Diffusing through spectres: ridge curves, ghost circles and a partition of phase space

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Abstract

The study of transport in Hamiltonian and related systems is greatly illuminated if one can construct a framework of "almost invariant" surfaces to organize the dynamics. This can be done in the case of area-preserving twist maps, using pieces of the stable and unstable manifolds of periodic orbits or cantori, as shown by MacKay, Meiss and Percival. The resulting surfaces are not, however, necessarily the most appropriate ones, as they need not be graphs, nor is it clear that they can always be chosen mutually disjoint. G. R. Hall proposed a choice based on "ridge curves" for the gradient flow of the associated variational problem, which C. Golé christened "ghost circles". They have the advantage that they are always graphs. In this letter, we present numerical experiments suggesting that ghost circles are mutually disjoint. Our work has subsequently led to a proof of this by Angenent and Golé. We propose that ghost circles form a convenient, natural skeleton around which to organize studies of transport.

1 Introduction

The ideas of partial barriers, turnstiles and flux, presented in [MMP84, MMP87] have had enormous success in the understanding of transport in area-preserving maps and related flows.

Three classes of partial barrier were constructed:
(i) type p/q, for p/q rational: those connecting minimising and minimax periodic orbits of rotation number p/q,
(ii) types $p/q+$ and $p/q-$, for p/q rational: those connecting minimising and minimax advancing (respectively, retreating) heteroclinic orbits to minimising periodic orbits of rotation number p/q,
(iii) type ω, ω irrational: those connecting cantori of rotation number ω and their minimax orbits.

There is considerable freedom in the choice of partial barriers of the above three classes. References [MMP84, MMP87] concentrated on those of classes (ii) and (iii) given by joining pieces of the stable and unstable manifolds of the minimizing orbits, but even here there is considerable freedom, for example, in the choice of gap in a cantorus or the point in a minimizing heteroclinic orbit at which to switch from stable to unstable manifold.

Furthermore, it is easy to make examples (see Appendix) in which there is no choice of switch-over point for which the resulting partial barrier and its image are graphs (*i.e.* cut each vertical line

*Nuffield Foundation Science Research Fellow precisely once). This can make the interpretation of the geometric flux complicated (some of the issues are addressed in [Mac93]).

Even worse, there is no proof that a consistent choice of switch-over point can be made such that the partial barriers are disjoint. Of course, this is false for partial barriers of types p/q, $p/q+$ and $p/q-$, which necessarily intersect in the minimising periodic orbits of rotation number p/q, but what we mean in this case by disjointness is that the partial barrier of type $p/q+$ should be contained in the closure of the component above that of type $p/q-$, etc. Disjointness is essential in order to interpret the partial barriers of classes (ii) and (iii) as creating a partition by rotation number. It was hoped that the turnstiles could be lined up in "chimneys" [MMP84] and that then disjointness would follow, but this has not been proved.

It was already clear in 1983, however, that the concept of partial barrier is not tied to piecewise stable and unstable manifolds of the minimising orbits. The construction of partial barriers of class (i) in [MMP84], for example, had nothing to do with stable and unstable manifolds. In classes (ii) and (iii), one could start from a partial barrier made from stable and unstable manifolds, W^s and W^u, divide its turnstile into an infinite sequence of "onion layers", (see figure 1) and then move the

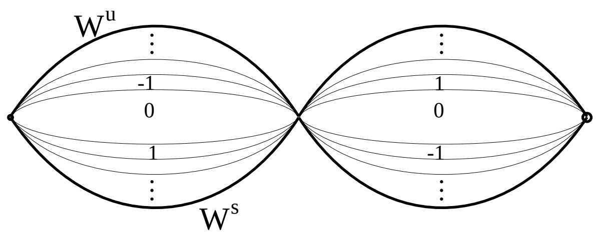

Figure 1: *A turnstile divided into arbitrary onion layers.*

n-th layer, $n \neq 0$, into another gap by applying f^n, the n-th iterate of the map. One thus obtains two curves, C^+ and C^-, such that $f(C^-) = C^+$ and the two intersect only at points of the minimax or minimizing orbits. The effect is to spread the turnstile over the whole orbit of gaps.

This abundance of choice poses the question: "Is there a best choice of partial barrier?" All choices that connect the same pair of minimax and minimizing orbits must yield the same flux (modulo difficulties in interpretation if the partial barrier or its image is not a graph), given by the difference in action between the orbits, and have the same area beneath them (see [MMP87]), but there may be other properties to optimize, *e.g.* to make successive steps in the vertical small in a least squares sense.

We have not yet formulated such a criterion, though [DM92] have proposed something similar. In any case, there are two properties that we would like any system of partial barriers to have:

1. the curves $C^\pm$ should be graphs over the angle variable x; and
2. the C^- corresponding to different rotation numbers should be disjoint (the same result would follow for the C^+ curves).

In 1987, Dick Hall suggested a very natural class of partial barriers. He observed that the minimizing and minimax orbits correspond to critical points of the action, W, in the space of sequences $x = \{x_n\}_{n\in\mathcal{Z}}$, that, generically, have indices 0 and 1, respectively. If one begins at a minimax point and flows in both directions down the gradient of W (that is, along both the positive and negative senses of the single unstable direction of the gradient flow near the minimax orbit) one obtains a curve ? in the sequence space, which Hall named a "ridge curve". By the "invariance principle", ? must end in critical points, which in the simplest case are minimising. The ridge curve induces a pair C^+, C^- of sequences of arcs in the phase space (details will be given in section 2), with the properties that C^+ is the image of C^- and they intersect precisely in the orbits corresponding to the minimax and minimising points of ?. If the minimising orbit is unique then $C^\pm$ form circles round the cylinder. By a result of Angenent [A89], ridge curves are monotone, in the sense that all components x_n increase or decrease together. It follows that $C^\pm$ are graphs.

Golé formalised these ideas. Firstly he showed that the set B of (forward and backward) bounded orbits under the gradient flow of given class c (e.g. periodic sequences of type (p,q)) (which necessarily contains all periodic orbits and the ridge curves of all minimax ones) always has at least the topology of the circle [G89]. Secondly, he showed that one can always choose a subset ? of B, homeomorphic to a circle, which contains a minimax point and all minimising points [G92]. Furthermore, he showed that the resulting curves $C^\pm$ are graphs. He called all these objects: $B, ?, C^+, C^-$, "ghost circles". In order to avoid confusion, we shall reserve the name for the curves C^-.

The big question for us was whether ghost circles for different types are disjoint (in the sense given previously). So we investigated them numerically. Our results suggest that, indeed, ghost circles are disjoint. In the course of searching for a proof, we heard that, motivated by our work, Angenent and Golé have proved this [AG92] — we publish our numerics as motivation and illustration.

2 Action, the gradient flow and ghost circles

The orbits of an area-preserving twist map have a variational characterization that underpins the whole subject of ridge curves. If the map is $f : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R}$ we will find it most convenient to choose a lifting of f, say $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$, where

$$F(x, p) \;=\; (x'(x, p), p'(x, p)).$$

Once we have fixed a particular lift F we can construct a generating function or single-step action, $h(u,v)$, $h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, such that for any three consecutive points in an orbit of F: (x_{j-1}, p_{j-1}), (x_j, p_j) and (x_{j+1}, p_{j+1}), we have

$$\begin{aligned} p_j &= p^+(x_{j-1}, x_j) \\ &= \partial_2 h(x_{j-1}, x_j) \end{aligned} \tag{1}$$

and

$$\begin{aligned} p_j &= p^-(x_j, x_{j+1}) \\ &= -\partial_1 h(x_j, x_{j+1}), \end{aligned} \tag{2}$$

where $\partial_i, i = 1, 2$, denote the partial derivatives with respect to the first and second arguments, respectively. Further, if f has zero net flux, we have

$$h(x+1, x'+1) = h(x, x').$$

From equations (1) and (2) one can see that an orbit of F is completely specified by the sequence of positions, $\{x_j\}$. Indeed, it appears to be over-specified, but the orbits correspond precisely to the sequences for which both specifications of $\{p_j\}$ agree. It follows that a sequence $\{x_j\}$ corresponds to an orbit if and only if

$$W_{m,n} = \sum_{j=m}^{n-1} h(x_j, x_{j+1})$$

is stationary for every $m < n$ with respect to variations fixing x_m and x_n. This is because if $m < j < n$, then:

$$\begin{aligned} \frac{\partial W_{m,n}}{\partial x_j} &= \partial_2 h(x_{j-1}, x_j) + \partial_1 h(x_j, x_{j+1}) \\ &= p^+(x_{j-1}, x_j) - p^-(x_j, x_{j+1}). \end{aligned}$$

In the numerical experiments that follow, we will be interested in partial barriers of class (i), that is, which connect minimising and minimax periodic orbits of rotation number p/q. They are not the most relevant ones for transport, but they are the most amenable to numerical work, and the others can be obtained from them as limits. We will work in the class of sequences of type (p, q), that is sequences x_j such that for all j

$$x_{j+q} = x_j + p.$$

The variational problem for periodic orbits is finite dimensional; we only need to make $W_{0,q}$ stationary subject to $x_q = x_0 + p$. We define $W_{p/q}$ to be $W_{0,q}$ restricted to this space.

The variational problem has a lot of symmetry. $W_{p/q}$ is invariant under the following group of translations $(T_{m,n}x)_j = x_{j+m} + n$. So we will frequently regard sequences as equivalent if they differ only by such a translation.

We construct a ridge curve of type p/q by first identifying a minimax periodic orbit of type (p, q). These are defined by Mather's construction [Ma86], which is not easy to implement. But they are generically of index 1, that is the Hessian $D^2W_{p/q}$ is non-degenerate and has just one negative eigenvalue. Also they are rotationally ordered, that is, for all $m, n, j \in \mathcal{Z}$ if $mp < nq$ then $x_{j+m} < x_j + n$. In practice, any orbit with these two properties will suffice. So we look for critical points of $W_{p/q}$ by Newton's method and choose ones with these two properties.

Next we take an eigenvector $\boldsymbol{u} \in \mathbf{R}^q$ corresponding to the negative eigenvalue. One can do this especially speedily for reversible maps like the standard map if the periodic orbit is symmetric, because then the Hessian can be reduced to a pair of tridiagonal matrices. By lemma 2.2 of [A89], we know that all the entries in $\boldsymbol{u}$ should have the same sign and so this provides a check on the validity of the computation. Then we flow down the gradient of $W_{p/q}$ in initial directions $\pm\mathbf{u}$. We integrated the gradient flow with an adaptive-step Runge-Kutta algorithm [PFTV87], stopping when the trajectory entered a small neighborhood of another critical point. The resulting trajectories must terminate at critical points. The argument is that $W_{p/q}$ decreases along trajectories and the set of sequences for which $W_{p/q}$ is less than or equal to its initial value is compact (after identifying sequences which differ only by a translation $T_{m,n}$). The critical points at the ends of the trajectory must have lower action than the minimax point. Often they are two representatives of the same globally minimising orbit, in which case we are done. If not, then the minimax theory ensures that there is another "local" minimax between the two endpoints. If this is located, another ridge curve can be constructed in the gap. The argument can be repeated until the gaps are all closed, except in exceptional cases when there is a whole arc of critical points. But in this case, such arcs can be used to close the gaps. This way we obtain a circle ? in sequence space, composed of ridge curves.

Applying Angenent's lemma again, we deduce that all components of x increase together along ?. Thus it can be parametrised by any one of the components, e.g. x_0.

Given a union of ridge curves ? forming a circle in sequence space, we produce two circles $C^{\pm}$ in phase space as follows:

$$\begin{aligned} C^- &= \left\{ \left(x_j, \, p^-(x_j, \, x_{j+1})\right) : x \in \, ? \, , j = 0, \ldots, q-1 \right\} \\ C^+ &= \left\{ \left(x_j, \, p^+(x_{j-1}, \, x_j)\right) : x \in \, ? \, , j = 1, \ldots, q \right\} \end{aligned} \tag{3}$$

By construction, $f(C^-) = C^+$ and the two curves intersect only at points of rotationally ordered periodic orbits of type (p, q).

3 Numerical results

Here we illustrate the notions of ridge curve and ghost circle with orbits of the standard map:

$$\begin{aligned} p' &= p - \frac{k}{2\pi}\sin(2\pi x) \\ x' &= x + p' \end{aligned}$$

which is generated by

$$h(x, x') = \frac{1}{2}(x' - x)^2 + \frac{k}{4\pi^2}\cos(2\pi x).$$

Figure 2 shows the sequence space for orbits of type (1,2); the thin lines are contours of the action

$$\begin{aligned} W_{1/2} &= h(x_0, x_1) + h(x_1, x_2) \\ &= h(x_0, x_1) + h(x_1, x_0 + 1) \\ &= \frac{1}{2}((x_1 - x_0)^2 + (x_0 + 1 - x_1)^2) + \frac{k}{4\pi^2}(\cos(2\pi x_0) + \cos(2\pi x_1)) \end{aligned}$$

The sequences corresponding to the minimax orbit are at points of the form $(j, j + 1/2)$, where j is an integer. We chose $k = 12.0$ to show what happens when, besides the minimax and globally minimizing sequences, there are other critical points as well. In this case, they correspond to badly-ordered orbits (see [LM86]), and the set B of bounded orbits under the gradient flow would contain them plus in fact a whole 2-dimensional region as indicated. Nonetheless, as the figure illustrates, it is possible to choose a skeleton of ridge curves that connects minimax to minimizing points. The corresponding ghost circle and its image are shown in figure 3.

The main point of the numerics was to see whether ghost circles corresponding to distinct rotation numbers are disjoint. Figure 4 illustrates our results: the ghost circles for periodic orbits of types (2,5) and (5,13), even though they come very close to each other, do not cross. These two rotation numbers are Farey neighbours with a difference of only 1/65. In no case did we ever find an intersection of ghost circles.

This strongly suggests that ghost circles corresponding to distinct rotation numbers are disjoint.

4 Discussion

We have shown that the ghost circles of [G92] can be constructed numerically and that they appear to be disjoint. Angenent and Golé have subsequently announced a proof that it is possible to choose ghost circles to be disjoint [AG92], though it is not clear whether every choice of ghost circles is automatically disjoint.

The main question remaining is whether ghost circles have some optimality property as almost invariant curves.

There is still some freedom in the construction of ghost circles. In particular, they depend on the choice of metric in sequence space, used to define the gradient operator. It is clear one should choose a translation invariant one, in which case the standard l_2 metric looks most natural, but this could be prejudice.

Moser [Mo91] has recently proposed an alternative family of almost-invariant curves for area preserving twist maps. In fact, he obtains a complete foliation by graphs. These, however, bear no particular relation to the minimizing and minimax orbits.

The notion of ridge curves has a natural generalization to 2N-dimensional symplectic maps. Golé [G89] proved that the set of bounded orbits, B, of given type for the gradient flow, has at least the

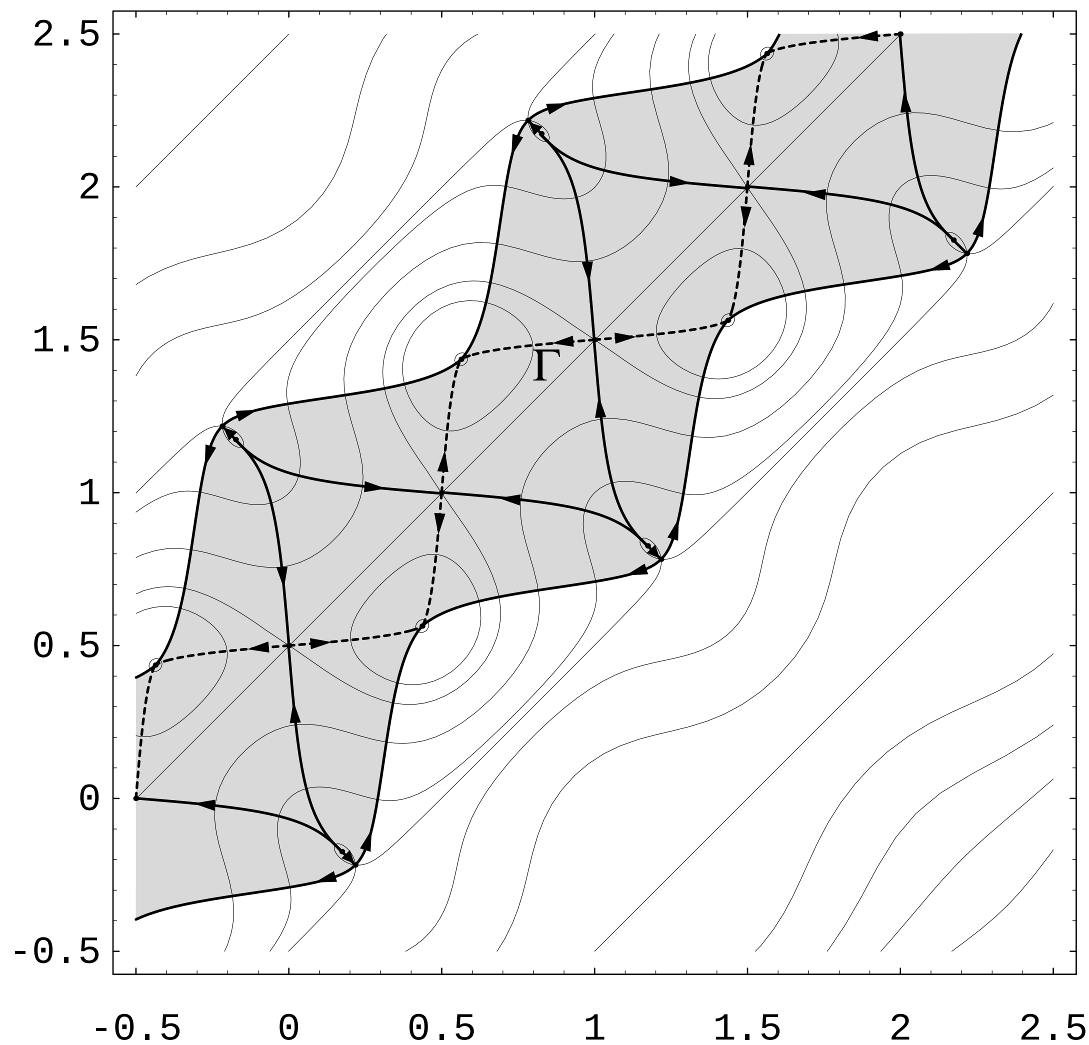

Figure 2: *A portion of the sequence space for period 2 orbits showing contours (thin lines) of the function $W_{1/2}$ for $k = 12.0$. The heavy black curves and the dashed curve are made up of trajectories of the gradient flow that connect critical points; saddle points lie on the line $x_1 = x_0 + \frac{1}{2}$ while the global minima are at the bottoms of the wells lying to either side. The dashed curve ? is our desired ridge curve, connecting the minimising and minimax points monotonically. The pair of critical points near (1.25, 0.75), a maximum and a saddle, correspond to badly-ordered periodic orbits; they are not part of the ridge curve and do not affect the ghost circles. The shaded region is the set B of (backward and forward) bounded trajectories of the gradient flow.*

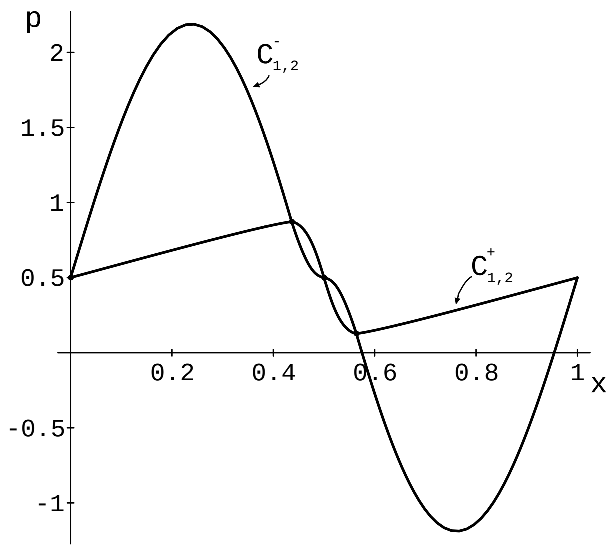

Figure 3: *The ghost circle corresponding to the ridge curve in figure 2, and its image.*

topology of an N-torus, meaning there is an injective map from the cohomology of $\mathcal{T}^N$ into that of B. It is plausible that we could make an invariant subset homeomorphic to an N-torus by taking the N-dimensional unstable manifold of index-N critical points in sequence space. This would give rise to N-dimensional "ghost tori" in phase space. Unfortunately, the monotonicity results of Angenent have no analogue in higher dimensions, so it is not easy to give a watertight construction, but we intend to investigate their construction, properties and potential uses for understanding transport.

Finally, note that the notion of a ridge curve plays a fundamental role in the theory of Floer homology (for a review, see [Sal91]). The idea there is to consider the gradient flow of

$$\int p\,dq - H\,dt \tag{4}$$

on the space of period-1 loops in phase space, for a time periodic Hamiltonian H (of period 1) on a symplectic manifold M. The critical points of (4) correspond to period-1 periodic orbits. Floer obtains a lower bound on the number of these orbits by establishing relations between the set of connecting trajectories for the gradient flow (the ridge curves) and the homology groups of M. What has yet to be exploited is that the ridge curves induce "almost invariant" cylinders in the extended phase space, $M \times S^1$, that connect pairs of period-1 orbits. These cylinders are the continuous-time analogues of our ghost circles.

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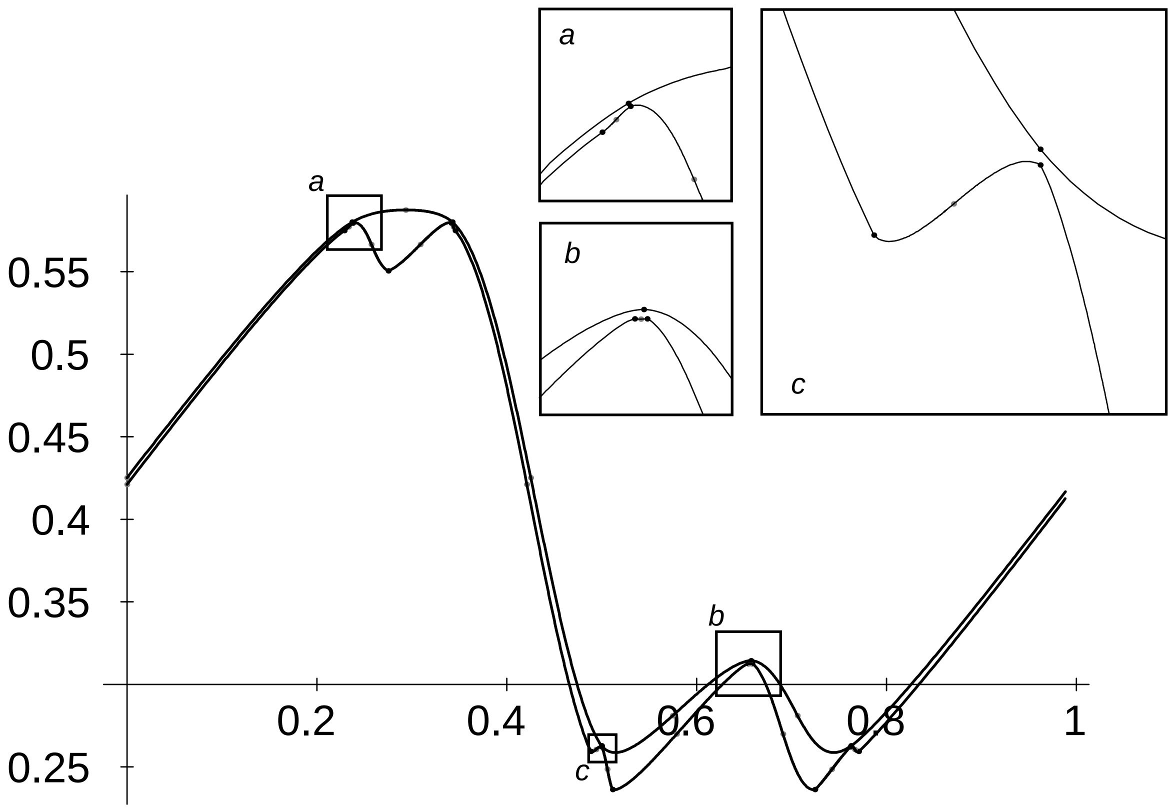

Figure 4: *The ghost circles* $C^-_{2,5}$ *and* $C^-_{5,13}$ *when* $k = 2.0$*. The insets (a)-(c) are enlargements of the regions where the two curves approach each other.*

Appendix

We show here how to construct examples of area-preserving twist maps with a hyperbolic minimising fixed point for which there is no minimising advancing homoclinic point such that the unstable and stable manifolds are both graphs up to this point.

Take a map of the standard form

$$y' = y + V'(x), x' = x + y', \tag{5}$$

V of period-1, with a pair of neighbouring hyperbolic minimising fixed points (which are typically translates of each other) with x-coordinates $X_0 < X_1$, and with a transverse minimising heteroclinic orbit with x-coordinates x_n, from X_0 to X_1 (e.g. $V(x) = \frac{k}{4\pi^2}\cos 2\pi x, k > 2\sqrt{\pi^2+1}$). The x_n are ordered in the same order as their indices, but the stable and unstable manifolds $W^s(X_1), W^u(X_0)$ are not graphs, because they accumulate along $W^s(X_0), W^u(X_1)$, respectively, with oscillations (Palis's inclination lemma).

Suppose $W^u(X_0)$ is a graph from X_0 up to x_0, say $y = Y(x)$. We will modify V on $x_{-2} < x < x_{-1}$ to make $W^u(X_0)$ have a turning point before it reaches the vertical through x_0. This does not affect $W^u(X_0)$ before x_{-1}, but the image of the piece from $x_{-2} < x < x_{-1}$ is given by

$$x' = x + Y(x) + V'(x), y' = Y(x) + V'(x). \tag{6}$$

By making $V''(x) < -1 - Y'(x)$ somewhere in $x_{-2} < x < x_{-1}$, taking care to keep $x + Y(x) + V'(x) < x_0$ up to there, x' becomes a non-monotone function of x, and $W^u(X_0)$ acquires a turning point before reaching the vertical through x_0.

Similarly, suppose $W^s(X_1)$ is a graph from X_1 down to x_0, say $y = Y(x)$. We will modify V on $x_1 < x < x_2$ to introduce a turning point in $W^s(X_1)$ before it crosses the vertical through x_0. This leaves $W^s(X_1)$ unchanged on $x_2 < x < X_1$, but modifies Y to $\tilde{Y}(x) = Y(x) - \Delta V'(x)$ on $x_1 < x < x_2$, where ΔV is the change to V. By making $\tilde{Y}$ have sufficiently large positive slope somewhere in this interval, a turning point will be introduced in the preimage (by twist). Taking care to keep $x - \tilde{Y}(x) > x_0$, this turning point will occur before $W^s(X_1)$ reaches the vertical through x_0.

Thus we can introduce turning points in each of $W^u(X_0)$ and $W^s(X_1)$ to the left and right, respectively of the vertical through x_0, if either did not already have one. Then it is clear that for this modified map, there is no switchover point possible from unstable to stable manifold such that both are graphs up to this point.

References

[A89] S.A. Angenent, "The periodic orbits of an area preserving twist map," *Comm. Math. Phys.* **115**, no. 3, (1989).

[AG92] S. Angenent and C. Golé, manuscript in preparation.

[DM92] R.L. Dewar and J.D. Meiss, "Flux-minimizing curves in area-preserving twist maps," *Physica* **D 57**, pp. 476-506, (1992).

[G89] C. Golé, "Periodic Points for Monotone Symplectomorphisms of $T^n \times R^n$," PhD Thesis, Boston University, (1989).

[G92] C. Golé, "Ghost Circles for twist maps," *Journal of Diff. Eqns.* **97**, pp. 140-173, (1992).

[LM86] I. Leage and R.S. MacKay, "Badly ordered periodic orbits of the standard map," *Phys. Lett. A*, **118**, pp. 274-278, (1986).

[Mac93] R.S.MacKay, Hyperbolic structure in classical chaos, in: Quantum Chaos, eds G.Casati, I.Guarneri, U.Smilansky (Ital Phys Soc) in press.

[MMP84] R.S. MacKay, J.D. Meiss and I.C. Percival, "Transport in Hamiltonian systems," *Physica* **13D**, pp. 55-81, (1984).

[MMP87] R.S. MacKay, J.D. Meiss and I.C. Percival, "Resonances in area preserving maps," *Physica* **27D**, pp. 1-20 , (1987).

[Ma86] J.N. Mather, "A criterion for non-existence of invariant circles", *Publ. Math. IHES*, **63**, pp. 153-204, (1986).

[Mo91] J.K. Moser, "An unusual variational problem connected with Mather's theory for monotone twist mappings," in Proc. Euler Inst. Dynamical Systems Semester, (St. Petersburg, Autumn 1991), to appear.

[PFTV87] W.H. Press, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling, *Numerical Recipes in C*, (Cambridge University Press, Cambridge, 1987).

[Sal91] D.A. Salamon, "Morse Theory, the Conley Index and Floer Homology," *Bull. London Math. Soc.* **22**, pp. 113-140, (1990).