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Two-Loop Master Integrals for $\gamma^* \to 3$ Jets: the Non-Planar Topologies

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Abstract

The calculation of the two-loop corrections to the three-jet production rate and to event shapes in electron–positron annihilation requires the computation of a number of two-loop four-point master integrals with one off-shell and three on-shell legs. Up to now, only those master integrals corresponding to planar topologies were known. In this paper, we compute the yet outstanding non-planar master integrals by solving differential equations in the external invariants which are fulfilled by these master integrals. We obtain the master integrals as expansions in $\epsilon = (4-d)/2$, where d is the space-time dimension. The fully analytic results are expressed in terms of the two-dimensional harmonic polylogarithms already introduced in the evaluation of the planar topologies.

1 Introduction

Precision applications of particle physics phenomenology often demand theoretical predictions at the next-to-next-to-leading order in perturbation theory. Corrections to this order are known for many inclusive observables, such as total cross sections or sum rules. For $2 \to 2$ scattering and $1 \to 3$ decay processes, the calculation of next-to-next-to-leading order corrections is a yet outstanding task. One of the major ingredients for these calculations consists the two-loop virtual corrections to the corresponding four-point Feynman amplitudes.

Using dimensional regularization [1–3] with $d = 4 - 2\epsilon$ dimensions as regulator for ultraviolet and infrared divergences, the large number of different integrals appearing in the two-loop Feynman amplitudes for $2 \to 2$ scattering or $1 \to 3$ decay processes can be reduced to a small number of master integrals. The techniques used in these reductions are integration-by-parts identities [3–5] and Lorentz invariance [6]. A computer algorithm for the automatic reduction of all two-loop four-point integrals was described in [6].

For two-loop four-point functions with massless internal propagators and all legs on-shell, which are relevant for example to the next-to-next-to-leading order calculation of two-jet production at hadron colliders, all master integrals have been calculated over the past year [7–12]. Very recently, these master integrals were already applied in the calculation of two-loop virtual corrections to Bhabha scattering [13] in the limit of vanishing electron mass and to quark–quark scattering [14].

A different class of master integrals is required for the computation of next-to-next-to-leading order corrections to observables such as the three-jet production rate in electron–positron annihilation, two-plus-one-jet production in deep inelastic electron–proton scattering or vector-boson-plus-jet production at hadron colliders: two-loop four-point functions with massless internal propagators and one external leg off-shell. These functions involve one more scale than their on-shell counterparts, and are therefore more complicated and also more numerous. Up to now, only the master integrals for planar topologies [15,16] as well as one of the non-planar master integrals were known [17]. In [15,17], Smirnov used a Mellin–Barnes countour integral technique to derive one planar and one non-planar master integral, obtaining analytic expressions for the divergent parts and an integral representation (one-dimensional in the planar case and three-dimensional in the non-planar case) for the finite part. Using the differential equation technique derived in [6], we computed the full set of planar master integrals in [16]. Complementary work on the purely numerical evaluation of this type of master integrals has been presented recently by Binoth and Heinrich [18].

Our results for the planar topologies [16] were given in fully analytic form, in terms of a new class of functions, two-dimensional harmonic polylogarithms (2dHPL). The 2dHPL are an extension of the harmonic polylogarithms (HPL) of [19]. All HPL and 2dHPL that appear in the divergent parts of the planar master integrals have weight ≤ 3 and can be related to the more commonly known Nielsen's generalized polylogarithms [20,21] of suitable arguments. The functions of weight 4 appearing in the finite parts of the master integrals can all be represented, by the very definition, as one-dimensional integrals over Nielsen's polylogarithms of weight 3, hence of Nielsen's generalized polylogarithms of suitable arguments according to the above remark. A table with all relations is included in the appendix of [16].

It is the purpose of the present paper to extend our work of [16] towards the computation of non-planar four-point master integrals with one off-shell leg from their differential equations. In Section 2, we discuss the use of the differential equation method in the computation of master integrals. We point out in particular the differences to the calculation of the planar master integrals. We then list our analytical results for the complete set of non-planar master integrals in Section 3. Our results are, like in the planar case, expressed in terms of 2dHPL. In Section 3 we also compare our results with the existing partial ones available in the literature. Finally, Section 4 contains conclusions and an outlook on future applications.

2 Solving Differential Equations for Master Integrals

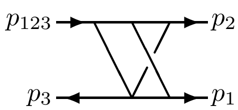
The use of differential equations for the computation of master integrals was first suggested in [22] as a means of relating integrals with massive internal propagators to their massless counterparts. The method was developed in detail in [23], where it was also extended towards differential equations in the external invariants. As a first application of this method, the two-loop sunrise diagram with arbitrary internal masses was studied in [24]. We have developed the differential equation formalism for two-loop four-point functions with massless internal propagators, three external legs on-shell and one external leg off-shell in [6]. We derived an algorithm for the automatic reduction, by means of computer algebra (using FORM [25] and Maple [26]), of any two-loop four-point integral to a small set of master integrals, for which we derived differential equations in the external invariants.

Four-point functions depend on three linearly independent momenta: p_1, p_2 and p_3. In our calculation, we take all these momenta on-shell ($p_i^2 = 0$), while the fourth momentum $p_{123} = (p_1 + p_2 + p_3)$ is taken off-shell. The kinematics of the four-point function is then fully described by specifying the values of the three Lorentz invariants $s_{12} = (p_1 + p_2)^2$, $s_{13} = (p_1 + p_3)^2$ and $s_{23} = (p_2 + p_3)^2$.

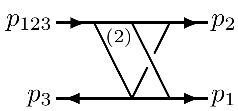
Expressing the system of differential equations obtained in [6] for any master integral in the variables $s_{123} = s_{12} + s_{13} + s_{23}$, $y = s_{13}/s_{123}$ and $z = s_{23}/s_{123}$, we obtain a homogeneous equation in s_{123}, and inhomogeneous equations in y and z. Since s_{123} is the only quantity carrying a mass dimension, the corresponding differential equation is nothing but the rescaling relation obtained by investigating the behaviour of the master integral under a rescaling of all external momenta by a constant factor.

In the reduction of non-planar two-loop four-point functions with one off-shell leg, one obtains (in addition to master integrals corresponding to planar subtopologies) several genuinely non-planar master integrals, corresponding to three- and four-point topologies. The two master integrals corresponding to three-point topologies are the crossed vertex integral with one leg off-shell, two legs on-shell and the crossed vertex integral with two legs off-shell, one leg on-shell. The former obeys only a homogeneous differential equation, which is of no use for its complete computation, but it is otherwise relatively simple and already known in the literature [27, 28]. We list its value in Section 3 for reference. For the latter vertex integral with two off-shell legs, one obtains one homogeneous and one inhomogeneous differential equation. The computation of this integral from its inhomogeneous differential equation is straightforward, following the procedure used for the planar integrals [16].

Of the four non-planar four-point topologies with one off-shell leg, only one is reducible to planar subtopologies; the remaining three contain two master integrals each [6]. For the six-propagator topology, we use the following basis of master integrals:

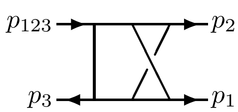


$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 (k - p_{123})^2 (k - l)^2 (k - l - p_1)^2 l^2 (l - p_2)^2}, \tag{2.1}$$

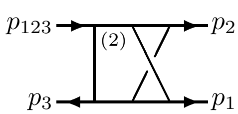


$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{2k \cdot p_2}{k^2 (k - p_{123})^2 (k - l)^2 (k - l - p_1)^2 l^2 (l - p_2)^2}. \tag{2.2}$$

For the two seven-propagator topologies, we choose:

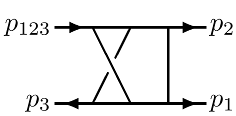


$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 (k - p_{12})^2 (k - p_{123})^2 (k - l)^2 (k - l - p_1)^2 l^2 (l - p_2)^2}, \tag{2.3}$$

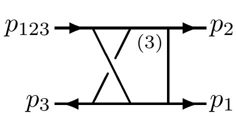


$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{2k \cdot p_2}{k^2 (k - p_{12})^2 (k - p_{123})^2 (k - l)^2 (k - l - p_1)^2 l^2 (l - p_2)^2}, \tag{2.4}$$

and,



$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 (k - p_{123})^2 (k - l)^2 (k - l - p_3)^2 l^2 (l - p_2)^2 (l - p_{12})^2}, \tag{2.5}$$



$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{2l \cdot p_3}{k^2 (k - p_{123})^2 (k - l)^2 (k - l - p_3)^2 l^2 (l - p_2)^2 (l - p_{12})^2}. \tag{2.6}$$

This choice of basis is not unambiguous. Our main motivation for this particular basis is the decoupling, in the $d \to 4$ limit, of the homogeneous part of the differential equations for each topology, which is crucial for the applicability of our algorithm to the solution of the differential equations outlined below.

Quite in general, the analytic expression of all planar and non-planar amplitudes is found to consist of several terms, each equal to a prefactor (a simple rational polynomial in the external invariants) times a combination of 2dHPL multiplied by simple coefficients (integer numbers or rational coefficients). The major difference between the planar and non-planar master integrals is in the number of such terms. Indeed, all planar master integrals can be written as the product of a single prefactor times a Laurent series in $\epsilon = (4-d)/2$ containing as coefficients only logarithms and polylogarithms of ratios of the invariants. It has become clear already in the on-shell case [8] that the non-planar master integrals do not take such a simple form. This expectation was confirmed for the off-shell non-planar master integrals in [18], where the $1/\epsilon^4$ and $1/\epsilon^3$ terms of (2.3) and (2.5) were calculated analytically. All these non-planar integrals contain the sum of several different prefactors, each multiplied by a different Laurent series in ϵ.

The algorithm employed in our calculation of the planar master integrals from their differential equations [16] relied on the possibility of determining a unique rational prefactor for each master integral. Once this prefactor was found, the coefficients in the Laurent series were determined by inserting a general ansatz containing all possible harmonic polylogarithms, up to a given weight, and a subsequent matching of the individual terms. This procedure has to be modified to account for the more involved structure of the non-planar master integrals.

With two master integrals $T_1(y,z,\epsilon)$ and $T_2(y,z,\epsilon)$ for a given topology, the differential equations in y take the following form, exact in ϵ:

$$
\begin{aligned}
\frac{\partial}{\partial y} T_1(y,z,\epsilon) &= R_{11}(y,z,\epsilon) T_1(y,z,\epsilon) + R_{12}(y,z,\epsilon) T_2(y,z,\epsilon) + I_1(y,z,\epsilon) \,, \\
\frac{\partial}{\partial y} T_2(y,z,\epsilon) &= R_{21}(y,z,\epsilon) T_1(y,z,\epsilon) + R_{22}(y,z,\epsilon) T_2(y,z,\epsilon) + I_2(y,z,\epsilon) \,,
\end{aligned}
\tag{2.7}
$$

where the $R_{ij}(y,z,\epsilon)$ are rational functions of y, z and ϵ, of order 1 in ϵ; $I_1(y,z,\epsilon)$ and $I_2(y,z,\epsilon)$ are the inhomogeneous terms, containing the simpler and known master integrals of subtopologies of the topology under consideration. The first master integral $T_1(y,z,\epsilon)$ is always chosen to contain no scalar products in the numerator and all propagators only to unit power in the denominator, see (2.1),(2.3),(2.5). The second master integral $T_2(y,z,\epsilon)$ can be chosen to contain one or more irreducible scalar products in the numerator, or one or more squared propagators in the denominator, or even a combination of both. The only requirement on $T_2(y,z,\epsilon)$ is its independence of $T_1(y,z,\epsilon)$ under integration-by-parts and Lorentz invariance identities. Any integral of the topology under consideration can then be expressed as a linear combination of T_1 and T_2 plus simpler subtopologies. By exploiting the freedom to choose $T_2(y,z,\epsilon)$, we always succeeded in finding a $T_2(y,z,\epsilon)$ for which $R_{21}(y,z,\epsilon)$ in (2.7) is of order ϵ (this condition is fulfilled for the basis of master integrals (2.1)–(2.6) introduced above). As a consequence, the y-differential equation for $T_2(y,z,\epsilon)$, when Laurent-expanded in ϵ, decouples from $T_1(y,z,\epsilon)$ and the coefficients of the expansions can be obtained from the differential equations by recursive quadrature.

Let us illustrate the algorithm when $T_1(y,z,\epsilon)$ and $T_2(y,z,\epsilon)$ are the amplitudes corresponding to (2.5) and (2.6), respectively. The explicit form of the equations, with $d = 4 - 2\epsilon$, in that case is

$$
\begin{aligned}
\frac{\partial}{\partial y} T_1(y,z,\epsilon) &= \left(-\frac{d-4}{y} + \frac{d-5}{y+z} + \frac{1}{1-y-z} \right) T_1(y,z,\epsilon) \\
&\quad + \frac{2d-9}{z} \left(\frac{1}{y} - \frac{1}{y+z} \right) T_2(y,z,\epsilon) + I_1(y,z,\epsilon) \,; \\
\frac{\partial}{\partial y} T_2(y,z,\epsilon) &= -(d-4) \frac{z}{2y} T_1(y,z,\epsilon) \\
&\quad + \left(\frac{3d-14}{2y} + \frac{1}{1-y-z} \right) T_2(y,z,\epsilon) + I_2(y,z,\epsilon) \,.
\end{aligned}
\tag{2.8}
$$

We now expand everything in Laurent series in ϵ, up to terms of order 0, as

$$T_k(y,z,\epsilon) = \sum_{i=0}^{4} \frac{1}{\epsilon^i} T_{k,i}(y,z) \ , \qquad\qquad I_k(y,z,\epsilon) = \sum_{i=0}^{4} \frac{1}{\epsilon^i} I_{k,i}(y,z) \ ; \tag{2.9}$$

the equations then read

$$\begin{aligned} \frac{\partial}{\partial y} T_{1,i}(y,z) &= \left(\frac{1}{1-y-z} - \frac{1}{y+z} \right) T_{1,i}(y,z) \\ &\quad + \frac{1}{z} \left(-\frac{1}{y} + \frac{1}{y+z} \right) T_{2,i}(y,z) + J_{1,i}(y,z) \ , \\ \frac{\partial}{\partial y} T_{2,i}(y,z) &= \left(-\frac{1}{y} + \frac{1}{1-y-z} \right) T_{2,i}(y,z) + J_{2,i}(y,z) \ , \end{aligned} \tag{2.10}$$

where the new inhomogeneous terms are

$$\begin{aligned} J_{1,i}(y,z) &= \left(-\frac{1}{y} + \frac{1}{y+z} \right) T_{1,i+1}(y,z) + \frac{2}{z} \left(\frac{1}{y} - \frac{1}{y+z} \right) T_{2,i+1}(y,z) + I_{1,i}(y,z) \ , \\ J_{2,i}(y,z) &= -\frac{z}{2y} T_{1,i+1}(y,z) + \frac{3}{2y} T_{2,i+1}(y,z) + I_{2,i}(y,z) \ ; \end{aligned} \tag{2.11}$$

and it is understood that quantities whose second index is greater than 4 vanish, as the greatest singularity of the Laurent expansion in ϵ is 4, according to the analysis of the soft and collinear structure [29]. The system is solved bottom up by starting from the coefficients of greatest singularity corresponding to $i = 4$, $T_{k,4}(y,z)$, whose equations involve as inhomogeneous terms only the $I_{k,4}(y,z)$, which are known. One then proceeds to the equations for $T_{k,3}(y,z)$, which involve as inhomogeneous terms $T_{k,4}(y,z)$ and $I_{k,3}(y,z)$, which are known by now and so on until the $T_{k,0}(y,z)$ are obtained.

According to the preceding general remarks, the equation for $T_{2,i}(y,z)$ does not involve $T_{1,i}(y,z)$ (it involves $T_{1,i+1}(y,z)$, which is known when considering the i-th term of the Laurent expansion); the homogeneous part of the equation is almost trivial, having as a solution $1/[y(1-y-z)]$; by following the standard method of the variation of the constants, the solution of the inhomogeneous equation is immediately given by the quadrature formula

$$T_{2,i}(y,z) = \frac{1}{y(1-y-z)} \int^y \mathrm{d}y' \ y' \ (1-y'-z) J_{2,i}(y',z) \ . \tag{2.12}$$

The above result can be inserted into the equation for $T_{1,i}(y,z)$, which is then solved in the same manner; starting from $i = 4$, the whole procedure can be iterated for a smaller value of i, till $i = 0$ is reached (one could in principle arrive at any other desired value of i). Note that the whole process involves only the y-dependent denominators appearing in the differential equations in the variable y, namely $1/y$, $1/(1-y)$, $1/(y+z)$ and $1/(1-y-z)$. As the homogeneous terms are also equal to products of those denominators and 2dHPL functions, after careful partial fractioning of the y-dependent factors the integrations in (2.12) can always be evaluated analytically: when a square denominator, say $1/y^2$, occurs, they can be carried out almost trivially by integrating by parts, while when the integrand is a single denominator times any 2dHPL function of some weight w, the primitive is nothing but another 2dHPL function of weight $w+1$ (see the Appendix of [16] for more details on 2dHPL).

The quadrature formula (2.12) is definite up to a constant of integration in y, so one still needs one constant per master integral per order in ϵ. These constants of integration (which are functions of z) can be determined only by evaluating the master integrals for some boundary condition. In the planar case [16], we have used the fact that all planar master integrals, as well as their derivatives, are regular in the whole kinematical plane, with the exception of two branch points at $y = 0$ and $z = 0$. As a consequence, any factor $(1-y)$, $(1-y-z)$, or $(y+z)$ appearing in the denominator of the homogeneous term of the differential equations for a master integral could be used to determine the boundary condition in $y = 1$, $y = (1-z)$, or $y = -z$, respectively: multiplying the differential equation with one of these factors and taking the limit where the factor vanishes, the derivative and all the other terms not having that factor as a denominator drop out and one obtains a linear relation expressing the master integral in this special kinematical point in terms of the known master integrals of its subtopologies. The non-planar master integrals possess a different analytic structure: they have three branch points: in $y = 0$, $z = 0$ and $y = 1-z$. Consequently, only factors $(1-y)$ or $(y+z)$ in the denominator of the homogeneous term can be used for the determination of the boundary conditions. It turns out that these boundary conditions alone are not sufficient for a complete determination of the constants of integration in all the master integrals: for each topology, one finds that one constant of integration per order in ϵ remains unconstrained. We determine this (z-dependent) constant by solving the system of z-differential equations for each topology at the point $y = 1$ and requiring this solution to be regular at the point $z = 1$. Some care has to be taken in the analytic continuation of the master integrals across the cut in $y = 1-z$ to the point $y = 1$; details on the analytic continuation of the 2dHPL can be found in the appendix of [16].

Using the integration procedure described here, we have determined all non-planar two-loop four-point master integrals with one off-shell leg. The results are summarized in the following section.

3 Master Integrals

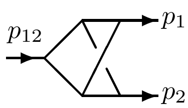
In the reduction of a non-planar two-loop four-point function with one off-shell leg, one will in general encounter master integrals with non-planar topology in combination with master integrals corresponding to planar subtopologies. These planar master integrals were all computed in [16]. In this section, we tabulate all genuinely non-planar master integrals relevant to the computation of two-loop four-point functions with one off-shell leg. We classify the integrals, according to the number of different kinematical scales on which they depend, into one-scale integrals, two-scale integrals and three-scale integrals.

The common normalization factor of all master integrals is

$$S_\epsilon = \left[(4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] . \tag{3.1}$$

3.1 One-Scale Integrals

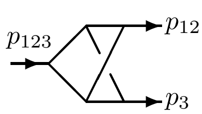
The only non-planar one-scale integral is the crossed on-shell vertex integral, which was first computed in [27, 28]. We list the result only for completeness:



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{12})^{-2-2\epsilon} \left[-\frac{1}{\epsilon^4} + \frac{5\pi^2}{6\epsilon^2} + \frac{23}{\epsilon}\zeta_3 + \frac{103\pi^4}{180} + \mathcal{O}(\epsilon) \right] . \tag{3.2}$$

3.2 Two-Scale Integrals

The crossed vertex integral with two off-shell legs is the only non-planar two-scale master integral. This integral was, to the best of our knowledge, not known up to now. It fulfils one inhomogeneous differential equation in the ratio of the two external invariants, which can be employed for its computation. We obtain:



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{(s_{123}-s_{12})^2} \sum_{i=0}^{4} \frac{g_{6.1,i}\left(\frac{s_{12}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{3.3}$$

with:

$$g_{6.1,4}(x) = 0 , \tag{3.4}$$

$$g_{6.1,3}(x) = 0 , \tag{3.5}$$

$$g_{6.1,2}(x) = -2H(0,0;x) , \tag{3.6}$$

$$g_{6.1,1}(x) = +6H(0,0,0;x) + 4H(0,1,0;x) - 4H(1,0,0;x) + \frac{2}{3}\pi^2 H(0;x) + 12\zeta_3 \ , \tag{3.7}$$

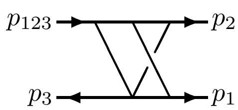
$$\begin{aligned} g_{6.1,0}(x) &= -14H(0,0,0,0;x) - 2H(0,0,1,0;x) - 6H(0,1,0,0;x) + 4H(0,1,1,0;x) \\ &\quad +12H(1,0,0,0;x) + 8H(1,0,1,0;x) - 8H(1,1,0,0;x) + \frac{23\pi^4}{90} \\ &\quad +\zeta_3 \left[-2H(0;x) + 24H(1;x)\right] + \frac{\pi^2}{6}\left[-2H(0,0;x) + 4H(0,1;x) + 8H(1,0;x)\right] \ . \end{aligned} \tag{3.8}$$

3.3 Three-Scale Integrals

There are four different topologies of non-planar two-loop four-point functions with one off-shell leg, two topologies with six different propagators, and two others with seven different propagators. Only one of the six-propagator topologies is fully reducible to simpler (planar) subtopologies [6]. All other topologies contain two master integrals each. The two master integrals for each topology fulfil a coupled system of inhomogeneous differential equations in $y = s_{13}/s_{123}$ and another coupled system in $z = s_{23}/s_{123}$. As explained in Section 2, solving one of these systems is already sufficient to compute the master integrals up to a constant of integration. The second system is used to constrain the boundary conditions and hence the constants of integration, as well as a check on the result.

The choice of the basis of master integrals for each topology is described in Section 2. In particular, we always choose the integral without scalar products in the numerator and no squared propagator as one of the master integrals. This integral is symmetric under the interchange of two external momenta, which we take to be p_1 and p_2.

The two master integrals for the six-propagator topology are:



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{12}s_{123}} \sum_{i=0}^{4} \frac{f_{6.2,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \tag{3.9}$$

with:

$$f_{6.2,4}(y,z) = -\frac{1}{4} \ , \tag{3.10}$$

$$f_{6.2,3}(y,z) = -\frac{1}{2}H(0;y) - \frac{1}{2}H(0;z) - \frac{1}{2}H(1;z) - \frac{1}{2}H(1-z;y) \ , \tag{3.11}$$

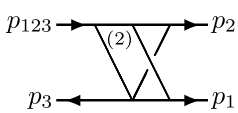
$$\begin{aligned} f_{6.2,2}(y,z) &= -H(0;y)H(0;z) - H(0;y)H(1;z) - H(0;z)H(1-z;y) + H(0,0;y) + H(0,0;z) \\ &\quad -H(0,1;z) - H(0,1-z;y) - H(1;z)H(1-z;y) + 2H(1,0;y) + H(1,0;z) \\ &\quad -H(1,1;z) - H(1-z,0;y) - H(1-z,1-z;y) + \frac{\pi^2}{2} \ , \end{aligned} \tag{3.12}$$

$$\begin{aligned} f_{6.2,1}(y,z) &= +2H(0;y)H(1,0;z) - 2H(0;y)H(1,1;z) + 4H(0;z)H(1,0;y) - 2H(0;z)H(1-z,0;y) \\ &\quad -2H(0;z)H(1-z,1-z;y) + 2H(0,0;y)H(0;z) + 2H(0,0;y)H(1;z) + 2H(0,0;z)H(0;y) \\ &\quad +2H(0,0;z)H(1-z;y) - 2H(0,0,0;y) - 2H(0,0,0;z) + 2H(0,0,1;z) + 2H(0,0,1-z;y) \\ &\quad -2H(0,1;z)H(0;y) - 2H(0,1;z)H(1-z;y) + H(0,1,0;y) + 3H(0,1,0;z) - 2H(0,1,1;z) \\ &\quad -2H(0,1-z;y)H(0;z) - 2H(0,1-z;y)H(1;z) + 2H(0,1-z,0;y) \\ &\quad -2H(0,1-z,1-z;y) - 2H(1;z)H(1-z,0;y) - 2H(1;z)H(1-z,1-z;y) \\ &\quad +4H(1,0;y)H(1;z) + 2H(1,0;z)H(1-z;y) - 4H(1,0,0;y) - 2H(1,0,0;z) \\ &\quad +2H(1,0,1;z) + 4H(1,0,1-z;y) - 2H(1,1;z)H(1-z;y) - 3H(1,1,0;y) + 3H(1,1,0;z) \\ &\quad -2H(1,1,1;z) + 4H(1,1-z,0;y) + 2H(1-z,0,0;y) - 2H(1-z,0,1-z;y) \end{aligned}$$

$$
\begin{aligned}
& +4H(1-z,1,0;y)-2H(1-z,1-z,0;y)-2H(1-z,1-z,1-z;y) \\
& +\frac{23}{2}\zeta_3+\frac{\pi^2}{6}\Big[+3H(0;y)+3H(0;z)-3H(1;y)+3H(1;z)+6H(1-z;y)\Big]\,, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
f_{6.2,0}(y,z) \;=\; & -4H(0;y)H(1,0,0;z)-2H(0;y)H(1,0,1;z)-4H(0;y)H(1,1,0;z) \\
& -4H(0;y)H(1,1,1;z)-8H(0;z)H(1,0,0;y)+2H(0;z)H(1,0,1-z;y) \\
& -8H(0;z)H(1,1-z,0;y)+4H(0;z)H(1-z,0,0;y)-4H(0;z)H(1-z,0,1-z;y) \\
& +8H(0;z)H(1-z,1,0;y)-4H(0;z)H(1-z,1-z,0;y) \\
& -4H(0;z)H(1-z,1-z,1-z;y)-4H(0,0;y)H(0,0;z)+4H(0,0;y)H(0,1;z) \\
& -4H(0,0;y)H(1,0;z)+4H(0,0;y)H(1,1;z)+4H(0,0;z)H(0,1-z;y) \\
& -8H(0,0;z)H(1,0;y)+4H(0,0;z)H(1-z,0;y)+4H(0,0;z)H(1-z,1-z;y) \\
& -4H(0,0,0;y)H(0;z)-4H(0,0,0;y)H(1;z)-4H(0,0,0;z)H(0;y) \\
& -4H(0,0,0;z)H(1-z;y)+4H(0,0,0,0;y)+4H(0,0,0,0;z)+2H(0,0,0,1;z) \\
& -4H(0,0,0,1-z;y)-2H(0,0,1;z)H(0;y)-6H(0,0,1;z)H(1;y) \\
& +4H(0,0,1;z)H(1-z;y)-2H(0,0,1,0;y)-6H(0,0,1,0;z)+4H(0,0,1,1;z) \\
& -2H(0,0,1-z;y)H(0;z)+4H(0,0,1-z;y)H(1;z)-4H(0,0,1-z,0;y) \\
& +4H(0,0,1-z,1-z;y)+6H(0,0,z;y)H(1;z)+6H(0,0,z,1-z;y) \\
& -10H(0,1;z)H(0,1-z;y)+8H(0,1;z)H(1,0;y)-6H(0,1;z)H(1,1-z;y) \\
& -4H(0,1;z)H(1-z,0;y)-4H(0,1;z)H(1-z,1-z;y)+8H(0,1,0;y)H(0;z) \\
& +8H(0,1,0;y)H(1;z)-4H(0,1,0;z)H(0;y)-10H(0,1,0;z)H(1;y) \\
& +6H(0,1,0;z)H(1-z;y)-2H(0,1,0,0;y)-6H(0,1,0,0;z)+6H(0,1,0,1;z) \\
& +8H(0,1,0,1-z;y)-4H(0,1,1;z)H(0;y)-4H(0,1,1;z)H(1-z;y)-9H(0,1,1,0;y) \\
& -5H(0,1,1,0;z)-4H(0,1,1,1;z)+8H(0,1,1-z,0;y)-6H(0,1-z;y)H(1,0;z) \\
& -4H(0,1-z;y)H(1,1;z)-12H(0,1-z,0;y)H(0;z)+4H(0,1-z,0;y)H(1;z) \\
& -4H(0,1-z,0,0;y)+4H(0,1-z,0,1-z;y)-8H(0,1-z,1,0;y) \\
& -4H(0,1-z,1-z;y)H(0;z)-4H(0,1-z,1-z;y)H(1;z)+4H(0,1-z,1-z,0;y) \\
& -4H(0,1-z,1-z,1-z;y)-6H(0,1-z,z;y)H(1;z)-6H(0,1-z,z,1-z;y) \\
& +4H(1;z)H(1-z,0,0;y)-4H(1;z)H(1-z,0,1-z;y)+8H(1;z)H(1-z,1,0;y) \\
& -4H(1;z)H(1-z,1-z,0;y)-4H(1;z)H(1-z,1-z,1-z;y)-8H(1,0;y)H(1,0;z) \\
& +8H(1,0;y)H(1,1;z)-10H(1,0;z)H(1,1-z;y)+4H(1,0;z)H(1-z,0;y) \\
& +4H(1,0;z)H(1-z,1-z;y)-8H(1,0,0;y)H(1;z)-4H(1,0,0;z)H(1-z;y) \\
& +8H(1,0,0,0;y)+4H(1,0,0,0;z)+2H(1,0,0,1;z)-8H(1,0,0,1-z;y) \\
& -6H(1,0,1;z)H(1;y)+4H(1,0,1;z)H(1-z;y)-4H(1,0,1,0;y)-6H(1,0,1,0;z) \\
& +4H(1,0,1,1;z)+8H(1,0,1-z;y)H(1;z)-8H(1,0,1-z,0;y) \\
& +8H(1,0,1-z,1-z;y)+6H(1,0,z;y)H(1;z)+6H(1,0,z,1-z;y) \\
& -4H(1,1;z)H(1-z,0;y)-4H(1,1;z)H(1-z,1-z;y)-10H(1,1,0;z)H(1;y)
\end{aligned}
$$

$$
\begin{aligned}
&+6H(1,1,0;z)H(1-z;y)+6H(1,1,0,0;y)-6H(1,1,0,0;z)+6H(1,1,0,1;z)\\
&-4H(1,1,1;z)H(1-z;y)-3H(1,1,1,0;y)-5H(1,1,1,0;z)-4H(1,1,1,1;z)\\
&+8H(1,1-z,0;y)H(1;z)-8H(1,1-z,0,0;y)+8H(1,1-z,0,1-z;y)\\
&-16H(1,1-z,1,0;y)+8H(1,1-z,1-z,0;y)-6H(1,1-z,z;y)H(1;z)\\
&-6H(1,1-z,z,1-z;y)-4H(1-z,0,0,0;y)+4H(1-z,0,0,1-z;y)\\
&+2H(1-z,0,1,0;y)+4H(1-z,0,1-z,0;y)-4H(1-z,0,1-z,1-z;y)\\
&-8H(1-z,1,0,0;y)+8H(1-z,1,0,1-z;y)-6H(1-z,1,1,0;y)\\
&+8H(1-z,1,1-z,0;y)+4H(1-z,1-z,0,0;y)-4H(1-z,1-z,0,1-z;y)\\
&+8H(1-z,1-z,1,0;y)-4H(1-z,1-z,1-z,0;y)-4H(1-z,1-z,1-z,1-z;y)\\
&+\frac{31\pi^4}{360}+\zeta_3\Big[+10H(0;y)+10H(0;z)-13H(1;y)+10H(1;z)+23H(1-z;y)\Big]\\
&+\frac{\pi^2}{6}\Big[-4H(0;y)H(0;z)-4H(0;y)H(1;z)-10H(0;z)H(1;y)+6H(0;z)H(1-z;y)\\
&-6H(0,0;y)-6H(0,0;z)-9H(0,1;y)-7H(0,1;z)+2H(0,1-z;y)-10H(1;y)H(1;z)\\
&+6H(1;z)H(1-z;y)-12H(1,0;y)-6H(1,0;z)-3H(1,1;y)-7H(1,1;z)\\
&-10H(1,1-z;y)+6H(1-z,0;y)-6H(1-z,1;y)+12H(1-z,1-z;y)\Big]\ . \qquad (3.14)
\end{aligned}
$$



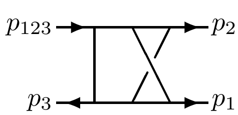
$$
= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{s_{12}+s_{13}} \sum_{i=0}^{4} \frac{f_{6.3,i}\left(\frac{s_{13}}{s_{123}},\frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon), \qquad (3.15)
$$

with:

$$
\begin{aligned}
f_{6.3,4}(y,z) &= 0\ , && (3.16)\\
f_{6.3,3}(y,z) &= -\frac{1}{2}H(0;z)\ , && (3.17)\\
f_{6.3,2}(y,z) &= -H(0;y)H(0;z)-H(0;z)H(1-z;y)+H(0,0;z)-H(0,1;z)+\frac{\pi^2}{6}\ , && (3.18)\\
f_{6.3,1}(y,z) &= +2H(0;y)H(1,0;z)+2H(0;z)H(1-z,0;y)-2H(0;z)H(1-z,1-z;y)\\
&\quad +2H(0,0;y)H(0;z)+2H(0,0;z)H(0;y)+2H(0,0;z)H(1-z;y)-2H(0,0,0;z)\\
&\quad +2H(0,1,0;y)+3H(0,1,0;z)-2H(0,1,1;z)-2H(0,1-z;y)H(0;z)\\
&\quad +2H(0,z;y)H(1;z)+2H(0,z,1-z;y)+2H(1;z)H(1-z,z;y)+2H(1,0;z)H(1-z;y)\\
&\quad +3H(1,1,0;z)+2H(1-z,1,0;y)+2H(1-z,z,1-z;y)\\
&\quad +\frac{\pi^2}{6}\Big[+2H(0;y)+3H(0;z)+3H(1;z)+2H(1-z;y)\Big]+5\zeta_3\ , && (3.19)\\
f_{6.3,0}(y,z) &= -4H(0;y)H(1,0,0;z)-4H(0;y)H(1,1,0;z)-4H(0;z)H(1-z,0,0;y)\\
&\quad -2H(0;z)H(1-z,0,1-z;y)+2H(0;z)H(1-z,1,0;y)-6H(0;z)H(1-z,1-z,0;y)\\
&\quad -4H(0;z)H(1-z,1-z,1-z;y)-4H(0,0;y)H(0,0;z)-4H(0,0;y)H(1,0;z)\\
&\quad +4H(0,0;z)H(0,1-z;y)-4H(0,0;z)H(1-z,0;y)+4H(0,0;z)H(1-z,1-z;y)\\
&\quad -4H(0,0,0;y)H(0;z)-4H(0,0,0;z)H(0;y)-4H(0,0,0;z)H(1-z;y)
\end{aligned}
$$

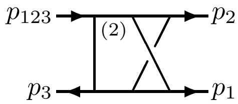
$$
\begin{aligned}
&+4H(0,0,0,0;z)+2H(0,0,0,1;z)-2H(0,0,1;z)H(1-z;y)-4H(0,0,1,0;y)\\
&-6H(0,0,1,0;z)-2H(0,0,1-z;y)H(0;z)+2H(0,0,z;y)H(1;z)+2H(0,0,z,1-z;y)\\
&-2H(0,1;z)H(0,1-z;y)-2H(0,1;z)H(0,z;y)+2H(0,1;z)H(1-z,0;y)\\
&-2H(0,1;z)H(1-z,z;y)+2H(0,1,0;y)H(0;z)+2H(0,1,0;y)H(1;z)\\
&-4H(0,1,0;z)H(0;y)-4H(0,1,0;z)H(1-z;y)-4H(0,1,0,0;y)-6H(0,1,0,0;z)\\
&+2H(0,1,0,1;z)+2H(0,1,0,1-z;y)-2H(0,1,1,0;y)-5H(0,1,1,0;z)\\
&-4H(0,1,1,1;z)+2H(0,1,1-z,0;y)-6H(0,1-z;y)H(1,0;z)\\
&-6H(0,1-z,0;y)H(0;z)-4H(0,1-z,1,0;y)-4H(0,1-z,1-z;y)H(0;z)\\
&+2H(0,1-z,z;y)H(1;z)+2H(0,1-z,z,1-z;y)+4H(0,z;y)H(1,1;z)\\
&+4H(0,z,1-z;y)H(1;z)+4H(0,z,1-z,1-z;y)-2H(0,z,z;y)H(1;z)\\
&-2H(0,z,z,1-z;y)+4H(1;z)H(1-z,0,z;y)+2H(1;z)H(1-z,1,0;y)\\
&+4H(1;z)H(1-z,1-z,z;y)+4H(1;z)H(1-z,z,1-z;y)-2H(1;z)H(1-z,z,z;y)\\
&-4H(1,0;z)H(1-z,0;y)-6H(1,0;z)H(1-z,1-z;y)-4H(1,0,0;z)H(1-z;y)\\
&+2H(1,0,1;z)H(1-z;y)-4H(1,0,1,0;z)+4H(1,1;z)H(1-z,z;y)\\
&-4H(1,1,0;z)H(1-z;y)-6H(1,1,0,0;z)-3H(1,1,1,0;z)-6H(1-z,0,1,0;y)\\
&+4H(1-z,0,z,1-z;y)-4H(1-z,1,0,0;y)+2H(1-z,1,0,1-z;y)\\
&-2H(1-z,1,1,0;y)+2H(1-z,1,1-z,0;y)-6H(1-z,1-z,1,0;y)\\
&+4H(1-z,1-z,z,1-z;y)+4H(1-z,z,1-z,1-z;y)-2H(1-z,z,z,1-z;y)\\
&-\frac{\pi^4}{120}+\zeta_3\Big[+10H(0;z)+H(1;z)+6H(1-z;y)\Big]+\frac{\pi^2}{6}\Big[-4H(0;y)H(0;z)\\
&-4H(0;y)H(1;z)-4H(0;z)H(1-z;y)-4H(0,0;y)-6H(0,0;z)-2H(0,1;y)\\
&-5H(0,1;z)-4H(0,1-z;y)-6H(1;z)H(1-z;y)-4H(1,0;z)-3H(1,1;z)\\
&-6H(1-z,0;y)-2H(1-z,1;y)-6H(1-z,1-z;y)\Big]\;. \qquad (3.20)
\end{aligned}
$$

The two master integrals for the first seven-propagator crossed topology can be expressed in the following form, which makes the $p_1 \leftrightarrow p_2$ interchange symmetry of (2.3) manifest:



$$
\begin{aligned}
= \;& \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{123})^{-2\epsilon} \sum_{i=0}^{4} \frac{1}{\epsilon^i} \left(\frac{s_{123}}{s_{12}^2 s_{13} s_{23}} f_{7.3,i}\left(\frac{s_{13}}{s_{123}},\frac{s_{23}}{s_{123}}\right)\left(1-4\epsilon+16\epsilon^2-64\epsilon^3\right)\right.\\
&+\frac{1}{s_{12}^2 s_{13}}\left[f_{7.4,i}\left(\frac{s_{13}}{s_{123}},\frac{s_{23}}{s_{123}}\right)\left(1-4\epsilon+16\epsilon^2\right)+f_{7.5,i}\left(\frac{s_{13}}{s_{123}},\frac{s_{23}}{s_{123}}\right)\right]\\
&+\frac{1}{s_{12}^2 s_{23}}\left[f_{7.4,i}\left(\frac{s_{23}}{s_{123}},\frac{s_{13}}{s_{123}}\right)\left(1-4\epsilon+16\epsilon^2\right)+f_{7.5,i}\left(\frac{s_{23}}{s_{123}},\frac{s_{13}}{s_{123}}\right)\right]\\
&\left.+\frac{1}{s_{12}^2 s_{123}} f_{6.2,i}\left(\frac{s_{13}}{s_{123}},\frac{s_{23}}{s_{123}}\right)\right)+\mathcal{O}(\epsilon), \qquad (3.21)
\end{aligned}
$$

and



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{123})^{-2\epsilon} \sum_{i=0}^{4} \frac{1}{\epsilon^i} \left(\frac{s_{123}}{s_{12}s_{13}s_{23}} f_{7.3,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) \left(1 - 4\epsilon + 16\epsilon^2 - 64\epsilon^3\right) \right.$$
$$+ \frac{1}{s_{12}s_{23}} \left[f_{7.4,i}\left(\frac{s_{23}}{s_{123}}, \frac{s_{13}}{s_{123}}\right) \left(1 - 4\epsilon + 16\epsilon^2\right) + f_{7.5,i}\left(\frac{s_{23}}{s_{123}}, \frac{s_{13}}{s_{123}}\right) \right]$$
$$\left. + \frac{1}{s_{12}s_{13}} f_{7.4,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) \left(1 - 4\epsilon + 16\epsilon^2\right) \right) + \mathcal{O}(\epsilon), \tag{3.22}$$

with

$$f_{7.3,4}(y,z) = 0 \ , \tag{3.23}$$

$$f_{7.3,3}(y,z) = -\frac{3}{2} \ , \tag{3.24}$$

$$f_{7.3,2}(y,z) = 3H(0;y) + 3H(0;z) \tag{3.25}$$

$$f_{7.3,1}(y,z) = -6H(0;y)H(0;z) - 6H(0,0;y) - 6H(0,0;z) - 6H(1,0;y) - 6H(1,0;z) - \pi^2 \ , \tag{3.26}$$

$$\begin{aligned} f_{7.3,0}(y,z) = {} & +12H(0;y)H(1,0,z) + 12H(0;z)H(1-z,0,y) + 12H(0,0;y)H(0,z) \\ & +12H(0,0;z)H(0,y) + 12H(0,0,0;y) + 12H(0,0,0;z) + 12H(0,1,0;y) + 12H(0,1,0;z) \\ & +12H(1,0;z)H(1-z;y) + 12H(1,0,0;y) + 12H(1,0,0;z) + 12H(1,1,0;z) \\ & +12H(1-z,1,0;y) + 9\zeta_3 + 2\pi^2\left[+H(0;y) + H(0;z) + H(1;z) + H(1-z;y)\right] \ , \end{aligned} \tag{3.27}$$

$$f_{7.4,4}(y,z) = 0 \ , \tag{3.28}$$

$$f_{7.4,3}(y,z) = 0 \ , \tag{3.29}$$

$$f_{7.4,2}(y,z) = -3H(0;z) - 3H(1;z) - 3H(1-z;y) \ , \tag{3.30}$$

$$\begin{aligned} f_{7.4,1}(y,z) = {} & +6H(0;y)H(0;z) + 6H(0;y)H(1;z) + 6H(0,0;z) - 6H(0,1;z) + 6H(0,1-z;y) \\ & -6H(1;z)H(1-z;y) - 6H(1;z)H(z;y) + 6H(1,0;z) - 6H(1,1;z) + 6H(1-z,0;y) \\ & -6H(1-z,1-z;y) - 6H(z,1-z;y) + \pi^2 \ , \end{aligned} \tag{3.31}$$

$$\begin{aligned} f_{7.4,0}(y,z) = {} & -12H(0;y)H(1,0;z) + 12H(0;y)H(1,1;z) - 12H(0;z)H(1-z,0;y) \\ & -12H(0,0;y)H(0;z) - 12H(0,0;y)H(1;z) - 12H(0,0;z)H(0;y) - 12H(0,0,0;z) \\ & -12H(0,0,1;z) - 12H(0,0,1-z;y) + 12H(0,1;z)H(0;y) - 12H(0,1;z)H(1-z;y) \\ & -12H(0,1,0;z) - 12H(0,1,1;z) + 12H(0,1-z;y)H(1;z) - 12H(0,1-z,0;y) \\ & +12H(0,1-z,1-z;y) + 12H(0,z;y)H(1;z) + 12H(0,z,1-z;y) \\ & +12H(1;z)H(1-z,0;y) - 12H(1;z)H(1-z,1-z;y) - 12H(1;z)H(1-z,z;y) \\ & -12H(1;z)H(z,1-z;y) - 12H(1,0;z)H(1-z;y) - 12H(1,0,0;z) - 12H(1,0,1;z) \\ & -12H(1,1;z)H(1-z;y) - 12H(1,1;z)H(z;y) - 12H(1,1,0;z) - 12H(1,1,1;z) \\ & -12H(1-z,0,0;y) + 12H(1-z,0,1-z;y) - 24H(1-z,1,0;y) + 12H(1-z,1-z,0;y) \\ & -12H(1-z,1-z,1-z;y) - 12H(1-z,z,1-z;y) - 12H(z,1-z,1-z;y) \end{aligned}$$

$$-2\pi^2\left[+H(0;y)+H(0;z)+H(1;z)+H(1-z;y)\right] \ , \quad (3.32)$$

$$f_{7.5,4}(y,z) = -\frac{1}{8} \ , \quad (3.33)$$

$$f_{7.5,3}(y,z) = +\frac{1}{4}H(0;y)-\frac{3}{4}H(0;z)-\frac{1}{4}H(1;z)-\frac{1}{4}H(1-z;y) \ , \quad (3.34)$$

$$\begin{aligned} f_{7.5,2}(y,z) = {} & +\frac{3}{2}H(0;y)H(0;z)+\frac{1}{2}H(0;y)H(1;z)-\frac{3}{2}H(0;z)H(1-z;y)-\frac{1}{2}H(0,0;y)+\frac{3}{2}H(0,0;z) \\ & +\frac{3}{2}H(0,1;z)+\frac{1}{2}H(0,1-z;y)-\frac{1}{2}H(1;z)H(1-z;y)+3H(1;z)H(z;y)-2H(1,0;y) \\ & -\frac{3}{2}H(1,0;z)-\frac{1}{2}H(1,1;z)+\frac{1}{2}H(1-z,0;y)-\frac{1}{2}H(1-z,1-z;y)+3H(z,1-z;y) \\ & +\frac{\pi^2}{3} \ , \quad (3.35) \end{aligned}$$

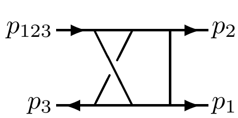
$$\begin{aligned} f_{7.5,1}(y,z) = {} & +3H(0;y)H(1,0;z)+H(0;y)H(1,1;z)-4H(0;z)H(1,0;y)+3H(0;z)H(1-z,0;y) \\ & -3H(0;z)H(1-z,1-z;y)-6H(0;z)H(z,1-z;y)-3H(0,0;y)H(0;z) \\ & -H(0,0;y)H(1;z)-3H(0,0;z)H(0;y)+3H(0,0;z)H(1-z;y)+H(0,0,0;y) \\ & -3H(0,0,0;z)+6H(0,0,1;z)-H(0,0,1-z;y)-3H(0,1;z)H(0;y) \\ & +3H(0,1;z)H(1-z;y)+9H(0,1;z)H(z;y)+H(0,1,0;y)-9H(0,1,0;z)+3H(0,1,1;z) \\ & +3H(0,1-z;y)H(0;z)+H(0,1-z;y)H(1;z)-H(0,1-z,0;y)+H(0,1-z,1-z;y) \\ & -6H(0,z;y)H(1;z)-6H(0,z,1-z;y)+H(1;z)H(1-z,0;y) \\ & -H(1;z)H(1-z,1-z;y)+6H(1;z)H(1-z,z;y)-6H(1;z)H(z,0;y) \\ & +6H(1;z)H(z,1-z;y)+15H(1;z)H(z,z;y)-4H(1,0;y)H(1;z) \\ & -3H(1,0;z)H(1-z;y)-6H(1,0;z)H(z;y)+4H(1,0,0;y)+3H(1,0,0;z) \\ & +3H(1,0,1;z)-4H(1,0,1-z;y)-H(1,1;z)H(1-z;y)+6H(1,1;z)H(z;y) \\ & +3H(1,1,0;y)-3H(1,1,0;z)-H(1,1,1;z)-4H(1,1-z,0;y)-H(1-z,0,0;y) \\ & +H(1-z,0,1-z;y)-4H(1-z,1,0;y)+H(1-z,1-z,0;y)-H(1-z,1-z,1-z;y) \\ & +6H(1-z,z,1-z;y)-6H(z,0,1-z;y)-6H(z,1-z,0;y)+6H(z,1-z,1-z;y) \\ & +15H(z,z,1-z;y)+\frac{17}{4}\zeta_3+\frac{\pi^2}{6}\big[-4H(0;y)-3H(0;z)+3H(1;y)+4H(1;z) \\ & +4H(1-z;y)\big] \ , \quad (3.36) \end{aligned}$$

$$\begin{aligned} f_{7.5,0}(y,z) = {} & -6H(0;y)H(1,0,0;z)-6H(0;y)H(1,0,1;z)+6H(0;y)H(1,1,0;z) \\ & +2H(0;y)H(1,1,1;z)+8H(0;z)H(1,0,0;y)-2H(0;z)H(1,0,1-z;y) \\ & +8H(0;z)H(1,1-z,0;y)-6H(0;z)H(1-z,0,0;y)+6H(0;z)H(1-z,0,1-z;y) \\ & -8H(0;z)H(1-z,1,0;y)+6H(0;z)H(1-z,1-z,0;y) \\ & -6H(0;z)H(1-z,1-z,1-z;y)-12H(0;z)H(1-z,z,1-z;y) \\ & -6H(0;z)H(z,0,1-z;y)+8H(0;z)H(z,1-z,0;y)-12H(0;z)H(z,1-z,1-z;y) \\ & -6H(0;z)H(z,z,1-z;y)+6H(0,0;y)H(0,0;z)+6H(0,0;y)H(0,1;z) \\ & -6H(0,0;y)H(1,0;z)-2H(0,0;y)H(1,1;z)-6H(0,0;z)H(0,1-z;y) \end{aligned}$$

$$+8H(0,0;z)H(1,0;y)-6H(0,0;z)H(1-z,0;y)+6H(0,0;z)H(1-z,1-z;y)$$
$$+12H(0,0;z)H(z,1-z;y)+6H(0,0,0;y)H(0;z)+2H(0,0,0;y)H(1;z)$$
$$+6H(0,0,0;z)H(0;y)-6H(0,0,0;z)H(1-z;y)-2H(0,0,0,0;y)$$
$$+6H(0,0,0,0;z)+15H(0,0,0,1;z)+2H(0,0,0,1-z;y)-12H(0,0,1;z)H(0;y)$$
$$+6H(0,0,1;z)H(1;y)+12H(0,0,1;z)H(1-z;y)+9H(0,0,1;z)H(z;y)$$
$$+4H(0,0,1,0;y)-6H(0,0,1,0;z)+12H(0,0,1,1;z)-6H(0,0,1-z;y)H(0;z)$$
$$-2H(0,0,1-z;y)H(1;z)+2H(0,0,1-z,0;y)-2H(0,0,1-z,1-z;y)$$
$$+12H(0,0,z;y)H(1;z)+12H(0,0,z,1-z;y)-6H(0,1;z)H(0,1-z;y)$$
$$-12H(0,1;z)H(0,z;y)-8H(0,1;z)H(1,0;y)+6H(0,1;z)H(1,1-z;y)$$
$$-6H(0,1;z)H(1-z,0;y)+6H(0,1;z)H(1-z,1-z;y)+18H(0,1;z)H(1-z,z;y)$$
$$-12H(0,1;z)H(z,0;y)+12H(0,1;z)H(z,1-z;y)+3H(0,1;z)H(z,z;y)$$
$$+8H(0,1,0;y)H(0;z)+8H(0,1,0;y)H(1;z)+18H(0,1,0;z)H(0;y)$$
$$+10H(0,1,0;z)H(1;y)-18H(0,1,0;z)H(1-z;y)-8H(0,1,0;z)H(z;y)$$
$$-2H(0,1,0,0;y)+18H(0,1,0,0;z)+12H(0,1,0,1;z)+8H(0,1,0,1-z;y)$$
$$-6H(0,1,1;z)H(0;y)+6H(0,1,1;z)H(1-z;y)+18H(0,1,1;z)H(z;y)$$
$$-3H(0,1,1,0;y)+6H(0,1,1,0;z)+6H(0,1,1,1;z)+8H(0,1,1-z,0;y)$$
$$+6H(0,1-z;y)H(1,0;z)+2H(0,1-z;y)H(1,1;z)-6H(0,1-z,0;y)H(0;z)$$
$$-2H(0,1-z,0;y)H(1;z)+2H(0,1-z,0,0;y)-2H(0,1-z,0,1-z;y)$$
$$+2H(0,1-z,1,0;y)+6H(0,1-z,1-z;y)H(0;z)+2H(0,1-z,1-z;y)H(1;z)$$
$$-2H(0,1-z,1-z,0;y)+2H(0,1-z,1-z,1-z;y)-12H(0,1-z,z;y)H(1;z)$$
$$-12H(0,1-z,z,1-z;y)+12H(0,z;y)H(1,0;z)-12H(0,z;y)H(1,1;z)$$
$$+6H(0,z,0;y)H(1;z)+6H(0,z,0,1-z;y)+12H(0,z,1-z;y)H(0;z)$$
$$-12H(0,z,1-z;y)H(1;z)+6H(0,z,1-z,0;y)-12H(0,z,1-z,1-z;y)$$
$$-24H(0,z,z;y)H(1;z)-24H(0,z,z,1-z;y)-2H(1;z)H(1-z,0,0;y)$$
$$+2H(1;z)H(1-z,0,1-z;y)-12H(1;z)H(1-z,0,z;y)-8H(1;z)H(1-z,1,0;y)$$
$$+2H(1;z)H(1-z,1-z,0;y)-2H(1;z)H(1-z,1-z,1-z;y)$$
$$+12H(1;z)H(1-z,1-z,z;y)-12H(1;z)H(1-z,z,0;y)$$
$$+12H(1;z)H(1-z,z,1-z;y)+30H(1;z)H(1-z,z,z;y)+12H(1;z)H(z,0,0;y)$$
$$-12H(1;z)H(z,0,1-z;y)-6H(1;z)H(z,0,z;y)-12H(1;z)H(z,1-z,0;y)$$
$$+12H(1;z)H(z,1-z,1-z;y)+24H(1;z)H(z,1-z,z;y)-6H(1;z)H(z,z,0;y)$$
$$+30H(1;z)H(z,z,1-z;y)+9H(1;z)H(z,z,z;y)+8H(1,0;y)H(1,0;z)$$
$$-8H(1,0;y)H(1,1;z)+10H(1,0;z)H(1,1-z;y)+6H(1,0;z)H(1-z,0;y)$$
$$-6H(1,0;z)H(1-z,1-z;y)-12H(1,0;z)H(1-z,z;y)+8H(1,0;z)H(z,0;y)$$
$$+2H(1,0;z)H(z,1-z;y)-6H(1,0;z)H(z,z;y)+8H(1,0,0;y)H(1;z)$$
$$+6H(1,0,0;z)H(1-z;y)+12H(1,0,0;z)H(z;y)-8H(1,0,0,0;y)$$

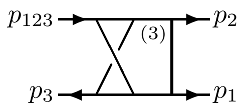
$$
\begin{aligned}
&-6H(1,0,0,0;z)+12H(1,0,0,1;z)+8H(1,0,0,1-z;y)+6H(1,0,1;z)H(1;y)\\
&+6H(1,0,1;z)H(1-z;y)+12H(1,0,1;z)H(z;y)+4H(1,0,1,0;y)\\
&-18H(1,0,1,0;z)+6H(1,0,1,1;z)-8H(1,0,1-z;y)H(1;z)+8H(1,0,1-z,0;y)\\
&-8H(1,0,1-z,1-z;y)-6H(1,0,z;y)H(1;z)-6H(1,0,z,1-z;y)\\
&+2H(1,1;z)H(1-z,0;y)-2H(1,1;z)H(1-z,1-z;y)+12H(1,1;z)H(1-z,z;y)\\
&-12H(1,1;z)H(z,0;y)+12H(1,1;z)H(z,1-z;y)+30H(1,1;z)H(z,z;y)\\
&+10H(1,1,0;z)H(1;y)-6H(1,1,0;z)H(1-z;y)+2H(1,1,0;z)H(z;y)\\
&-6H(1,1,0,0;y)+6H(1,1,0,0;z)+6H(1,1,0,1;z)-2H(1,1,1;z)H(1-z;y)\\
&+12H(1,1,1;z)H(z;y)+3H(1,1,1,0;y)-6H(1,1,1,0;z)-2H(1,1,1,1;z)\\
&-8H(1,1-z,0;y)H(1;z)+8H(1,1-z,0,0;y)-8H(1,1-z,0,1-z;y)\\
&+16H(1,1-z,1,0;y)-8H(1,1-z,1-z,0;y)+6H(1,1-z,z;y)H(1;z)\\
&+6H(1,1-z,z,1-z;y)+2H(1-z,0,0,0;y)-2H(1-z,0,0,1-z;y)\\
&+2H(1-z,0,1,0;y)-2H(1-z,0,1-z,0;y)+2H(1-z,0,1-z,1-z;y)\\
&-12H(1-z,0,z,1-z;y)+8H(1-z,1,0,0;y)-8H(1-z,1,0,1-z;y)\\
&+6H(1-z,1,1,0;y)-8H(1-z,1,1-z,0;y)-2H(1-z,1-z,0,0;y)\\
&+2H(1-z,1-z,0,1-z;y)-8H(1-z,1-z,1,0;y)+2H(1-z,1-z,1-z,0;y)\\
&-2H(1-z,1-z,1-z,1-z;y)+12H(1-z,1-z,z,1-z;y)\\
&-12H(1-z,z,0,1-z;y)-12H(1-z,z,1-z,0;y)+12H(1-z,z,1-z,1-z;y)\\
&+30H(1-z,z,z,1-z;y)+12H(z,0,0,1-z;y)-14H(z,0,1,0;y)\\
&+12H(z,0,1-z,0;y)-12H(z,0,1-z,1-z;y)-6H(z,0,z,1-z;y)\\
&+12H(z,1-z,0,0;y)-12H(z,1-z,0,1-z;y)+14H(z,1-z,1,0;y)\\
&-12H(z,1-z,1-z,0;y)+12H(z,1-z,1-z,1-z;y)+24H(z,1-z,z,1-z;y)\\
&-6H(z,z,0,1-z;y)-6H(z,z,1-z,0;y)+30H(z,z,1-z,1-z;y)\\
&+9H(z,z,z,1-z;y)+\frac{71\pi^4}{240}+\frac{\zeta_3}{2}\big[-17H(0;y)+9H(0;z)+26H(1;y)+17H(1;z)\\
&+17H(1-z;y)\big]+\frac{\pi^2}{6}\big[+6H(0;y)H(0;z)-8H(0;y)H(1;z)+10H(0;z)H(1;y)\\
&-6H(0;z)H(1-z;y)+8H(0,0;y)+6H(0,0;z)-3H(0,1;y)+18H(0,1;z)\\
&-8H(0,1-z;y)+10H(1;y)H(1;z)+8H(1;z)H(1-z;y)+20H(1;z)H(z;y)\\
&+12H(1,0;y)-6H(1,0;z)+3H(1,1;y)+8H(1,1;z)+10H(1,1-z;y)\\
&-8H(1-z,0;y)+6H(1-z,1;y)+8H(1-z,1-z;y)+20H(z,1-z;y)\big]\,. \qquad (3.37)
\end{aligned}
$$

The two master integrals for the second seven-propagator crossed topology are expressed as follows. Again, the $p_1 \leftrightarrow p_2$ interchange symmetry of (2.5) becomes apparent:



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{123})^{-2\epsilon} \sum_{i=0}^{4} \frac{1}{\epsilon^i} \left(\frac{1}{s_{13}s_{23}s_{123}} f_{7.6,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) + \frac{1}{s_{12}s_{13}s_{123}} f_{7.7,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) + \frac{1}{s_{12}s_{23}s_{123}} f_{7.7,i}\left(\frac{s_{23}}{s_{123}}, \frac{s_{13}}{s_{123}}\right) + \frac{1}{s_{13}(s_{13}+s_{23})s_{123}} f_{7.8,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) + \frac{1}{s_{23}(s_{13}+s_{23})s_{123}} f_{7.8,i}\left(\frac{s_{23}}{s_{123}}, \frac{s_{13}}{s_{123}}\right) \right) + \mathcal{O}(\epsilon), \tag{3.38}$$

and



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-s_{123})^{-2\epsilon} \sum_{i=0}^{4} \frac{1}{\epsilon^i} \left(\frac{1}{s_{12}s_{13}} f_{7.7,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) - \frac{1}{s_{13}s_{123}} f_{6.2,i}\left(\frac{s_{12}}{s_{123}}, \frac{s_{23}}{s_{123}}\right) \right) + \mathcal{O}(\epsilon), \tag{3.39}$$

with

$$f_{7.6,4}(y,z) = -1 \;, \tag{3.40}$$

$$f_{7.6,3}(y,z) = 2H(0;y) + 2H(0;z) \;, \tag{3.41}$$

$$\begin{aligned} f_{7.6,2}(y,z) = & -4H(0;y)H(0;z) - 4H(0,0;y) - 4H(0,0;z) + 3H(1;z)H(1-z;y) - 3H(1,0;y) \\ & -3H(1,0;z) + 3H(1,1;z) + 3H(1-z,1-z;y) - \frac{\pi^2}{3} \;, \end{aligned} \tag{3.42}$$

$$\begin{aligned} f_{7.6,1}(y,z) = & +6H(0;y)H(1,0;z) - 4H(0;y)H(1,1;z) + 4H(0;z)H(1,0;y) + 2H(0;z)H(1-z,0;y) \\ & -4H(0;z)H(1-z,1-z;y) + 8H(0,0;y)H(0;z) + 8H(0,0;z)H(0;y) + 8H(0,0,0;y) \\ & +8H(0,0,0;z) + 6H(0,1;z)H(1-z;y) + 6H(0,1,0;y) + 6H(0,1,0;z) \\ & -2H(0,1,1;z) - 4H(0,1-z;y)H(1;z) - 4H(0,1-z,1-z;y) - 4H(1;z)H(1-z,0;y) \\ & +9H(1;z)H(1-z,1-z;y) + 10H(1;z)H(1-z,z;y) + 2H(1;z)H(z,1-z;y) \\ & +4H(1,0;y)H(1;z) + 2H(1,0;z)H(1-z;y) + 6H(1,0,0;y) + 6H(1,0,0;z) \\ & +10H(1,0,1;z) + 4H(1,0,1-z;y) + 9H(1,1;z)H(1-z;y) + 2H(1,1;z)H(z;y) \\ & +H(1,1,0;y) + 7H(1,1,0;z) + 9H(1,1,1;z) + 4H(1,1-z,0;y) \\ & -4H(1-z,0,1-z;y) + 6H(1-z,1,0;y) - 4H(1-z,1-z,0;y) \\ & +9H(1-z,1-z,1-z;y) + 10H(1-z,z,1-z;y) + 2H(z,1-z,1-z;y) \\ & +16\zeta_3 + \frac{\pi^2}{6}\big[+ 4H(0;y) + 4H(0;z) + H(1;y) + 3H(1;z) + 2H(1-z;y)\big] \;, \end{aligned} \tag{3.43}$$

$$f_{7.6,0}(y,z) = -12H(0;y)H(1,0,0;z) - 16H(0;y)H(1,0,1;z) - 14H(0;y)H(1,1,0;z)$$

$$-12H(0;y)H(1,1,1;z) - 8H(0;z)H(1,0,0;y) - 2H(0;z)H(1,0,1-z;y)$$
$$-8H(0;z)H(1,1-z,0;y) - 4H(0;z)H(1-z,0,0;y) - 14H(0;z)H(1-z,0,1-z;y)$$
$$-4H(0;z)H(1-z,1,0;y) - 2H(0;z)H(1-z,1-z,0;y)$$
$$-12H(0;z)H(1-z,1-z,1-z;y) - 16H(0,0;y)H(0,0;z) - 12H(0,0;y)H(1,0;z)$$
$$+8H(0,0;y)H(1,1;z) - 8H(0,0;z)H(1,0;y) - 4H(0,0;z)H(1-z,0;y)$$
$$+8H(0,0;z)H(1-z,1-z;y) - 16H(0,0,0;y)H(0;z) - 16H(0,0,0;z)H(0;y)$$
$$-16H(0,0,0,0;y) - 16H(0,0,0,0;z) - 12H(0,0,1;z)H(1;y) + 14H(0,0,1;z)H(1-z;y)$$
$$-12H(0,0,1,0;y) - 12H(0,0,1,0;z) + 4H(0,0,1,1;z) + 8H(0,0,1-z;y)H(1;z)$$
$$+8H(0,0,1-z,1-z;y) - 12H(0,1;z)H(0,1-z;y) + 10H(0,1;z)H(1,0;y)$$
$$-12H(0,1;z)H(1,1-z;y) - 14H(0,1;z)H(1-z,0;y) + 13H(0,1;z)H(1-z,1-z;y)$$
$$+6H(0,1;z)H(1-z,z;y) + 4H(0,1;z)H(z,1-z;y) - 4H(0,1,0;y)H(0;z)$$
$$-4H(0,1,0;y)H(1;z) - 12H(0,1,0;z)H(0;y) - 6H(0,1,0;z)H(1;y)$$
$$+2H(0,1,0;z)H(1-z;y) - 12H(0,1,0,0;y) - 12H(0,1,0,0;z) - 4H(0,1,0,1;z)$$
$$-4H(0,1,0,1-z;y) + 11H(0,1,1;z)H(1-z;y) - 4H(0,1,1;z)H(z;y)$$
$$+2H(0,1,1,0;y) - 6H(0,1,1,0;z) - 6H(0,1,1,1;z) - 4H(0,1,1-z,0;y)$$
$$-12H(0,1-z;y)H(1,0;z) - 12H(0,1-z;y)H(1,1;z) - 8H(0,1-z,0;y)H(0;z)$$
$$+8H(0,1-z,0;y)H(1;z) + 8H(0,1-z,0,1-z;y) - 12H(0,1-z,1,0;y)$$
$$-12H(0,1-z,1-z;y)H(1;z) + 8H(0,1-z,1-z,0;y) - 12H(0,1-z,1-z,1-z;y)$$
$$-12H(0,1-z,z;y)H(1;z) - 12H(0,1-z,z,1-z;y) + 8H(1;z)H(1-z,0,0;y)$$
$$-12H(1;z)H(1-z,0,1-z;y) - 4H(1;z)H(1-z,1,0;y) - 12H(1;z)H(1-z,1-z,0;y)$$
$$+21H(1;z)H(1-z,1-z,1-z;y) + 25H(1;z)H(1-z,1-z,z;y)$$
$$+23H(1;z)H(1-z,z,1-z;y) + 6H(1;z)H(1-z,z,z;y) + 6H(1;z)H(z,1-z,1-z;y)$$
$$+4H(1;z)H(z,1-z,z;y) - 4H(1;z)H(z,z,1-z;y) - 8H(1,0;y)H(1,0;z)$$
$$+10H(1,0;y)H(1,1;z) - 6H(1,0;z)H(1,1-z;y) - 6H(1,0;z)H(1-z,0;y)$$
$$-4H(1,0;z)H(1-z,1-z;y) - 8H(1,0,0;y)H(1;z) - 4H(1,0,0;z)H(1-z;y)$$
$$-12H(1,0,0,0;y) - 12H(1,0,0,0;z) + 18H(1,0,0,1;z) - 8H(1,0,0,1-z;y)$$
$$-12H(1,0,1;z)H(1;y) + 9H(1,0,1;z)H(1-z;y) + 4H(1,0,1;z)H(z;y)$$
$$+2H(1,0,1,0;y) - 4H(1,0,1,0;z) + 21H(1,0,1,1;z) + 10H(1,0,1-z;y)H(1;z)$$
$$-8H(1,0,1-z,0;y) + 10H(1,0,1-z,1-z;y) + 12H(1,0,z;y)H(1;z)$$
$$+12H(1,0,z,1-z;y) - 12H(1,1;z)H(1-z,0;y) + 21H(1,1;z)H(1-z,1-z;y)$$
$$+23H(1,1;z)H(1-z,z;y) + 6H(1,1;z)H(z,1-z;y) - 4H(1,1;z)H(z,z;y)$$
$$-6H(1,1,0;z)H(1;y) - 6H(1,1,0;z)H(1-z;y) - 2H(1,1,0,0;y) - 14H(1,1,0,0;z)$$
$$+7H(1,1,0,1;z) + 21H(1,1,1;z)H(1-z;y) + 6H(1,1,1;z)H(z;y) - H(1,1,1,0;y)$$
$$-15H(1,1,1,0;z) + 21H(1,1,1,1;z) + 10H(1,1-z,0;y)H(1;z) - 8H(1,1-z,0,0;y)$$
$$+10H(1,1-z,0,1-z;y) - 18H(1,1-z,1,0;y) + 10H(1,1-z,1-z,0;y)$$

$$
\begin{aligned}
&-12H(1,1-z,z;y)H(1;z)-12H(1,1-z,z,1-z;y)+8H(1-z,0,0,1-z;y)\\
&-6H(1-z,0,1,0;y)+8H(1-z,0,1-z,0;y)-12H(1-z,0,1-z,1-z;y)\\
&-12H(1-z,1,0,0;y)-4H(1-z,1,0,1-z;y)+2H(1-z,1,1,0;y)\\
&-4H(1-z,1,1-z,0;y)+8H(1-z,1-z,0,0;y)-12H(1-z,1-z,0,1-z;y)\\
&+8H(1-z,1-z,1,0;y)-12H(1-z,1-z,1-z,0;y)\\
&+21H(1-z,1-z,1-z,1-z;y)+25H(1-z,1-z,z,1-z;y)\\
&+23H(1-z,z,1-z,1-z;y)+6H(1-z,z,z,1-z;y)\\
&+6H(z,1-z,1-z,1-z;y)+4H(z,1-z,z,1-z;y)-4H(z,z,1-z,1-z;y)\\
&+\frac{2\pi^4}{5}+\zeta_3\big[-32H(0;y)-32H(0;z)-11H(1;y)-15H(1;z)-4H(1-z;y)\big]\\
&+\frac{\pi^2}{6}\big[-8H(0;y)H(0;z)-6H(0;y)H(1;z)-6H(0;z)H(1;y)-8H(0,0;y)-8H(0,0;z)\\
&+2H(0,1;y)-6H(0,1;z)-8H(0,1-z;y)-6H(1;y)H(1;z)-4H(1;z)H(1-z;y)\\
&-6H(1,0;y)-6H(1,0;z)-H(1,1;y)-11H(1,1;z)-6H(1,1-z;y)\\
&+2H(1-z,1;y)-6H(1-z,1-z;y)\big]\ , \qquad (3.44)
\end{aligned}
$$

$$
f_{7.7,4}(y,z) = -\frac{5}{4}\ , \qquad (3.45)
$$

$$
f_{7.7,3}(y,z) = +\frac{5}{2}H(0;y)+\frac{3}{2}H(0;z)+\frac{1}{2}H(1;z)+\frac{1}{2}H(1-z;y)\ , \qquad (3.46)
$$

$$
\begin{aligned}
f_{7.7,2}(y;z) = &-3H(0;y)H(0;z)-H(0;y)H(1;z)+H(0;z)H(1-z;y)-5H(0,0;y)-3H(0,0;z)\\
&+3H(0,1;z)-H(0,1-z;y)+4H(1;z)H(1-z;y)+2H(1;z)H(z;y)-3H(1,0;y)\\
&+4H(1,1;z)-H(1-z,0;y)+4H(1-z,1-z;y)+2H(z,1-z;y)-\frac{\pi^2}{6}\ , \qquad (3.47)
\end{aligned}
$$

$$
\begin{aligned}
f_{7.7,1}(y;z) = &-8H(0;y)H(1,1;z)+4H(0;z)H(1,0;y)+4H(0;z)H(z,1-z;y)+6H(0,0;y)H(0;z)\\
&+2H(0,0;y)H(1;z)+6H(0,0;z)H(0;y)-2H(0,0;z)H(1-z;y)+10H(0,0,0;y)\\
&+6H(0,0,0;z)+9H(0,0,1;z)+2H(0,0,1-z;y)-6H(0,1;z)H(0;y)\\
&+9H(0,1;z)H(1-z;y)+7H(0,1;z)H(z;y)+6H(0,1,0;y)+9H(0,1,0;z)\\
&+6H(0,1,1;z)-2H(0,1-z;y)H(0;z)-8H(0,1-z;y)H(1;z)+2H(0,1-z,0;y)\\
&-8H(0,1-z,1-z;y)-4H(0,z;y)H(1;z)-4H(0,z,1-z;y)-8H(1;z)H(1-z,0;y)\\
&+11H(1;z)H(1-z,1-z;y)+9H(1;z)H(1-z,z;y)-4H(1;z)H(z,0;y)\\
&+6H(1;z)H(z,1-z;y)+3H(1;z)H(z,z;y)+4H(1,0;y)H(1;z)+2H(1,0;z)H(1-z;y)\\
&+4H(1,0;z)H(z;y)+6H(1,0,0;y)+9H(1,0,1;z)+4H(1,0,1-z;y)\\
&+11H(1,1;z)H(1-z;y)+6H(1,1;z)H(z;y)+H(1,1,0;y)+11H(1,1,1;z)\\
&+4H(1,1-z,0;y)+2H(1-z,0,0;y)-8H(1-z,0,1-z;y)+6H(1-z,1,0;y)\\
&-8H(1-z,1-z,0;y)+11H(1-z,1-z,1-z;y)+9H(1-z,z,1-z;y)\\
&-4H(z,0,1-z;y)-4H(z,1-z,0;y)+6H(z,1-z,1-z;y)+3H(z,z,1-z;y)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{37}{2}\zeta_3+\frac{\pi^2}{6}\Big[+2H(0;y)+3H(0;z)+H(1;y)-2H(1;z)\Big]\ , \qquad (3.48)\\
f_{7.7,0}(y;z) \;=\; &-18H(0;y)H(1,0,1;z)-22H(0;y)H(1,1,1;z)-8H(0;z)H(1,0,0;y)\\
&-2H(0;z)H(1,0,1-z;y)-8H(0;z)H(1,1-z,0;y)-2H(0;z)H(1-z,0,1-z;y)\\
&-2H(0;z)H(1-z,1-z,1-z;y)+8H(0;z)H(z,0,1-z;y)-2H(0;z)H(z,1-z,0;y)\\
&+4H(0;z)H(z,1-z,1-z;y)-12H(0,0;y)H(0,0;z)+12H(0,0;y)H(0,1;z)\\
&+16H(0,0;y)H(1,1;z)+4H(0,0;z)H(0,1-z;y)-8H(0,0;z)H(1,0;y)\\
&-8H(0,0;z)H(z,1-z;y)-12H(0,0,0;y)H(0;z)-4H(0,0,0;y)H(1;z)\\
&-12H(0,0,0;z)H(0;y)+4H(0,0,0;z)H(1-z;y)-20H(0,0,0,0;y)\\
&-12H(0,0,0,0;z)+21H(0,0,0,1;z)-4H(0,0,0,1-z;y)-18H(0,0,1;z)H(0;y)\\
&-12H(0,0,1;z)H(1;y)+13H(0,0,1;z)H(1-z;y)+5H(0,0,1;z)H(z;y)\\
&-12H(0,0,1,0;y)+18H(0,0,1,1;z)+4H(0,0,1-z;y)H(0;z)\\
&+16H(0,0,1-z;y)H(1;z)-4H(0,0,1-z,0;y)+16H(0,0,1-z,1-z;y)\\
&+8H(0,0,z;y)H(1;z)+8H(0,0,z,1-z;y)-18H(0,1;z)H(0,1-z;y)\\
&-14H(0,1;z)H(0,z;y)+10H(0,1;z)H(1,0;y)-12H(0,1;z)H(1,1-z;y)\\
&-18H(0,1;z)H(1-z,0;y)+21H(0,1;z)H(1-z,1-z;y)-3H(0,1;z)H(1-z,z;y)\\
&-8H(0,1;z)H(z,0;y)+4H(0,1;z)H(z,1-z;y)-3H(0,1;z)H(z,z;y)\\
&-8H(0,1,0;y)H(0;z)-8H(0,1,0;y)H(1;z)-18H(0,1,0;z)H(0;y)\\
&-6H(0,1,0;z)H(1;y)+2H(0,1,0;z)H(1-z;y)+2H(0,1,0;z)H(z;y)\\
&-12H(0,1,0,0;y)-18H(0,1,0,0;z)-8H(0,1,0,1-z;y)-12H(0,1,1;z)H(0;y)\\
&+19H(0,1,1;z)H(1-z;y)+6H(0,1,1;z)H(z;y)-2H(0,1,1,0;y)-21H(0,1,1,0;z)\\
&+12H(0,1,1,1;z)-8H(0,1,1-z,0;y)-4H(0,1-z;y)H(1,0;z)\\
&-22H(0,1-z;y)H(1,1;z)+16H(0,1-z,0;y)H(1;z)-4H(0,1-z,0,0;y)\\
&+16H(0,1-z,0,1-z;y)-12H(0,1-z,1,0;y)-22H(0,1-z,1-z;y)H(1;z)\\
&+16H(0,1-z,1-z,0;y)-22H(0,1-z,1-z,1-z;y)-18H(0,1-z,z;y)H(1;z)\\
&-18H(0,1-z,z,1-z;y)-8H(0,z;y)H(1,0;z)-12H(0,z;y)H(1,1;z)\\
&+8H(0,z,0;y)H(1;z)+8H(0,z,0,1-z;y)-8H(0,z,1-z;y)H(0;z)\\
&-12H(0,z,1-z;y)H(1;z)+8H(0,z,1-z,0;y)-12H(0,z,1-z,1-z;y)\\
&-6H(0,z,z;y)H(1;z)-6H(0,z,z,1-z;y)+16H(1;z)H(1-z,0,0;y)\\
&-22H(1;z)H(1-z,0,1-z;y)-16H(1;z)H(1-z,0,z;y)\\
&-22H(1;z)H(1-z,1-z,0;y)+25H(1;z)H(1-z,1-z,1-z;y)\\
&+23H(1;z)H(1-z,1-z,z;y)+21H(1;z)H(1-z,z,1-z;y)\\
&-3H(1;z)H(1-z,z,z;y)+8H(1;z)H(z,0,0;y)-4H(1;z)H(z,0,1-z;y)\\
&-16H(1;z)H(z,0,z;y)-4H(1;z)H(z,1-z,0;y)+14H(1;z)H(z,1-z,1-z;y)\\
&+2H(1;z)H(z,z,1-z;y)-3H(1;z)H(z,z,z;y)-8H(1,0;y)H(1,0;z)
\end{aligned}
$$

$$
\begin{aligned}
&+10H(1,0;y)H(1,1;z)-6H(1,0;z)H(1,1-z;y)-2H(1,0;z)H(z,0;y)\\
&-6H(1,0;z)H(z,1-z;y)-8H(1,0,0;y)H(1;z)-4H(1,0,0;z)H(1-z;y)\\
&-8H(1,0,0;z)H(z;y)-12H(1,0,0,0;y)+15H(1,0,0,1;z)\\
&-8H(1,0,0,1-z;y)-12H(1,0,1;z)H(1;y)+21H(1,0,1;z)H(1-z;y)\\
&+4H(1,0,1;z)H(z;y)+2H(1,0,1,0;y)+21H(1,0,1,1;z)\\
&+10H(1,0,1-z;y)H(1;z)-8H(1,0,1-z,0;y)+10H(1,0,1-z,1-z;y)\\
&+12H(1,0,z;y)H(1;z)+12H(1,0,z,1-z;y)-22H(1,1;z)H(1-z,0;y)\\
&+25H(1,1;z)H(1-z,1-z;y)+21H(1,1;z)H(1-z,z;y)-4H(1,1;z)H(z,0;y)\\
&+14H(1,1;z)H(z,1-z;y)+2H(1,1;z)H(z,z;y)-6H(1,1,0;z)H(1;y)\\
&+4H(1,1,0;z)H(1-z;y)-6H(1,1,0;z)H(z;y)-2H(1,1,0,0;y)\\
&+21H(1,1,0,1;z)+25H(1,1,1;z)H(1-z;y)+14H(1,1,1;z)H(z;y)\\
&-H(1,1,1,0;y)+25H(1,1,1,1;z)+10H(1,1-z,0;y)H(1;z)\\
&-8H(1,1-z,0,0;y)+10H(1,1-z,0,1-z;y)-18H(1,1-z,1,0;y)\\
&+10H(1,1-z,1-z,0;y)-12H(1,1-z,z;y)H(1;z)-12H(1,1-z,z,1-z;y)\\
&-4H(1-z,0,0,0;y)+16H(1-z,0,0,1-z;y)-12H(1-z,0,1,0;y)\\
&+16H(1-z,0,1-z,0;y)-22H(1-z,0,1-z,1-z;y)-16H(1-z,0,z,1-z;y)\\
&-12H(1-z,1,0,0;y)+6H(1-z,1,1,0;y)+16H(1-z,1-z,0,0;y)\\
&-22H(1-z,1-z,0,1-z;y)+18H(1-z,1-z,1,0;y)-22H(1-z,1-z,1-z,0;y)\\
&+25H(1-z,1-z,1-z,1-z;y)+23H(1-z,1-z,z,1-z;y)\\
&+21H(1-z,z,1-z,1-z;y)-3H(1-z,z,z,1-z;y)+8H(z,0,0,1-z;y)\\
&-6H(z,0,1,0;y)+8H(z,0,1-z,0;y)-4H(z,0,1-z,1-z;y)\\
&-16H(z,0,z,1-z;y)+8H(z,1-z,0,0;y)-4H(z,1-z,0,1-z;y)\\
&+6H(z,1-z,1,0;y)-4H(z,1-z,1-z,0;y)+14H(z,1-z,1-z,1-z;y)\\
&+2H(z,z,1-z,1-z;y)-3H(z,z,z,1-z;y)+\frac{31\pi^4}{60}\\
&+\zeta_3\big[-37H(0;y)-24H(0;z)-11H(1;y)-17H(1;z)-9H(1-z;y)\big]\\
&+\frac{\pi^2}{6}\big[-6H(0;y)H(0;z)+4H(0;y)H(1;z)-6H(0;z)H(1;y)+2H(0;z)H(1-z;y)\\
&-4H(0,0;y)-6H(0,0;z)-2H(0,1;y)-15H(0,1;z)-6H(1;y)H(1;z)\\
&-2H(1;z)H(z;y)-6H(1,0;y)-H(1,1;y)-4H(1,1;z)-6H(1,1-z;y)\\
&+4H(1-z,0;y)+6H(1-z,1;y)-4H(1-z,1-z;y)-2H(z,1-z;y)\big]\,, \qquad (3.49)
\end{aligned}
$$

$$f_{7.8,4}(y,z) = 0\,, \qquad (3.50)$$

$$f_{7.8,3}(y,z) = 0\,, \qquad (3.51)$$

$$f_{7.8,2}(y,z) = 0\,, \qquad (3.52)$$

$$f_{7.8,1}(y,z) = -2H(0;y)H(1,1;z)+2H(0;z)H(1-z,1-z;y)+2H(0,1;z)H(1-z;y)+2H(0,1,1;z)$$

$$
\begin{aligned}
& -2H(0,1-z;y)H(1;z) - 2H(0,1-z,1-z;y) - 2H(1;z)H(1-z,0;y) \\
& +2H(1,0;z)H(1-z;y) + 2H(1,0,1;z) + 2H(1,1,0;z) - 2H(1-z,0,1-z;y) \\
& -2H(1-z,1-z,0;y) \ , \qquad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
f_{7.8,0}(y;z) \;=\; & -8H(0;y)H(1,0,1;z) - 4H(0;y)H(1,1,0;z) - 6H(0;y)H(1,1,1;z) \\
& +4H(0;z)H(1-z,1,0;y) + 6H(0;z)H(1-z,1-z,1-z;y) \\
& +8H(0;z)H(1-z,z,1-z;y) - 4H(0;z)H(z,1-z,1-z;y) + 4H(0,0;y)H(1,1;z) \\
& -4H(0,0;z)H(1-z,1-z;y) + 12H(0,0,1;z)H(1-z;y) - 4H(0,0,1,1;z) \\
& +4H(0,0,1-z;y)H(1;z) + 4H(0,0,1-z,1-z;y) - 4H(0,1;z)H(0,1-z;y) \\
& -8H(0,1;z)H(1-z,0;y) + 6H(0,1;z)H(1-z,1-z;y) + 8H(0,1;z)H(1-z,z;y) \\
& -4H(0,1;z)H(z,1-z;y) - 4H(0,1,0;y)H(0;z) - 4H(0,1,0;y)H(1;z) \\
& +4H(0,1,0;z)H(1-z;y) + 4H(0,1,0,1;z) - 4H(0,1,0,1-z;y) \\
& +6H(0,1,1;z)H(1-z;y) - 4H(0,1,1;z)H(z;y) - 4H(0,1,1,0;y) + 6H(0,1,1,1;z) \\
& -4H(0,1,1-z,0;y) + 4H(0,1-z;y)H(1,0;z) - 6H(0,1-z;y)H(1,1;z) \\
& +4H(0,1-z,0;y)H(0;z) + 4H(0,1-z,0;y)H(1;z) + 4H(0,1-z,0,1-z;y) \\
& +4H(0,1-z,1-z;y)H(0;z) - 6H(0,1-z,1-z;y)H(1;z) + 4H(0,1-z,1-z,0;y) \\
& -6H(0,1-z,1-z,1-z;y) - 8H(0,1-z,z;y)H(1;z) - 8H(0,1-z,z,1-z;y) \\
& -4H(0,z;y)H(1,1;z) - 4H(0,z,1-z;y)H(1;z) - 4H(0,z,1-z,1-z;y) \\
& +4H(1;z)H(1-z,0,0;y) - 6H(1;z)H(1-z,0,1-z;y) - 8H(1;z)H(1-z,0,z;y) \\
& +4H(1;z)H(1-z,1,0;y) - 6H(1;z)H(1-z,1-z,0;y) - 8H(1;z)H(1-z,z,0;y) \\
& +4H(1;z)H(z,0,1-z;y) + 4H(1;z)H(z,1-z,0;y) - 4H(1,0;z)H(1-z,0;y) \\
& +2H(1,0;z)H(1-z,1-z;y) + 8H(1,0;z)H(1-z,z;y) - 4H(1,0;z)H(z,1-z;y) \\
& -4H(1,0,0;z)H(1-z;y) + 12H(1,0,0,1;z) + 2H(1,0,1;z)H(1-z;y) \\
& -4H(1,0,1;z)H(z;y) + 4H(1,0,1,0;z) + 6H(1,0,1,1;z) - 6H(1,1;z)H(1-z,0;y) \\
& +4H(1,1;z)H(z,0;y) - 6H(1,1,0;z)H(1-z;y) - 4H(1,1,0;z)H(z;y) - 4H(1,1,0,0;z) \\
& +2H(1,1,0,1;z) - 6H(1,1,1,0;z) + 4H(1-z,0,0,1-z;y) + 4H(1-z,0,1-z,0;y) \\
& -6H(1-z,0,1-z,1-z;y) - 8H(1-z,0,z,1-z;y) + 4H(1-z,1,0,1-z;y) \\
& +4H(1-z,1,1,0;y) + 4H(1-z,1,1-z,0;y) + 4H(1-z,1-z,0,0;y) \\
& -6H(1-z,1-z,0,1-z;y) + 4H(1-z,1-z,1,0;y) - 6H(1-z,1-z,1-z,0;y) \\
& -8H(1-z,z,0,1-z;y) - 8H(1-z,z,1-z,0;y) + 4H(z,0,1-z,1-z;y) \\
& +4H(z,1-z,0,1-z;y) + 4H(z,1-z,1-z,0;y) + \frac{\pi^2}{6}\big[-4H(0,1;y) + 4H(0,1-z;y) \\
& -4H(1;z)H(1-z;y) - 4H(1,1;z) + 4H(1-z,1;y)\big] \ . \qquad (3.54)
\end{aligned}
$$

Both seven-propagator master integrals without scalar products in the numerator (3.21), (3.38) were first considered by Binoth and Heinrich [18]. Using a dedicated algebraic procedure to separate the structure of infrared divergencies, the $1/\epsilon^4$ and $1/\epsilon^3$ terms of these integrals were obtained analytically in [18], the remaining terms were computed numerically order by order in ϵ. We agree with the results for the $1/\epsilon^4$

and $1/\epsilon^3$ terms of [18].

More recently, Smirnov has considered [17] the problem of evaluating one of the seven-propagator master integrals (3.21). The result, quoted in [17], contains explicit expressions for all divergent parts of (3.21), which we confirm. The finite part of (3.21), however, is expressed as a three-dimensional Mellin–Barnes integral, which we are unable to transform into two-dimensional harmonic polylogarithms. A direct comparison with the finite part of [17] is therefore not possible. Based on general considerations about the analytic structure of the results obtained in [17], Smirnov moreover argues that two-dimensional harmonic polylogarithms could be insufficient to describe the finite part of (3.21) and calls for the introduction of three-dimensional harmonic polylogarithms – but that is not the case, as shown by our results.

4 Conclusion

Two-loop four-point functions with one off-shell leg are an important ingredient for the calculation of next-to-next-to-leading order corrections to three-jet production and related observables in electron–positron annihilation. By exploiting integration-by-parts [3–5] and Lorentz invariance [6] identities, one can express the loop integrals appearing in these functions as a linear combination of a small set of master integrals, which are scalar functions of the external invariants. Their determination was up to now a major obstacle to further progress in next-to-next-to-leading order calculations.

These master integrals fulfil inhomogeneous differential equations [6] in their external invariants, which can be used for their computation. In this approach, one solves the differential equations by starting from an appropriate ansatz, which is determined from the homogeneous part of the equation. Once this ansatz is found, the equations are expanded in $\epsilon = (4 - d)/2$ and then solved by repeated quadratures, yielding a general solution that still contains free constants of integration. These constants are determined from boundary conditions to the same differential equations. Using this method, we have computed the master integrals for planar two-loop four-point functions with one off-shell leg and presented it in a previous publication [16]. In the present work, we have now derived the corresponding non-planar master integrals as well. The results are expressed as a Laurent series in ϵ, with coefficients containing 2dHPL, which were introduced in [16] as generalization of harmonic polylogarithms [19] and of Nielsen's generalized polylogarithms [20]. All 2dHPL up to weight 3, as those appearing in the divergent parts of the master integrals, can be expressed in terms of generalized polylogarithms of complicated arguments, see [16], while the 2dHPL of weight 4 appearing in the finite parts of the master integrals involve a one-dimensional integral over a combination of generalized polylogarithms of weight 3.

The results presented in this paper complete the calculation of master integrals needed for the computation of two-loop four-point functions with one off-shell leg. These functions are a crucial ingredient for the virtual next-to-next-to-leading order corrections to processes such as three-jet production in electron-positron annihilation, two-plus-one-jet production in deep inelastic scattering, and vector-boson-plus-jet production at hadron colliders. The results given in [16] and in the present work correspond to the kinematical situation of three-jet production in electron–positron annihilation, where all master integrals are real (up to a complex normalization factor). Using the prescription for the analytic continuation of two-dimensional harmonic polylogarithms derived in the appendix of [16], our results can be continued into any other kinematic region of interest.

Equipped with the complete set of master integrals, it should soon become possible to compute the two-loop virtual corrections to the above-mentioned processes. However, it must be kept in mind that these corrections form only part of a full next-to-next-to-leading order calculation, which also has to include the one-loop corrections to processes with one soft or collinear real parton [30–32], as well as tree-level processes with two soft or collinear partons [33]. Only after summing all these contributions (and including terms from the renormalization of parton distributions for processes with partons in the initial state), do the divergent terms cancel among one another. The remaining finite terms have to be combined into a numerical programme implementing the experimental definition of jet observables and event-shape variables. A first calculation involving the above features was presented for the case of photon-plus-one-jet final states in electron–positron annihilation in [33], thus demonstrating the feasibility of this type of calculations.

Note added: After this paper was submitted for publication, we have carried out a detailed comparison with the purely numerical results of [18]. We find full agreement with the values quoted in [18] for the master integrals (3.21) and (3.38) at specific points of the kinematical invariants, as well as with values for the master integral (3.9) provided by the authors of [18]. We would like to thank Thomas Binoth and Gudrun Heinrich for their assistance with this comparison.

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