several authors in [4], [5], [8] and [9]. Pseudo-BCI-algebras are algebraic models of some extension of a non-commutative version of the BCI-logic. These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras.

In this paper, the category \textit{psBCI} of pseudo-BCI-algebras and homomorphisms between them is considered. We prove that it has equalizers, coequalizers, products, pullbacks, limits, kernel pairs and it is complete. Moreover, we show that in \textit{psBCI} surjective morphisms and coequalizers coincide. Finally, the category \textit{psBCI}_p of p-semisimple pseudo-BCI-algebras and homomorphisms between them is studied. We show that it is a reflective subcategory of \textit{psBCI} and it is isomorphic with the category \textit{Grp} of groups and group homomorphisms.

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2. Preliminaries

We include some necessary material concerning pseudo-BCI-algebras, needed in the sequel.

A pseudo-BCI-algebra is a structure \((X, \leq, \rightarrow, \sim, 1)\), where \(\leq\) is a binary relation on a set \(X\), \(\rightarrow\) and \(\sim\) are binary operations on \(X\) and 1 is an element of \(X\) such that, for all \(x, y, z \in X\), we have

(a1) \(x \rightarrow y \leq (y \rightarrow z) \sim (x \rightarrow z)\), \(x \sim y \leq (y \sim z) \rightarrow (x \sim z)\),
(a2) \(x \leq (x \rightarrow y) \sim y\), \(x \leq (x \sim y) \rightarrow y\),
(a3) \(x \leq x\),
(a4) if \(x \leq y\) and \(y \leq x\), then \(x = y\),
(a5) \(x \leq y\) iff \(x \rightarrow y = 1\) iff \(x \sim y = 1\).

It is obvious that any pseudo-BCI-algebra \((X, \leq, \rightarrow, \sim, 1)\) can be regarded as a universal algebra \((X, \rightarrow, \sim, 1)\) of type \((2, 2, 0)\). Note that every pseudo-BCI-algebra satisfying \(x \rightarrow y = x \sim y\), for all \(x, y \in X\) is a BCI-algebra.

Every pseudo-BCI-algebra satisfying \(x \leq 1\), for all \(x \in X\) is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called proper.

Later in the paper, we will usually use the symbol \(X\) in place of \((X, \rightarrow, \sim, 1)\).

Any pseudo-BCI-algebra \((X, \rightarrow, \sim, 1)\) satisfies the following, for all \(x, y, z \in X\),

(b1) if \(1 \leq x\), then \(x = 1\),
(b2) if \(x \leq y\), then \(y \rightarrow z \leq x \rightarrow z\) and \(y \sim z \leq x \sim z\),
(b3) if \(x \leq y\) and \(y \leq z\), then \(x \leq z\),
(b4) \(x \rightarrow (y \sim z) = y \sim (x \rightarrow z)\),
(b5) \(x \leq y \rightarrow z\) iff \(y \leq x \sim z\),
(b6) \(x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)\), \(x \sim y \leq (z \sim x) \sim (z \sim y)\),
(b7) if \(x \leq y\), then \(z \rightarrow x \leq z \rightarrow y\) and \(z \sim x \leq z \sim y\),
(b8) \(1 \rightarrow x = 1 \sim x = x\),
(b9) \((x \rightarrow y) \sim y) \rightarrow y = x \rightarrow y\), \(((x \sim y) \rightarrow y) \sim y = x \sim y\),
(b10) \(x \rightarrow y \leq (y \rightarrow x) \sim 1\),
(b11) \(x \sim y \leq (y \sim x) \rightarrow 1\),
(b12) \((x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \sim (y \sim 1)\),
(b13) \((x \sim y) \sim 1 = (x \sim 1) \rightarrow (y \rightarrow 1)\),
(b14) \(x \rightarrow 1 = x \sim 1\).

If \((X, \leq, \rightarrow, \sim, 1)\) is a pseudo-BCI-algebra, then by (a3), (a4), (b3) and (b1), \((X, \leq)\) is a poset with 1 as a maximal element.
For any pseudo-BCI-algebra \((X, \rightarrow, \leadsto, 1)\) the set
\[
K(X) = \{ x \in X : x \leq 1 \}
\]
is a subalgebra of \(X\) (called pseudo-BCK-part of \(X\), see [3]).

Let \((X, \rightarrow, \leadsto, 1)\) be a pseudo-BCI-algebra. Then \(X\) is \(p\)-semisimple if it satisfies for all \(x \in X\),
\[
\text{if } x \leq 1, \text{ then } x = 1.
\]
Note that if \(X\) is a \(p\)-semisimple pseudo-BCI-algebra, then \(K(X) = \{ 1 \}\).

Hence, if \(X\) is a \(p\)-semisimple pseudo-BCK-algebra, then \(X = \{ 1 \}\). It is proved in [5] that \((X, \rightarrow, \leadsto, 1)\) is \(p\)-semisimple if and only if for all \(x, y \in X\),
\[
(x \rightarrow 1) \leadsto y = (y \leadsto 1) \rightarrow x.
\]

Let \((X, \rightarrow, \leadsto, 1)\) be a pseudo-BCI-algebra. We say that a subset \(D\) of \(X\) is a deductive system of \(X\) if it satisfies: (1) \(1 \in D\), (2) for all \(x, y \in X\), if \(x \in D\) and \(x \rightarrow y \in D\), then \(y \in D\). Under this definition, \(\{ 1 \}\) and \(X\) are the simplest examples of deductive systems. Note that the condition (2) can be replaced by (2') for all \(x, y \in X\), if \(x \in D\) and \(x \leadsto y \in D\), then \(y \in D\). It can be easily proved that for any \(x, y \in X\), if \(x \in D\) and \(x \leq y\), then \(y \in D\).

A deductive system \(D\) of a pseudo-BCI-algebra \((X, \rightarrow, \leadsto, 1)\) is called closed if \(D\) is closed under operations \(\rightarrow\) and \(\leadsto\), that is, if \(D\) is a subalgebra of \(X\). It is not difficult to show (see [4]) that a deductive system \(D\) of a pseudo-BCI-algebra \((X, \rightarrow, \leadsto, 1)\) is closed if and only if for any \(x, y \in X\), \(x \rightarrow 1 = x \leadsto 1 \in D\). Obviously, the pseudo-BCK-part \(K(X)\) is a closed deductive system of \(X\).

A deductive system \(D\) of a pseudo-BCI-algebra \((X, \rightarrow, \leadsto, 1)\) is said to be compatible if for all \(x, y \in X\),
\[
x \rightarrow y \in D \text{ iff } x \leadsto y \in D.
\]
Further, if \(D\) is a compatible deductive system of \(X\), then the relation \(\Theta_D\) defined by
\[
(x, y) \in \Theta_D \text{ iff } x \rightarrow y \in D \text{ and } y \rightarrow x \in D
\]
is a congruence. We say that \(\Theta \in \text{Con}(X)\) is a relative congruence of \((X, \rightarrow, \leadsto, 1)\) if the quotient algebra \((X/\Theta, \rightarrow, \leadsto, [1]_{\Theta})\) is a pseudo-BCI-algebra. It is proved in [4] that relative congruences of \(X\) correspond one-to-one to closed compatible deductive systems of \(X\), that is, every relative congruence of \(X\) is given by (1) for some closed compatible deductive system \(D\).

For every relative congruence \(\Theta_D\), the quotient algebra \((X/\Theta_D, \rightarrow, \leadsto, [1]_{\Theta_D})\) will be usually denoted by \((X/D, \rightarrow, \leadsto, 1/D)\) and then we will write \(x/D\) instead of \([x]_{\Theta_D}\).

We know that pseudo-BCK-part \(K(X)\) of a pseudo-BCI-algebra \((X, \rightarrow, \leadsto, 1)\) is a closed deductive system of \(X\). It is proved in [4] that
it is also compatible and we have that \((X/K(X), \rightarrow, \sim, 1/K(X))\) is a \(p\)-semisimple pseudo-BCI-algebra.

Moreover we will need the following fact.

**Lemma 2.1.** Let \(f : X \to Y\) be a homomorphism of pseudo-BCI-algebras \(X, Y\). Then \(\text{Ker}(f) = \{x \in X : f(x) = 1\}\) is a closed compatible deductive system of \(X\).

**Proof.** Routine. \(\blacksquare\)

### 3. The category \(\text{psBCI}\)

All notions from the category theory occurring in this section the reader can find in [1] or [11].

If we consider the class of all pseudo-BCI-algebras as the class of objects and the class of all homomorphisms between pseudo-BCI-algebras as the class of morphisms, then we obtain the category of pseudo-BCI-algebras. We denote it by \(\text{psBCI}\). In the section, we investigate this category.

First, remark that the class of objects in \(\text{psBCI}\) is not a set. Therefore, \(\text{psBCI}\) is not a small category. Moreover, we can define a forgetful functor \(F : \text{psBCI} \to \text{Set}\) which is faithful. Hence, the category \(\text{psBCI}\) is concrete and embedded in the category \(\text{Set}\) of sets and functions.

Observe yet that in \(\text{psBCI}\), \(\{1\}\) is a zero object because it is an initial object as well as a terminal object. Indeed, there is an unique morphism \(f : \{1\} \to X\) for any object \(X\), so \(\{1\}\) is an initial object. Similarly, there exists an unique morphism \(g : X \to \{1\}\) for any object \(X\), so \(\{1\}\) is also a terminal object. Further, note that \(0_{\{1\}} : X \to \{1\}\) is a zero morphism in \(\text{psBCI}\), since it is in the same time a constant morphism and coconstant morphism.

**Theorem 3.1.** For any morphism \(f : X \to Y\) in \(\text{psBCI}\) the following are equivalent:

1. \(f\) is injective,
2. for all morphisms \(g, h\), if \(f \circ g = f \circ h\), then \(g = h\),
3. \(\text{Ker}(f) = \{1\}\).

**Proof.** (i)\(\Rightarrow\)(ii): Assume that \(f\) is an injective morphism between objects \(X, Y\). Let \(Z\) be another object, and let \(g, h : Z \to X\) be morphisms such that \(f \circ g = f \circ h\). Then for all \(z \in Z\), \(f(g(z)) = f(h(z))\). Hence since \(f\) is injective, we get \(g(z) = h(z)\). Thus \(g = h\).

(ii)\(\Rightarrow\)(iii): Suppose that \(\text{Ker}(f) \neq \{1\}\). Then there exists \(x \in \text{Ker}(f)\) and \(x \neq 1\). Let us consider morphisms \(i : \text{Ker}(f) \to X\) and \(j : \text{Ker}(f) \to X\) such that \(i(x) = x\) and \(j(x) = 1\), for all \(x \in \text{Ker}(f)\). Then \(f \circ i = f \circ j\). Now, by (ii), \(i = j\). Thus we get a contradiction. Therefore \(\text{Ker}(f) = \{1\}\).
(iii)⇒(i): Let $\text{Ker}(f) = \{1\}$ and $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$. Then $f(x_1 \to x_2) = f(x_1) \to f(x_2) = 1$ and $f(x_2 \to x_1) = f(x_2) \to f(x_1) = 1$. Hence $x_1 \to x_2, x_2 \to x_1 \in \text{Ker}(f) = \{1\}$. Thus $x_1 \to x_2 = x_2 \to x_1 = 1$, so, $x_1 \leq x_2$ and $x_2 \leq x_1$. Now it is clear that $x_1 = x_2$ and $f$ is injective.

**Corollary 3.2.** In the category $\text{psBCI}$ injective morphisms and monomorphisms coincide.

**Proposition 3.3.** Let $f : X \to Y$ be a morphism in $\text{psBCI}$. If $f$ is surjective, then for all morphisms $g, h$, if $g \circ f = h \circ f$, then $g = h$.

**Proof.** Let $f : X \to Y$ be a surjective morphism, $Z$ be an object and $g, h : Y \to Z$ be morphisms such that $g \circ f = h \circ f$. Since $f$ is surjective, for any $y \in Y$ there exists $x \in X$ such that $y = f(x)$. Then $g(y) = g(f(x)) = h(f(x)) = h(y)$, for all $y \in Y$. Therefore $g = h$.

**Corollary 3.4.** A morphism in the category $\text{psBCI}$ is an epimorphism if it is surjective.

**Remark.** It is well-known that any Hilbert algebra is a pseudo-BCI-algebra (precisely, a BCK-algebra). In [2] there is given an example of an epimorphism between Hilbert algebras (so, pseudo-BCI-algebras) which is not surjective. Thus, in the category $\text{psBCI}$ isomorphisms and bimorphisms are not the same.

**Corollary 3.5.** The category $\text{psBCI}$ is not balanced.

Let $\mathbf{C}$ be a category and $(X_i)_{i \in I}$ a family of objects in $\mathbf{C}$. A direct product of a family $(X_i)_{i \in I}$ is a pair $(P, (p_i))_{i \in I}$, where $P$ is an object in $\mathbf{C}$ and $(p_i)_{i \in I}$ is a family of morphisms in $\mathbf{C}$, $p_i : P \to X_i$, such that for any other pair $(P', (p'_i))_{i \in I}$ composed by an object $P'$ and a family of morphisms $(p'_i)_{i \in I}$, $p'_i : P' \to X_i$, there is an unique morphism $u : P' \to P$ such that $p_i \circ u = p'_i$ for every $i \in I$, so that for every $i \in I$ the following diagram is commutative:

\[
\begin{array}{c}
P \\
p_i \downarrow \\
X_i \\
p'_i \downarrow \\
P'
\end{array}
\]

We say that a category $\mathbf{C}$ has products if there exists a direct product of any family of objects from $\mathbf{C}$.

**Theorem 3.6.** The category $\text{psBCI}$ has products.

**Proof.** Let $(X_i)_{i \in I}$ be a family of objects. Consider the set $P = \prod_{i \in I} X_i$ of all functions $f : I \to \bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$. A
function \( 1 : I \to \bigcup_{i \in I} X_i \) such that \( 1(i) = 1 \) for all \( i \in I \), is a special element of \( P \). Define binary operations \( \to \) and \( \simeq \) on \( P \) as follows: for \( f, g \in P \), 
\[(f \to g)(i) = f(i) \to g(i) \text{ and } (f \simeq g)(i) = f(i) \simeq g(i) \text{ for all } i \in I.\]
We can verify that the structure \((P, \to, \simeq, 1)\) forms a pseudo-BCI-algebra, that is \( P \) is an object in \( \text{psBCI} \).

For each \( i \in I \), there is a natural projection \( p_i : P \to X_i \) defined by 
\[p_i(f) = f(i) \text{ for all } f \in P.\]
Further, for all objects \( P' \) and morphisms \( p'_i : P' \to X_i \) for \( i \in I \) the map \( u : P' \to P \) defined by 
\[(u(x))(i) = p'_i(x) \text{ for all } x \in P' \text{ and } i \in I \]
is the unique morphism such that \( p_i \circ u = p'_i \). Thus the category \( \text{psBCI} \) has products.

By a couple of morphisms \((f, g)\) in a category \( C \) we understand two morphisms \( f, g : X \to Y \), where \( X, Y \) are objects in \( C \). A pair \((E, e)\) with \( E \) an object in \( C \) and \( e : E \to X \) a morphism in \( C \), will be called an equalizer of a couple \((f, g)\) if \( f \circ e = g \circ e \) and for every other pair \((E', e')\) with \( E' \) an object and \( e' : E' \to X \) a morphism such that \( f \circ e' = g \circ e' \), there exists an unique morphism \( u : E' \to E \) such that \( e' = e \circ u \):

\[
\begin{align*}
E & \xrightarrow{e} X \xrightarrow{f} Y \\
E' & \xrightarrow{u} X \xrightarrow{g} Y \\
& \xrightarrow{e'}
\end{align*}
\]

We say that a category \( C \) has equalizers if there exists an equalizer for any couple of morphisms in \( C \).

**Theorem 3.7.** The category \( \text{psBCI} \) has equalizers.

**Proof.** Let \((f, g)\) be a couple of morphisms, \( f, g : X \to Y \). Then nonempty set \( E = \{x \in X : f(x) = g(x)\} \) is a subalgebra of \( X \) and if we consider the empedding \( e : E \to X \), then \( f \circ e = g \circ e \).

Further, let \( E' \) be other object and let \( e' : E' \to X \) be a morphism such that \( f \circ e' = g \circ e' \). We define \( u : E' \to E \), \( u(x) = e'(x) \) for all \( x \in E' \). Then \( u \) is well defined, since from \( f \circ e' = g \circ e' \) we have \( e'(x) \in E \) for every \( x \in E' \). It is clear that \( u \) is a morphism and \( e \circ u = e' \).

The uniqueness of \( u \) follows from the fact that \( e \) is a monomorphism.

**Corollary 3.8.** The category \( \text{psBCI} \) has pullbacks, limits and it is complete.
Let \( f : X \to Y \) be a morphism in \( \mathbf{C} \). We say that \( f \) is an *equalizer* if there exists a couple of morphisms \((\alpha, \beta)\) such that \( \alpha, \beta : Y \to Z \) and \((X, f)\) is an equalizer of \((\alpha, \beta)\). Obviously, every equalizer in \( \mathbf{C} \) is a monomorphism.

Thus by Corollary 3.2, we have the following theorem.

**Theorem 3.9.** In the category \( \mathbf{psBCI} \) every equalizer is injective.

**Remark.** The converse of Theorem 3.9 is not true. In [6], there is given an example of an injective morphism between Hilbert algebras (so, pseudo-BCI-algebras) which can not be an equalizer for any couple of morphisms.

Let \( f, g : X \to Y \), where \( X, Y \) are objects in a category \( \mathbf{C} \). A pair \((Q, q)\) with \( Q \) an object in \( \mathbf{C} \) and \( q : Y \to Q \) a morphism in \( \mathbf{C} \), will be called a *coequalizer* of a couple \((f, g)\) if \( q \circ f = q \circ g \) and for every other pair \((Q', q')\) with \( Q' \) an object and \( q' : Y \to Q' \) a morphism such that \( q' \circ f = q' \circ g \), there exists an unique morphism \( u : Q \to Q' \) such that \( q' = u \circ q \):

\[
\begin{array}{ccc}
X @>{f}>> Y @>q>> Q \\
@VV{g}V \downarrow \quad @VV{u}V \\
\quad Q' \end{array}
\]

We say that a category \( \mathbf{C} \) has *coequalizers* if there exists a coequalizer for any couple of morphisms in \( \mathbf{C} \).

**Theorem 3.10.** The category \( \mathbf{psBCI} \) has coequalizers.

**Proof.** Let \((f, g)\) be a couple of morphisms, \( f, g : X \to Y \). Put

\[
R = \{(f(x), g(x)) \in Y \times Y : x \in X\}.
\]

Let \( \Theta \) be the intersection of all relative congruences on \( Y \) (that is, congruences determined by closed compatible deductive systems of \( Y \)) which contain \( R \). Then \( Q = Y/\Theta \) is an object in \( \mathbf{psBCI} \). Let \( q : Y \to Q \) be the canonical surjection. We show that \((Q, q)\) is a coequalizer of \((f, g)\). Since \((f(x), g(x)) \in \Theta\) for all \( x \in X \), we have \((q \circ f)(x) = q(f(x)) = [f(x)]_\Theta = [g(x)]_\Theta = q(g(x)) = (q \circ g)(x)\) for all \( x \in X \). Thus \( q \circ f = q \circ g \).

Let \( Q' \) be another object and let \( q' : Y \to Q' \) be a morphism such that \( q' \circ f = q' \circ g \). Let \( \Theta' = \{(y_1, y_2) \in Y \times Y : q'(y_1) = q'(y_2)\} \). Then \( \Theta' \) is a relative congruence determined by a closed compatible deductive system \( \text{Ker}(q') \). Since for every \( x \in X \) we have \( q'(f(x)) = q'(g(x)) \), we obtain \((f(x), g(x)) \in \Theta'\) for every \( x \in X \). Hence \( R \subset \Theta' \). Thus \( \Theta \subset \Theta' \). We can define now a morphism \( u : Q \to Q' \) such that \( u([y]_\Theta) = q'(y) \). Then \( u \) is well defined because for
[y_1]_{\Theta} = [y_2]_{\Theta} \text{ we have } (y_1, y_2) \in \Theta \subset \Theta' \text{ whence } q'(y_1) = q'(y_2). \text{ Clearly, } u \circ q = q'.

The uniqueness of u follows from the fact that q is an epimorphism. This completes the proof.

Let C be a category and f : X \rightarrow Y a morphism in C. A system (P; p_1, p_2) formed by an object P and two morphisms p_1, p_2 : P \rightarrow X, is called a kernel pair of f if f \circ p_1 = f \circ p_2 and for any other system (Q; q_1, q_2) with an object Q and morphisms q_1, q_2 : Q \rightarrow X such that f \circ q_1 = f \circ q_2, there exists an unique morphism u : Q \rightarrow P such that p_1 \circ u = q_1 and p_2 \circ u = q_2:

\[
\begin{array}{ccc}
Q & \xrightarrow{u} & X \\
\downarrow{q_1} & & \downarrow{f} \\
P & \xrightarrow{p_1} & X \\
\downarrow{p_2} & & \downarrow{f} \\
X & \xrightarrow{f} & Y
\end{array}
\]

We say that a category C has kernel pairs if every morphism in it has a kernel pair.

**Theorem 3.11.** The category psBCI has kernel pairs.

**Proof.** Let f : X \rightarrow Y be a morphism. Let us put

\[ P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}. \]

Obviously, P is a subalgebra of the product algebra X \times X. Let p_1, p_2 : P \rightarrow X be the canonical projections, that is, p_i(x_1, x_2) = x_i for i = 1, 2 and all (x_1, x_2) \in P. We show that (P; p_1, p_2) is a kernel pair of f. Clearly, f \circ p_1 = f \circ p_2. Let (Q; q_1, q_2) with an object Q and morphisms q_1, q_2 : Q \rightarrow X be another system such that f \circ q_1 = f \circ q_2. We take u : Q \rightarrow P as

\[ u(x) = (q_1(x), q_2(x)) \text{ for all } x \in Q. \]

Now, u is well defined because f \circ q_1 = f \circ q_2 implies f(q_1(x)) = f(q_2(x)) whence (q_1(x), q_2(x)) \in P. Further,

\[
u(x_1 \rightarrow x_2) = (q_1(x_1 \rightarrow x_2), q_2(x_1 \rightarrow x_2))
\]

\[= (q_1(x_1) \rightarrow q_1(x_2), q_2(x_1) \rightarrow q_2(x_2))
\]

\[= (q_1(x_1), q_2(x_1)) \rightarrow (q_1(x_2), q_2(x_2))
\]

\[= u(x_1) \rightarrow u(x_2)\]
and
\[ u(x_1 \sim x_2) = (q_1(x_1 \sim x_2), q_2(x_1 \sim x_2)) \]
\[ = (q_1(x_1) \sim q_1(x_2), q_2(x_1) \sim q_2(x_2)) \]
\[ = (q_1(x_1), q_2(x_1)) \sim (q_1(x_2), q_2(x_2)) \]
\[ = u(x_1) \sim u(x_2). \]

Thus \( u \) is a morphism in \( \text{psBCI} \). Moreover, it is easy to see that \( p_1 \circ u = q_1 \) and \( p_2 \circ u = q_2 \).

Now, let \( u' : Q \rightarrow P \) be another morphism such that \( p_1 \circ u' = q_1 \) and \( p_2 \circ u' = q_2 \). Let \( u'(x) = (x', x'') \). Then \( p_1 \circ u' = p_1 \circ u \) gives \( p_1(x', x'') = p_1(q_1(x), q_2(x)) \) whence \( x' = q_1(x) \) and \( x'' = q_2(x) \). Thus \( u'(x) = (x', x'') = (q_1(x), q_2(x)) = u(x) \) for all \( x \in Q \). Hence \( u \) is unique. Therefore, the system \( (P; p_1, p_2) \) is a kernel pair of \( f \). \( \blacksquare \)

Let \( f : X \rightarrow Y \) be a morphism in \( \mathbf{C} \). We say that \( f \) is a coequalizer if there exists a couple of morphisms \((\alpha, \beta)\) such that \( \alpha, \beta : Z \rightarrow X \) and \((Y, f)\) is a coequalizer of \((\alpha, \beta)\). Clearly, every coequalizer in \( \mathbf{C} \) is an epimorphism.

**Proposition 3.12.** Let \( f : X \rightarrow Y \) be a coequalizer in \( \text{psBCI} \). Then \( f \) is a coequalizer of its kernel pair.

**Proof.** Let \( \alpha, \beta : Z \rightarrow X \) be such that \( f \) is a coequalizer of \((\alpha, \beta)\) and let \((P; p_1, p_2)\) be a kernel pair of \( f \). Since \( f \circ p_1 = f \circ p_2 \), it is sufficient to prove that for any other morphism \( f' : X \rightarrow Y' \) such that \( f' \circ p_1 = f' \circ p_2 \), there exists an unique morphism \( u : Y \rightarrow Y' \) such that \( f' = u \circ f \).

Since \( f \circ \alpha = f \circ \beta \) and \((P; p_1, p_2)\) is a kernel pair of \( f \), we get the existence of an unique morphism \( v : Z \rightarrow P \) such that \( \alpha = p_1 \circ v \) and \( \beta = p_2 \circ v \):

\[
\begin{array}{ccc}
Z & \xrightarrow{\alpha} & X \\
\downarrow{v} & & \downarrow{f} \\
P & \xrightarrow{p_1} & Y \\
\end{array}
\]

Hence \( f' \circ \alpha = (f' \circ p_1) \circ v = (f' \circ p_2) \circ v = f' \circ \beta \). Thus since \( f \) is a coequalizer of \((\alpha, \beta)\), we obtain the existence of an unique \( u : Y \rightarrow Y' \) such that \( f' = u \circ f \). This completes the proof. \( \blacksquare \)

**Theorem 3.13.** Every surjective morphism in \( \text{psBCI} \) is a coequalizer.

**Proof.** Let \((P; p_1, p_2)\) be a kernel pair of a surjective morphism \( f : X \rightarrow Y \). Then as we know \( P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\} \) and \( p_1, p_2 : P \rightarrow X \) are the canonical projections. It is sufficient to prove that \((Y, f)\) is a coequalizer of \((p_1, p_2)\). Clearly, \( f \circ p_1 = f \circ p_2 \). Let \( f' : X \rightarrow Y' \) be a
morphism such that $f' \circ p_1 = f' \circ p_2$. Since $f$ is surjective, for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Now, take $u : Y \to Y'$ as follows: $u(y) = f'(x)$. It is well defined because if $f(x_1) = f(x_2) = y$, then $(x_1, x_2) \in P$ and $u(y) = f'(x_1) = (f' \circ p_1)(x_1, x_2) = (f' \circ p_2)(x_1, x_2) = f'(x_2)$. Next, let $y_1, y_2 \in Y$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$, and hence $f'(x_1) = u(y_1)$ and $f'(x_2) = u(y_2)$. Further, we have $y_1 \to y_2 = f(x_1) \to f(x_2) = f(x_1 \to x_2)$ and $y_1 \sim y_2 = f(x_1) \sim f(x_2) = f(x_1 \sim x_2)$. Hence $u(y_1 \to y_2) = f'(x_1 \to x_2) = f'(x_1) \to f'(x_2) = u(y_1) \to u(y_2)$ and $u(y_1 \sim y_2) = f'(x_1 \sim x_2) = f'(x_1) \sim f'(x_2) = u(y_1) \sim u(y_2)$. Thus $u$ is a morphism and obviously, $u \circ f = f'$. The uniqueness of $u$ follows from the fact that $f$ is an epimorphism. Therefore $f$ is a coequalizer.

**Proposition 3.14.** Let $X, Y, Z$ be objects in psBCI and $f : X \to Y$ and $g : X \to Z$ be morphisms in psBCI such that $f$ is surjective and $\text{Ker}(f) \subset \text{Ker}(g)$. Then there exists an unique morphism $h : Y \to Z$ such that $h \circ f = g$.

**Proof.** Let $(P; p_1, p_2)$ be a kernel pair of $f$, that is, $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \to X$ are the canonical projections. Since $f$ is surjective, by Theorem 3.13, we have that $f$ is a coequalizer, that is, $(Y, f)$ is a coequalizer of $(p_1, p_2)$. Let $(x_1, x_2) \in P$. Then $f(x_1) = f(x_2)$ which gives that $x_1 \to x_2, x_2 \to x_1 \in \text{Ker}(f) \subset \text{Ker}(g)$, so $g(x_1) = g(x_2)$. Hence, there exists an unique morphism $h : Y \to Z$ such that $h \circ f = g$: 

\[
\begin{array}{ccc}
P & \xrightarrow{p_1} & X & \xrightarrow{f} & Y \\
& & \downarrow{g} & \downarrow{h} & \\
& & & & Z
\end{array}
\]

This completes the proof. ■

**Theorem 3.15.** Every coequalizer in psBCI is surjective.

**Proof.** Let $f : X \to Y$ be a coequalizer. By Proposition 3.12, $f$ is a coequalizer of its kernel pair $(P; p_1, p_2)$, where $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \to X$ are the canonical projections. Note that $P = \{(x_1, x_2) \in X \times X : x_1 \to x_2, x_2 \to x_1 \in \text{Ker}(f)\}$. Hence $P$ is a relative congruence determined by the closed compatible deductive system $\text{Ker}(f)$. Let $X/\text{Ker}(f)$ be the corresponding quotient pseudo-BCI-algebra and let $p : X \to X/\text{Ker}(f)$ be the canonical surjection. Notice that $p \circ p_1 = p \circ p_2$. Indeed, for every $(x_1, x_2) \in P$ we have $(p \circ p_1)(x_1, x_2) = x_1/\text{Ker}(f) = x_2/\text{Ker}(f) = (p \circ p_2)(x_1, x_2)$. Since $(Y, f)$ is a coequalizer of $(p_1, p_2)$, there
exists an unique morphism \( u : Y \to X/Ker(f) \) such that \( u \circ f = p \):

\[
\begin{array}{c}
\text{\( P \)} \\
\downarrow \scriptstyle{p_1} \\
\downarrow \scriptstyle{p_2} \\
\text{\( X \)} \\
\downarrow \scriptstyle{u} \\
\text{\( Y \)} \\
\uparrow \scriptstyle{\leftarrow} \\
\text{\( X/Ker(f) \)}
\end{array}
\]

Let \( x \in Ker(p) \). Then \( p(x) = 1/Ker(f) \). Since \( p(x) = x/Ker(f) \), we get \( (x, 1) \in P \), so \( x \in Ker(f) \). This means that \( Ker(p) \subset Ker(f) \). Thus by Proposition 3.14, there exists an unique morphism \( v : X/Ker(f) \to Y \) such that \( v \circ p = f \):

\[
\begin{array}{c}
\text{\( X \)} \\
\downarrow \scriptstyle{\leftarrow} \\
\text{\( Y \)} \\
\uparrow \scriptstyle{\leftarrow} \\
\text{\( X/Ker(f) \)}
\end{array}
\]

Now,

\[
(u \circ v) \circ p = u \circ f = p = 1_{X/Ker(f)} \circ p
\]

and

\[
(v \circ u) \circ f = v \circ p = f = 1_Y \circ f.
\]

Since \( p \) is surjective and \( f \) is a coequalizer, both are epimorphisms. Hence

\[
u \circ v = 1_{X/Ker(f)} \quad \text{and} \quad v \circ u = 1_Y.
\]

Thus \( u \) and \( v \) are isomorphisms, one the inverse of the other. Now, we get that \( f = v \circ p \) is surjective, because both \( v \) and \( p \) are surjective.

**Corollary 3.16.** In the category \( \text{psBCI} \) surjective morphisms and coequalizers coincide.

**Remark.** In the category \( \text{psBCI} \) not every epimorphism is a coequalizer. Indeed, in [6] there is given an example of an epimorphism (not a surjective one) between Hilbert algebras (so, pseudo-BCI-algebras) which is not a coequalizer.

4. The category \( \text{psBCI}_p \)

The category formed by taking the class of objects as the class of all p-semisimple pseudo-BCI-algebras and the class of morphisms as the class of all homomorphisms between them is called the category of p-semisimple pseudo-BCI-algebras. We denote this category by \( \text{psBCI}_p \). We have an inclusion functor \( I : \text{psBCI}_p \hookrightarrow \text{psBCI} \), which is faithful and full. Hence
\textbf{psBCI}_p is a full subcategory of the category \textbf{psBCI}. Like \textbf{psBCI}, the category \textbf{psBCI}_p is not a small category, it is concrete and embedded in the category \textbf{Set}; it also has zero objects ($\{1\}$ is so) and zero morphisms ($0_{\{1\}} : X \to \{1\}$ is the one).

For p-semisimple pseudo-BCI-algebras we have the following nice fact from [5] (compare with [10] for p-semisimple BCI-algebras).

\textbf{Theorem 4.1.} A pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ is p-semisimple if and only if $(X, \cdot, \cdot^{-1}, e)$ is a group, where, for any $x, y \in X$, $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$, $x^{-1} = x \to 1 = x \rightsquigarrow 1$ and $e = 1$. In this case, $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$ for any $x, y \in X$.

Moreover, it is not difficult to prove that $f$ is a morphism in the category \textbf{psBCI}_p if and only if it is a morphism in the category \textbf{Grp} of groups and group homomorphisms. Thus we have the following theorem.

\textbf{Theorem 4.2.} The category \textbf{psBCI}_p is isomorphic with the category \textbf{Grp}.

\textbf{Remark.} From Theorem 4.2, it follows that the category \textbf{psBCI}_p has the same properties as the category \textbf{Grp}. For example, it has coproducts and it is balanced and cocomplete.

A subcategory $C'$ of a category $C$ is called \textit{reflective} if there is a covariant functor $R : C \to C'$, called \textit{reflector}, such that for every object $X$ from $C$ there is a morphism $\phi_R(X) : X \to R(X)$ in $C$ with the properties:

(i) if $f : X \to Y$ is a morphism in $C$, then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\bigg\downarrow \phi_R(X) & & \bigg\downarrow \phi_R(Y) \\
R(X) & \xrightarrow{R(f)} & R(Y)
\end{array}
\]

is commutative, that is, $\phi_R(Y) \circ f = R(f) \circ \phi_R(X)$,

(ii) if $X'$ is an object in $C'$ and $f : X \to X'$ is a morphism in $C$, then there is an unique morphism $f' : R(X) \to X'$ in $C'$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\bigg\downarrow \phi_R(X) & & \bigg\downarrow f' \\
R(X) & & 
\end{array}
\]

is commutative, that is, $f' \circ \phi_R(X) = f$. 
Remark. It is a well known fact that \( C' \) is a reflective subcategory of a category \( C \) if and only if there exist a function which assigns to every object \( X \) in \( C \), an object \( R(X) \) in \( C' \) and a function which assigns to every \( X \) in \( C \), a morphism \( \phi_R(X) : X \to R(X) \) in \( C \) such that for every object \( X' \) in \( C' \) and every morphism \( f : X \to X' \) in \( C \) there is an unique morphism \( f' : R(X) \to X' \) in \( C' \) such that \( f' \circ \phi_R(X) = f \).

Theorem 4.3. The category \( \text{psBCI}_p \) is a reflective subcategory of the category \( \text{psBCI} \).

Proof. Let \( X \) be an object in \( \text{psBCI} \). Then as we know \( X/K(X) \) is an object in \( \text{psBCI}_p \). Thus, we put \( R(X) = X/K(X) \). We define \( \phi_R(X) : X \to R(X) \) as follows

\[
(\phi_R(X))(x) = x/K(X), \text{ for all } x \in X,
\]

that is, \( \phi_R(X) \) is the canonical surjection.

Now, take a morphism \( f : X \to Y \), where \( Y \) is an object in \( \text{psBCI}_p \). First, note that \( f(x) = 1 \), for all \( x \in K(X) \). Indeed, \( x \to 1 = 1 \) gives \( 1 = f(1) = f(x \to 1) = f(x) \to f(1) = f(x) \to 1 \), that is, \( f(x) \in K(Y) = \{1\} \) whence \( f(x) = 1 \). We define \( f' : R(X) \to Y \) as follows

\[
f'(x/K(X)) = f(x), \text{ for all } x \in X.
\]

First of all, we prove that \( f' \) is well defined. Let \( x_1/K(X) = x_2/K(X) \). Then \( x_1 \to x_2 \in K(X) \) and \( x_2 \to x_1 \in K(X) \), which gives \( f(x_1 \to x_2) = 1 \) and \( f(x_2 \to x_1) = 1 \), that is, \( f(x_1) = f(x_2) \). This proves that \( f' \) is well defined. Further, it is easy to show that \( f' \) is a morphism in \( \text{psBCI}_p \) and \( f' \circ \phi_R(X) = f \).

The uniqueness of \( f' \) follows from the fact that \( \phi_R(X) \) is an epimorphism. This completes the proof. \( \blacksquare \)

Remark. The reflector \( R : \text{psBCI} \to \text{psBCI}_p \) is defined in the following way. If for \( X \) from \( \text{psBCI} \) we put

\[
R(X) = X/K(X),
\]

then we obtain the definition of \( R \) on objects. Now, let \( f : X \to Y \) be a morphism in \( \text{psBCI} \). If we define \( R(f) : R(X) \to R(Y) \) by

\[
(R(f))(x/K(X)) = f(x)/K(Y), \text{ for all } x \in X,
\]

then we obtain the definition of \( R \) on morphisms. Obviously, \( R \) is a left adjoint for the inclusion functor \( I : \text{psBCI}_p \hookrightarrow \text{psBCI} \). Moreover, \( R \) is faithfull.

5. Conclusions

In the category \( \text{psBCI} \), monomorphisms and injective morphisms coincide, but epimorphisms and surjective morphisms not. These imply that
psBCI is not balanced. Since in psBCI, not every monomorphism is an equalizer and not every epimorphism is a coequalizer, they are not normal, that is, psBCI is not abelian. In the same time, since it has arbitrary limits, it is complete. It is an open problem if it is cocomplete.

The category psBCI_p is a full and reflective subcategory of psBCI and it is isomorphic with the category Grp. This means that psBCI_p is among other things balanced and cocomplete.

References


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