Construction of High Performance Balanced Symmetric Multifilter Banks and Application in Image Processing

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Abstract—Multifilter banks having all properties of symmetry, balancing, optimum time-frequency resolution, arbitrary order vanishing moment, orthogonality and compactly supported together are constructed for the first time. Thus overcome the shortcoming of the present orthogonal optimum time-frequency resolution multifilter banks that have only the properties of symmetry and 2-order vanishing moment but not balanced. Balanced symmetric orthogonal multifilter banks with good regularity and time-frequency resolution are also provided. Due to the excellent properties, application experiments in image denoising and compression showed that, most of our multifilter banks outperform the best tools known as multiwavelet SA4 and wavelet CDF9-7 in denoising with texture image and smooth image. They also outperform multiwavelet SA4 or approximate wavelet DB4 in compression with texture image.

Index Terms—multifilter bank, symmetry, balanced, vanishing moment, time-frequency resolution, regularity

I. INTRODUCTION

Multiwavelet is attracting more and more attention for its good properties such as orthogonality, symmetry, compactly supported, high vanishing moment and balance contrast with the traditional wavelet. These properties, along with optimum time-frequency resolution (OPTFR) are all very important in image processing. Generally, multiwavelet with i) orthogonality can keep energy conserved and make less redundancy; ii) the property of symmetry adapts to the function of human eyes. Image distortion can be reduced by symmetric extension transform of signals with finite length; iii) higher vanishing moment can lead to energy concentration in the part of low-frequency, with more zero values in the high frequency part; iv) in image processing, smooth error is more tolerable for human eyes than non-regular error that has the same energy, which emphasizes the importance of regularity for multiwavelet to improve the quality of reconstructed image; v) the property of locational time-frequency also plays a vital role in the great task of coding with stable image compacting or digital video[1][2], especially when extracting high frequency components such as textures, edges and movements; vi) in image denoising with balanced multiwavelet, the aliasing of low-frequency and high-frequency signals can be avoid, and good performance can be expected without pre-filtering. Therefore, how to design the balanced multifilter banks with all the properties of symmetry, time-frequency resolution, high vanishing moment, regularity and orthogonality together is a valuable issue to pay attention to.

In 1998, Jiang[4] first introduced the concept of time-frequency resolution of multiwavelets. Using OPTFR orthogonal multiwavelets, Jiang got some ideal result in image compressing. To present, Jiang’s work is known as the best in dealing with OPTFR multiwavelet and its application. However, with only 2-order vanishing moment, the capability of approximating to smooth function for Jiang’s OPTFR orthogonal multiwavelets is limited. Moreover, the orthogonal OPTFR multiwavelets in Jiang[4] are not balanced. Although we can get balanced orthogonal scaling functions and multiwavelets by rotating them for π/4 angle, the symmetry of the scaling functions, multiwavelets and their multifilter banks would be lost, which are not optimum in its time-frequency property any longer. Later in 2000, Jiang[6] introduced the parameterized representation of the symmetric orthogonal multiwavelet with different length of filter banks. But he only provided the example of the smoothest multifilter banks, without considering the time-frequency property.

With the properties of symmetry, balancing, optimum time-frequency resolution, arbitrary order vanishing moment and orthogonality synthetically concerned, construction method of multiwavelets is improved, and balanced symmetric orthogonal multifilter banks with arbitrary vanishing moment and optimum in time-frequency resolution were obtained. Furthermore, application in image denoising and compression showed their advantages.

II. RELEVANT THEORY

Suppose a set of compactly supported scaling functions ϕ1,⋯,ϕℓ ∈ L(R) whose integer translates form an
orthogonal basis of \( V \), then \( \{ V_k \} \) is called an orthogonal MRA. If the integer translates of a set functions \( \psi, \cdot \psi, \cdot \psi, \cdot \psi \), form an orthogonal basis of \( W \), then \( \psi, \cdot \psi, \cdot \psi, \cdot \psi \) is a set of orthogonal multiwavelets. Assume that \( P, Q \) are \( r \times r \) matrix filters with matrix coefficient \( P_k \) and \( Q_k \) satisfying 
\[
P_k = 0, \quad Q_k = 0, \quad (k < 0 \text{ and } k > N, \quad N \in \mathbb{Z}) \quad \text{and} \quad \Phi = (\phi, \cdots \phi)^T, \quad \Psi = (\psi, \cdots \psi)^T \]
are compactly supported refinable vector-valued function satisfying
\[
\Phi(x) = 2 \sum_{k=0}^{\infty} P_k \Phi(2x - k), \quad \Psi(x) = 2 \sum_{k=0}^{\infty} Q_k \Phi(2x - k) \tag{1}
\]
or equivalently satisfying
\[
\hat{\Phi}(\omega) = P(\omega/2) \Phi(\omega/2), \quad \hat{\Psi}(\omega) = P(\omega/2) \Psi(\omega/2), \tag{2}
\]
where \( P(\omega) = \sum_{k=0}^{\infty} P_k e^{-i\omega \pi k} \), \( Q(\omega) = \sum_{k=0}^{\infty} Q_k e^{-i\omega \pi k} \). If \( \Psi \) is a compactly supported orthogonal multiwavelet, then \( P, Q \) generate an orthogonal multiwavelet basis. The pair \( \{ P, Q \} \) is called multfilter bank, and \( P \) and \( Q \) respectively) is called a matrix lowpass filter (matrix highpass filter, respectively). For a multfilter bank, \( P, Q \) are called finite impulse responses (FIR).

If the compactly supported refinable vector \( \Phi \) is stable, then \( \hat{\Phi}_x(\omega) = \lim_{n \to \infty} \Phi_x(n) \), where
\[
\hat{\Phi}_x(\omega) = \hat{\Psi}_x(\omega) v_x \sin(\omega/2^m)^{-\alpha} \tag{3}
\]
and \( v_x \) is the normalized right 1-eigenvector of \( P(0) \) [7][8]. If \( \Psi \) approximate \( \Psi \), then
\[
\hat{\Psi}_x(\omega) = Q(0/2) \hat{\Phi}_x(0/2). \tag{4}
\]

Let \( H_\omega(\omega) \) denote the modulation matrix of an FIR multfilter bank \( \{ P, Q \} \) defined by
\[
H_\omega(\omega) = \begin{bmatrix} P(\omega) & P(\omega + \pi) \\ Q(\omega) & Q(\omega + \pi) \end{bmatrix}. \tag{5}
\]

If \( \{ P, Q \} \) generates an orthogonal multiwavelet, then \( H_\omega(\omega) \) is lossless, i.e., \( H_\omega(\omega) \) is unitary for all \( \omega \):
\[
P(\omega)P^*(\omega) + P(\omega + \pi)P^*(\omega + \pi) = I, \quad \omega \in [-\pi, \pi],
\]
\[
Q(\omega)Q^*(\omega) + Q(\omega + \pi)Q^*(\omega + \pi) = I, \quad \omega \in [-\pi, \pi],
\]
\[
P(\omega)Q^*(\omega) + P(\omega + \pi)Q^*(\omega + \pi) = 0, \quad \omega \in [-\pi, \pi],
\]
where \( B^* \) denotes the Hermitian adjoint of the matrix \( B \), \( I \) and \( 0 \) denote the \( r \times r \) identity matrix and zero matrix respectively [13]. Let \( H_\omega \) denote the space of all \( r \times r \) matrices with trigonometric polynomial entries whose Fourier coefficients are real and supported in \([-N,N-1]\). The transition operator \( T_\omega \) corresponding to \( P \) is defined on \( H_\omega \) by
\[
T_\omega(\omega) = P(\omega/2)H(\omega/2)P(\omega/2) + P(\omega/2 + \pi)H(\omega/2 + \pi)P(\omega/2 + \pi), H \in H_\omega.
\]
Thus, the representation matrix \( T_\omega \) of the operator \( T_\omega \) is
\[
T_\omega := (2A_{\omega,\nu})_\nu, \nu \in \mathbb{Z}.
\]

If a matrix or an operator \( A \) satisfies Condition E, then the spectral radius of \( A \) is 1, 1 is the unique eigenvalue of \( A \) on the unit circle and 1 is simple.

If transition operator \( T_\omega \) associated with \( P \) satisfies Condition E, then there exists a unique compactly supported solution \( \Phi \) with \( \Phi(0) \neq 0 \). Furthermore \( \Phi \) is a scaling function, i.e. \( \Phi \) generates a multiwavelet \( \Psi \) [3]. In this case, \( \{ P, Q \} \) generates the scaling function \( \Phi \) and the multiwavelet \( \Psi \).

For \( s \geq 0 \), it is said that a function \( f \) is in Sobolev space \( W^s(\mathbb{R}) \) if \( |1 + |\hat{f}(\omega)|^2\Phi(\omega) \in L^s(\mathbb{R}) \), where \( \hat{f} \) denotes the Fourier transform of \( f \).

If the compactly supported refinable vector \( \Phi \) is stable, then the property that \( \Phi \) has approximation of order \( m \) is equivalent to that \( P \) satisfies the vanishing moment conditions of order \( m \) [10].

**Proposition 1:** It is said that \( P \) satisfies the vanishing moment conditions of order \( m \) if there exist real \( 1 \times r \) row vectors with \( e_j^x = 0, 0 \leq \beta < m \), such that
\[
\sum_{s=0}^{m-1} \left( \frac{\beta}{\alpha} \right)^{2s-\alpha} e_j^s D^{\alpha} P(\omega) = 2^d t_i^\omega, \tag{6}
\]
\[
\sum_{s=0}^{m-1} \left( \frac{\beta}{\alpha} \right)^{2s-\alpha} e_j^s D^{\alpha} P(\omega) = 0, \tag{7}
\]
where \( D^{\alpha}P(\omega) \) denotes the matrix formed by the \( (\beta - \alpha) \)th derivatives of the entries of \( P(\omega) \) [10].

**Lemma 1:** If \( P \) satisfies the vanishing moment conditions of order 1, then
\[
\sum_{s=0}^{m-1} \left( \frac{\beta}{\alpha} \right)^{2s-\alpha} e_j^s D^{\alpha} P(\omega) = 0, \tag{8}
\]
where the real \( 1 \times r \) row vector \( e_j^\omega \neq 0 \).

**Definition 1:** An orthonormal multiwavelet system is said to be balanced (of order 1) if and only if the lowpass synthesis operator \( \hat{L} \) preserves the constant signals, i.e.,
\[
\hat{L} u_0 = u_0, \quad u_0 = (\cdots, 1, 1, 1, \cdots)^T.
\]

Symmetric properties of multfilter banks are very important in image applications. For symmetric filters, symmetric extension transforms of the finite length signals can be carried out, which will improve the rate-distortion performance in image compression. This paper only discuss symmetric filters with the coefficients \( h, g \), having the form
\[
[\begin{array}{c} h_0, h_1, \cdots, h_{N-1}, h_1, h_2, h_3, \cdots \end{array}] = \begin{bmatrix} a_0 & a_1 & \cdots & a_{N-1} \ \ h_0 & h_1 & \cdots & h_{N-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \ 0 & 0 \end{bmatrix},
\]
\[ \left[ g_0, g_1, \ldots, g_{2^j-1}, g_{2^j} \right] = \\
\left[ c_0, c_1, \cdots, c_{2^j-1}, -c_{2^j-1}, -c_{2^j} \right] \left[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
d_0 & d_1 & d_2 & d_3 \\
d_0 & d_1 & d_2 & d_3 \\
-d_0 & -d_1 & -d_2 & -d_3 \\
-d_0 & -d_1 & -d_2 & -d_3 
\end{array} \right] \] (6)

for some \( a, b, c, d \in R \), \( h, g, s = 0 , j < 0 \), \( j > 2^r + 1 \). The corresponding scaling functions \( \Phi \) and multiwavelets \( \Psi \) are not symmetric or antisymmetric. However, as multifilter banks, the rows of \([h_0, h_1, \ldots, h_{2^j-1}, h_{2^j}, h_{2^j+1}]^{T}\) and \([g_0, g_1, \ldots, g_{2^j-1}, g_{2^j}, g_{2^j+1}]^{T}\) are symmetric and antisymmetric respectively, which are more important than the symmetry of \( \Phi \), \( \Psi \) in image applications.

In this paper, we will discuss orthogonal multifilter banks \([P_0, Q]\) with \([P_0, Q]\) having the form of (6). It is obtained that (6) is equivalent to

\[ z^{-(2^r+1)} \left[ \begin{array}{c}
1 \\
0 \\
1 \\
0 
\end{array} \right], P(-\omega)J_z = P(\omega), \\
Q(-\omega)J_z = Q(\omega), \] (7)

for some integer \( l \) and that \( s_1 = \pm 1 \), \( s_2 = \pm 1 \).

**Lemma 2.** Assume that the multifilter bank \([P_0, Q]\) is orthogonal and that \([P_0, Q]\) satisfies (7), then \( s_1 = s_2 = -1 \).

**Proposition 2.** A Causal FIR filter bank \([P_0, Q]\) is orthogonal and satisfies (7) if and only if it can be factorized as

\[ \left[ \begin{array}{c}
P(\omega) \\
Q(\omega) 
\end{array} \right] = \frac{\sqrt{5}}{4} \left( I_n - B_x + B_y z^{-1} \right) \]

\[ U_{r, s}(z) \cdots U_1(z) \left[ \begin{array}{c}
w_0 \\
v_0 
\end{array} \right] \left[ \begin{array}{c}
I_1 \\
-1
\end{array} \right] \] (8)

where \( w_0, v_0 \in O(2) \) and \( B_x = \left[ \begin{array}{cc} 0 & 0 \\
0 & b_0 \end{array} \right] \)

\[ b_0 = \frac{1}{2} \left( 1, 0, 0, 0 \right)^T, \]

\[ U_0(z) = \frac{1}{2} I_2 + \frac{1}{2} I_4 \]

\[ U_1(z) = \left[ \begin{array}{cc}
1 & u_1 \\
1 & u_1 
\end{array} \right] \]

\[ U_{r, s}(z) = \left[ \begin{array}{cc}
1 & u_{r, s} \\
1 & u_{r, s} 
\end{array} \right] z^{-1} u_s \in O(2) [6]. \]

**Lemma 3.** Suppose \( P_0, Q_0 \) are the orthogonal filters defined by (8), then \( P \) satisfies the balanced conditions of order \( 1 \) if and only if

\[ w_0 = I_n, \quad v_0 = -r_y D_y, \]

where \( D_y = \left[ \begin{array}{cc} 1 & 0 \\
0 & -1 \end{array} \right] \), \( r_y = \frac{1}{2} \left( \alpha_y \cos \theta, \sin \theta \right) \), \( \alpha_y = \pm 1 \).

**Definition 2.** For a real \( r \times 1 \) vector function \( F = (f_1, \cdots, f_r) \in L^2(R) \) supported in \([0, N]\), define the energy moments of \( F \) in the time domain by

\[ I_F^r(\gamma) := \int_0^N |F(x)|^2 dx, \beta \in Z. \]

**Definition 3.** For a real \( r \times 1 \) vector function \( F = (f_1, \cdots, f_r) \in L^2(R) \) supported in \([0, N]\), if \( F \) is in Sobolev space \( W^s(R) \) for some \( s \geq 0 \), define for \( \beta \), \( 0 \leq \beta \leq 2s \), the energy moments of \( F \) in the frequency domain by

\[ D_F^r(\xi) := \frac{1}{2\pi} \int_0^\infty |\hat{F}(\omega)|^2 e^{i\beta \omega} d\omega. \]

**Definition 4.** For a window function \( f \) (with some smoothness and decay at infinity), the center in the time domain \( \tau \) and the time-duration \( \Delta \) of \( f \) are defined by

\[ \tau := \int_0^\infty |f(t)| dt / E, \quad \Delta := \int_0^\infty (t-\tau)^2 |f(t)| dt / E, \]

where \( E := \int_0^\infty |f(t)| dt \). If the Fourier transform \( \hat{f} \) of \( f \) satisfies \( \omega f(\omega) \in L^2(R) \), then the center in the frequency domain \( \hat{\tau} \) and the frequency-bandwidth \( \Delta_f \) of \( f \) are defined by

\[ \hat{\tau} := \int_0^\infty \omega |\hat{f}(\omega)| d\omega / E, \quad \Delta_f := \int_0^\infty (\omega-\hat{\tau})^2 |\hat{f}(\omega)| d\omega / E, \]

where \( E := \int_0^\infty |\hat{f}(\omega)| d\omega \).

The product of time-duration and frequency-bandwidth \( \Delta \Delta_f \) is called the resolution cell. As the resolution of time and frequency can not be infinitely improved together, in order to get good property of time and frequency, it will have to make \( \Delta \Delta_f \) minimized within the scope of which the Heisenberg uncertainty principle confined.

If every component of \( \Phi = (\phi_1, \cdots, \phi_r) \) and \( \Psi = (\psi_1, \cdots, \psi_r) \) is normalized, i.e., \( \|\phi_r\| = 1 \) and \( \|\psi_r\| = 1 \), then the frequency-bandwidth \( \Delta_{\phi_r} \) and \( \Delta_{\psi_r} \) are given by

\[ \Delta_{\phi_r}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |\hat{\phi}(\omega)|^2 d\omega, \quad \Delta_{\phi_r} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega |\hat{\phi}(\omega)| d\omega \]

**Definition 4.** The sum of resolution cells \( \Delta_{\phi_r}^2 \) and \( \Delta_{\psi_r}^2 \) is called the areas of time-frequency window( also called the window area)

\[ S := \sum_{r=1}^m \Delta_{\phi_r}^2 + \Delta_{\psi_r}^2 \]

To construct multiwavelets with good time-frequency properties means to find scaling functions and multiwavelets that are less in window area \( S \).

**III. MAIN ALGORITHM AND RESULT**

This paper aims to construct the multiwavelets with good properties of arbitrary order vanishing moment, time-frequency property and regularity through synthetic method of the genetic algorithm and parameterization.

**A. The algorithm for computing time-frequency window area**

It is not difficult to conclude that the frequency-bandwidth of scaling function and multiwavelet function with high order \( m \geq 2 \) of Jiang's[4] cannot be obtained, which will furthermore influence the calculation of window area under the high order condition. The 2-order energy...
moments vector in the frequency domain of scaling function \( \Phi \) and multiwavelet \( \Psi \) calculated with the method of Jiang's [4] is the crucial factor in working out the frequency-bandwidth. While the 2-order energy moments vector in the frequency domain is the right \( 1/4 \) eigenvalue of the representation matrix \( T_\rho \) which satisfies specific condition. The order of eigenvalue of representation matrix \( T_\rho \) with its \( 1/2 \) integer power is related with that of the vanishing moment \( m \). If \( m < 2 \), there is no eigenvalue \( 1/4 \) for \( T_\rho \). If \( m > 2 \), \( 1/4 \) may not be the single eigenvalue of \( T_\rho \), which means the eigenvalue vector cannot be uniquely destined.

After deep investigating into the relationship between the properties of energy moments in the time and frequency domain of scaling function, the window area of the scaling function and the multiwavelet function that have vanishing moments of arbitrary order can be obtained by using the algorithm of calculating the time-frequency window area, which can choose the method of calculating the frequency-bandwidth according to the count of \( 1/4 \) eigenvalue of representation matrix \( T_\rho \) of the multiwavelet. Finally, the window area of the multiwavelet is worked out. If the \( 1/4 \) eigenvalue of representation matrix \( T_\rho \) is simple, then the frequency-bandwidth \( \Delta \phi \) and \( \Delta \psi \) for the components of scaling functions and multiwavelet functions can be worked out using the 2-order energy moments vector in the frequency domain. Otherwise, \( \Delta \phi \) and \( \Delta \psi \) can be obtained by the formula of (3), (4) and compound Simpson. The window area \( S \) of scaling function and multiwavelets is finally worked out by using formula (10).

The new algorithm for the area of time-frequency window is as follows:

Step 1: Find the time-duration \( \Delta x \) and \( \Delta \psi \) of the components of scaling functions and multiwavelet functions with 1 or 2-order energy moments vector in the time domain.

Step 2: Work out the eigenvalues of the representation matrix \( T_\rho \)

a) If \( 1/4 \) is a single eigenvalue, then find the frequency-bandwidths \( \Delta x \) and \( \Delta \psi \) of scaling functions and multiwavelet functions with the 2-order energy moment vector in the frequency domain, else

b) Use the cascade formula (3), (4) and compound Simpson formula to find the frequency-bandwidths (9).

Step 3: Compute the window area \( S \) of the scaling functions and multiwavelet functions with formula (10).

The main body of cascade formula (3) is \( \hat{\Phi}(\omega) \). If \( \omega = \pi \) multiplicate \( n_1 \) times \( (n_1 = 8) \), where \( \omega \in [-2^n \pi, 2^n \pi] \), and \( \omega \) is equally separated to \( 2^{n+1} \) parts with the step length of \( \frac{\pi}{2^n} \), then the time complexity of the cascade formula (3) is \( O((n_1 + 1)2^{n+1-\gamma}) \). During the calculation of compound Simpson formula, the lower limit of integration \( a = -2^n \pi \), the upper limit \( b = 2^n \pi \), with the step length \( h = \pi/2^n \). Thus \( n = 2^n + 5 \). In order to control the error within \( 10^{-5} \), let \( n_1 = 6 \).

In fact, \( \hat{\phi}(i) \) and \( \hat{\psi}(i) \) are aperiodic continuous functions in their time-duration \([0, N]\), and the Fourier transform \( \hat{\phi}(\omega) \) and \( \hat{\psi}(\omega) \) of them are aperiodic continuous functions in their frequency-bandwidths \((-\infty, +\infty)\). As the attenuation of the window function \( W \) is a compulsory condition, which means if \( |\omega| \to \infty \), then \(|\hat{W}(\omega)| \to 0 \), and the attenuation of \( \hat{W}(\omega) \) gets faster with smoother window function \( W \). It is feasible letting the lower limit and upper limit of integration in compound Simpson formula to be \(-2^n \pi \) and \( 2^n \pi \) respectively.

B. New algorithm of designing balanced symmetric orthogonal multifilter banks

This paper aims to construct the balanced symmetric orthogonal multifilter banks with arbitrary order vanishing moments through synthetic method of parameterization and solving the nonlinear equations of vanishing moment conditions.

Step 1: Construct the parameterized representation of multifilter banks \( P \) and \( Q \) which satisfy the conditions of orthogonality and balanced of order 1. \( P \) and \( Q \) can be expressed as (6):

\[
|\hat{W}(\omega)| \to 0 ,
\]
and the attenuation of \( \hat{W}(\omega) \) gets faster with smoother window function \( W \). It is feasible letting the lower limit and upper limit of integration in compound Simpson formula to be \(-2^n \pi \) and \( 2^n \pi \) respectively.

Step 7: Verify that whether the representation matrix \( T_\rho \) of lowpass filter corresponding to the selecting parameters can satisfy the condition E [9].

Although these multifilter banks only produce scaling functions and multiwavelet functions with approximate symmetry, the parameterized form of \( P \) can not only satisfy the condition of 1-order balance, but also has \( \gamma \) free parameters, which is suitable to use in constructing the balanced symmetric orthogonal multifilter banks with vanishing moment of high order and smoothness.

During the \( 4^{th} \) step of the algorithm, the vanishing moment condition of each order with parameters are
obtained. Thus the problem turns to the solving of nonlinear equations. In this paper, the classical Newton algorithm and secant algorithm are used, during which the error \( |\epsilon| < 10^{-14} \).

### C. Construction result

The scaling function \( \phi = (\phi_1, \phi_2) \) corresponding to the balanced symmetric orthogonal multifilter banks constructed in this paper has the properties such as:

- a) \( \phi \) is balanced and orthogonal;
- b) The matrix lowpass filter of \( \phi_1 \) and \( \phi_2 \) are symmetric;
- c) The length difference of matrix lowpass filter of \( \phi_1 \) and \( \phi_2 \) is 4.

Table I shows the properties of the balanced symmetric orthogonal multifilter banks constructed in this paper when \( \gamma = 2, 3, 4 \). When the supported in \([0, 2\pi\gamma+1]\) and the vanishing moment is N-order, \( O_{\gamma-VN} \) and \( S_{\gamma-VN} \) represent those multifilter banks of optimum time-frequency resolution and optimum regularity, and \( OS_{\gamma-VN} \) represent those of good time-frequency resolution and regularity properties concerned together. Here, \( O_{\gamma-VN} \) and \( OS_{\gamma-VN} \) are provided by this paper for the first time, and \( S2_V2 \), \( S3_V3 \), \( S4_V3 \) are the same in properties to Jiang’s [6].

<table>
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<tr>
<th>Multifilter</th>
<th>Support</th>
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<th>Window area</th>
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<td>OS4_V3</td>
<td>9</td>
<td>2.0460</td>
<td>3</td>
<td>7.3696</td>
</tr>
</tbody>
</table>

We representatively list the coefficients of \( OS3_V3 \) as follows, which is good in regularity and time-frequency resolution with 3-order vanishing moment.

<table>
<thead>
<tr>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.065003778072000</td>
<td>0.49140113677500</td>
<td>0.000456198608000</td>
<td>-0.00158930692000</td>
</tr>
<tr>
<td>0.001315153417000</td>
<td>0.00458353690100</td>
<td>0.015615744518000</td>
<td>-0.099761226813000</td>
</tr>
<tr>
<td>0.100036004088000</td>
<td>0.479344520784000</td>
<td>-0.004593885023000</td>
<td>0.016104082380000</td>
</tr>
<tr>
<td>-0.017491086410000</td>
<td>-0.060959447599900</td>
<td>0.253257093680000</td>
<td>-0.426098721350000</td>
</tr>
<tr>
<td>-0.004561986080000</td>
<td>0.015893069200000</td>
<td>-0.045835360910000</td>
<td>0.218502606270000</td>
</tr>
<tr>
<td>-0.049685520123000</td>
<td>0.015151534170000</td>
<td>-0.045835360910000</td>
<td>0.048156488126000</td>
</tr>
</tbody>
</table>

Contrast with the OPTFR multiwavelets of Jiang’s [4], our multifilter banks in this paper have the advantages as follows

- a) The multifilter banks with optimum time-frequency resolution constructed in this paper have the vanishing moment of 3-order, comparing with the 2-order of Jiang’s.
- b) \( J_{EX2} \) and \( J_{EX4} \) are the multiwavelets listed in Jiang’s [4] as example 2 and 4. These orthogonal OPTFR multiwavelets are not balanced. Although we can further make them balanced by reversing the original multiwavelets for \( \pi/4 \) angle, the symmetry of the balanced multiwavelets and the multifilter banks would be lost, which are not optimum in its time-frequency property any longer. The multifilter banks constructed in this paper are not only balanced, but also have the balanced optimum time-frequency resolution and the symmetry properties as concerned in formula (6).
- c) The window area of multifilter banks we constructed is smaller than those of Jiang’s [4]. For example, the window area of \( O2_V2 \) is 3.2 smaller than \( J_{EX4} \), with their support being both 5. Although the support of \( O2_V2 \), \( OS3_V2 \), \( O3_V3 \), \( O4_V2 \), \( OS4_V2 \), \( O4_V3 \) and \( OS4_V3 \) are longer than Jiang’s, the window areas of them are all smaller by contraries.

### IV. APPLICATION IN IMAGE PROCESSING

In order to test the performances of the balanced symmetric orthogonal OPTFR multiwavelets constructed in our research, we use the standard \( 512 \times 512 \) grey image named Lena, Baboon and Barbara to do the simulating experiment, in which Barbara and Baboon are texture images, and Lena is the typical smooth image. As if wavelets CDF9-7, DB4 and multiwavelet SA4 performing well in image processing, i) multiwavelet GHM being similar in form of multifilter bank with ours, ii) \( J_{EX2} \) and \( J_{EX4} \) being the best two OPTFR multiwavelets of Jiang’s[4], we choose these wavelets/multiwavelets to do our experiment.

#### A. Application in image denoising

During the experiment, we first produced three kind of noise image of different intensity (\( \sigma = 10, 20, 30 \)). Then, with each kind of noise image decomposed to three degree, the denoising is pursued with Donoho’s SureShrink [9] threshold value program. Finally, the denoising images are reconstructed with multiwavelet
inverse transformation. Table II shows the peak signal to noise ratio (PSNR) of noise image and reconstructed denoising image. As shown in fig.1, the summary of denoising appreciation index (adding three PSNRs in the same row of Table II) to every multiwavelet for Barbara, Baboon and Lena are listed. In fig.2, a) is the original image of Lena, b), c) and d) are the denoising image processed by OS3_V3, J_EX2 and SA4 separately, based on the denoising image of Lena with $\sigma = 20$ (not showed here). Likewise, in fig.3, a) is the original image of Barbara, b),c) and d) are the denoising image processed by OS3_V3, J_EX2 and SA4 separately.

The experiment results show that:

a) The multifilter banks we have constructed are obviously better than multiwavelet GHM in image denoising. Moreover, the advantages become bigger with more noise. For texture image Barbara and Baboon, the value of PSNR is higher for 0.71 to 1.61dB; for image Lena, it is also higher for 1.04 to 3.14dB. Especially for multiwavelet O2_V2, it is lower in regularity for 0.5 and smaller in time-frequency window area for 2.4 contrasting with GHM when their vanishing moment is equivalent.

---

**TABLE II. PSNR OF WAVELETS/MULTIWAVELETS UNDER DIFFERENT NOISING VARIANCE**

<table>
<thead>
<tr>
<th>NO.</th>
<th>Wavelet/multiwavelet</th>
<th>Barbara</th>
<th>Baboon</th>
<th>Lena</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma = 10$</td>
<td>$\sigma = 20$</td>
<td>$\sigma = 30$</td>
<td>$\sigma = 10$</td>
</tr>
<tr>
<td>10</td>
<td>S3_V2</td>
<td>31.3730</td>
<td>27.5580</td>
<td>25.5221</td>
</tr>
<tr>
<td>19</td>
<td>S4_V3</td>
<td>30.9655</td>
<td>27.3312</td>
<td>25.4186</td>
</tr>
</tbody>
</table>

---

**Figure 1.** Denoising performance of different Wavelet/Multiwavelet

**Figure 2.** Denoising performance of different Wavelet/Multiwavelet with Lena ($\sigma = 20$)
Further more, its performance with Lena and Barbara are greatly better than GHM.

b) Contrast with multiwavelet J_EX4(NO.3), multiwavelet O2_V2(NO.7) has the same support, but lower in regularity for 0.7 and smaller in time-frequency window area for 3.2, which makes good greatly for its performance in image denoising. In fig. 1, we can see that O2_V2 perform better in denoising for all the three images than J_EX2 (NO.2), which was the best multiwavelet of Jiang's.

c) As shown in fig. 1, from multiwavelet SA4(NO.4) to wavelet CDF9-7(NO.6) and DB4(NO.5), the denoising performance decrease in turn, and SA4 is the best one at present. However, the synthetic performances of our multiwavelets(NO10. TO NO.14) in image denoising are better than SA4.

In summary, the multifilter banks constructed in this paper have good performances in both texture image and smooth image. Especially for S3_V2, O3_S V2, O3_V3, S3_V3 and OS3_V3, their performances in image denoising are better than not only the best multiwavelet SA4, but also the best wavelet CDF9-7 at present.

### Table III. PSNR of Wavelets/Multiwavelets in Compressing with Different Compressing Ratio

<table>
<thead>
<tr>
<th>NO.</th>
<th>Wavelet/ multiwavelet</th>
<th>Barbara</th>
<th>Baboon</th>
<th>Lena</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>16:1</td>
<td>32:1</td>
<td>64:1</td>
</tr>
<tr>
<td>1</td>
<td>GHM</td>
<td>28.1833</td>
<td>25.1393</td>
<td>23.3500</td>
</tr>
<tr>
<td>2</td>
<td>J_EX2</td>
<td>29.7339</td>
<td>26.1370</td>
<td>23.6241</td>
</tr>
<tr>
<td>3</td>
<td>J_EX4</td>
<td>26.5333</td>
<td>24.0995</td>
<td>22.2643</td>
</tr>
<tr>
<td>4</td>
<td>SA4</td>
<td>29.5402</td>
<td>26.0751</td>
<td>23.6423</td>
</tr>
<tr>
<td>5</td>
<td>DB4</td>
<td>29.9789</td>
<td>26.2356</td>
<td>23.6681</td>
</tr>
<tr>
<td>6</td>
<td>CDF9-7</td>
<td>30.3255</td>
<td>26.4018</td>
<td>23.7980</td>
</tr>
<tr>
<td>7</td>
<td>O2_V2</td>
<td>29.7333</td>
<td>26.1491</td>
<td>23.5683</td>
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<tr>
<td>8</td>
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<td>O3_V2</td>
<td>29.5660</td>
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<tr>
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<tr>
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<tr>
<td>13</td>
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<tr>
<td>14</td>
<td>OS3_V3</td>
<td>29.8723</td>
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<td>23.7158</td>
</tr>
<tr>
<td>15</td>
<td>O4_V2</td>
<td>29.9025</td>
<td>26.2353</td>
<td>23.6407</td>
</tr>
<tr>
<td>16</td>
<td>S4_V2</td>
<td>29.7660</td>
<td>26.2629</td>
<td>23.6102</td>
</tr>
<tr>
<td>17</td>
<td>OS4_V2</td>
<td><strong>30.0384</strong></td>
<td>26.3106</td>
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<tr>
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<td>O4_V3</td>
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</tr>
<tr>
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<td>26.2381</td>
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<tr>
<td>20</td>
<td>OS4_V3</td>
<td>29.8185</td>
<td>26.4303</td>
<td>23.7256</td>
</tr>
</tbody>
</table>

**Figure 3.** Denoising performance of different Wavelet/Multiwavelet with Barbara (σ = 20)

![Image](image3.png)

**Figure 4.** Compressing performance of different Wavelet/Multiwavelet with Barbara, Baboon, Lena (σ = 20)

![Image](image4.png)
B. Application in image compression

During the experiment, the multiwavelet is decomposed to 5 degree first. Then, job of quantification and compression are done by the improved SPIHT algorithm [12]. Finally, decoding for SPIHT bit current and image reconstructing through multiwavelet inverse transformation are performed. As the aim of our experiment is to contrast with the different performances of various multiwavelets/wavelets, the process of arithmetic coding for bit current which is coded by SPIHT is neglected. The contrasting experiment based on SPIHT bit current can well demonstrate the difference performances of the multifilter/filter banks. Table III shows the values of PSNR for the multiwavelets/wavelets dealing with Barbara, Baboon and Lena with the compression rate of 16:1, 32:1 and 64:1.

From the experiment we found that:

a) The performances of the multifilter banks we constructed in image compression are obviously better than that of GHM. As for image Barbara, Baboon and Lena, the value of PSNR of our result is higher than that of GHM for 0.42 to 1.86dB, 0.23 to 0.70dB and 0.34 to 1.04dB separately.

b) From fig. 4 we know, as for the three images we chose, most of our multifilter banks performed better than J_EX4 in image compression. As J_EX2 being the OPTFR multiwavelet of Jiang’s[4] with shortest supported in [0,3], most of our multifilter banks performed better than J_EX2 in compression with Barbara, and approximate J_EX2 with Baboon.

c) As shown in a) and b) of fig. 4, the best performances in compressing with texture image Barbara and Baboon are made by wavelet CDF9-7, wavelet DB4 and multiwavelet SA4. However, most of our multiwavelets are synthetically better performed than the best multiwavelet SA4, and better than or approximate that of wavelet DB4.

Image compression experiments show that, the performances of most of our multifilter banks are better than that of SA4 and approximate wavelet DB4.

It is not difficult to see that, the performances of OS3_V2, S3_V2, O3_V3, S3_V3, OS3_V3 in image compression are better than that of the others especially when dealing with texture images.

V. CONCLUSION

In this paper, construction method is improved to obtain the multifilter banks that have good properties and better performances in image processing. By introduce the time-frequency property to the construction of balanced symmetric orthogonal multifilter banks, we first obtained the optimum time-frequency resolution multifilter banks which have arbitrary vanishing moment. With the properties of vanishing moment, regularity, time-frequency resolution synthetically concerned, we first constructed the multifilter banks which have good performances both in regularity and time-frequency resolution. Furthermore, the smoothest balanced symmetric orthogonal multifilter bank is obtained. Application experiments in image processing show that, the performances of most of our multifilter banks in image denoising are better than that of multiwavelet SA4 and wavelet CDF9-7. As for image compression, their performances are better than that of SA4 and approximate wavelet DB4.

REFERENCES

[12] Zeng Zhaohui was born in Hunan, China, on Aug. 26, 1977, received her B.S. degree and M.S. degree in computer software from Xianyang University, Xiangtan, China, in 2000 and 2007 respectively. She is a lecture in Xiangtan University. Her current research interests are in wavelet/multiwavelet and image processing.