A PROOF OF MENGER’S THEOREM
BY CONTRACTION

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Abstract

A short proof of the classical theorem of Menger concerning the number of disjoint $AB$-paths of a finite graph for two subsets $A$ and $B$ of its vertex set is given. The main idea of the proof is to contract an edge of the graph.

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Proofs of Menger’s Theorem are given in [7, 6, 4, 8, 2]. A short proof is given by T. Böhme, F. Göring and J. Harant in [1]; another short proof based on edge deletion is given by the author in [5]. The new idea here is to get a short proof by contracting an arbitrary edge of the original graph.

For terminology and notation not defined here we refer to [3]. A graph with no edges is denoted by its vertex set. Let $G$ be a finite graph (loops and multiple edges being allowed). For an edge $e$ of $G$ let $G - e$ and $G/e$ denote the graphs obtained from $G$ by removing $e$ and contracting $e$ to one vertex $v_e$, respectively. For (possibly empty) sets of vertices $A$ and $B$ of $G$ let an $AB$-separator be a set of vertices of $G$ such that the graph obtained from $G$ by deleting these vertices contains no path from $A$ to $B$. Note that a single vertex of $A \cap B$ is considered as a path from $A$ to $B$, too. An $AB$-connector is a subgraph of $G$ such that each of its components is a path from $A$ to $B$ having only one vertex in common with $A$ and $B$, respectively. In particular the empty graph is also an $AB$-connector. If we contract an edge incident with a vertex of $A$ or $B$ then the resulting vertex is considered to be a vertex of $A$ or $B$, respectively.
Theorem (Menger, 1927). Let $G$ be a finite graph, $A$ and $B$ sets of vertices of $G$, and $s$ the minimum number of vertices forming an $AB$-separator. Then there is an $AB$-connector $C$ with $|C \cap A| = s$.

Proof. If $G$ is edgeless then set $C = A \cap B$. Suppose, $G$ is a counterexample with $|E(G)|$ minimal. Then $G$ contains an edge $e$ from $x$ to $y$ and $G/e$ has an $AB$-separator $S$ with $|S| < s$, otherwise we are done. Obviously, $v_e \in S$. Then $P = (S \setminus \{v_e\}) \cup \{x, y\}$ is an $AB$-separator of $G$ with $|P| = |S| + 1 = s$. An $AP$-separator (as well as an $PB$-separator) of $G - e$ is an $AB$-separator of $G$. Consequently, $G - e$ has an $AP$-connector $X$ and a $PB$-connector $Y$ containing $P$. Since $X \cap Y = P$, the set $C = (X \cup Y)$ is an $AB$-connector of $G$ with $|C \cap A| = s$, a contradiction. ■

References


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