Supply chain model for deteriorating items with stock-dependent consumption rate and shortages under inflation and permissible delay in payment

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Abstract: In today’s world the financial decisions of any business/retail enterprise are very crucial, as from the financial standpoint, an inventory i.e., stock on display represents a capital investment and must compete with other assets for a firm’s limited capital funds. Moreover, rising inflation rate directly affects the financial situation of an organization. On the contrary, the today's market is totally customer oriented which forces the retailer to invest more on stock in order to attract more and more customers. Further, this situation becomes more challenging when the retailer is dealing with deteriorating items. Apart from all this, the trade credit also plays a vital role in the financial decision making, as it helps in reducing the costs of holding stock. Keeping such a scenario in mind, this paper develops an inventory model for deteriorating items with stock-dependent consumption rate and allowable shortages under inflation and permissible delay in payments. Finally the model has been validated through numerical examples.

Keywords: inventory; inflation; shortages; permissible delay; stock-dependent demand; deterioration.

1 Introduction

Nowadays the retail stores have to display consumer goods in large quantities to attract new customers, as the market is totally customer-oriented and the demand rate is dependent on the stock on display. Thus, in order to survive in the market and to increase the demand, the retailer has to invest more on stock. The situation is more sensitive when the stock of displayed items is of deteriorating in nature. In general, deterioration is defined as damage, spoilage, decay, obsolescence, evaporation, pilferage, etc., that results in decreasing the usefulness of the original one. There is another form of deterioration, in which the utility and usefulness of items decreases due to change in market trends or technological trends viz. clothes, toys, household items, mobile phones, refrigerators, LCD and DVD players, and so on. And all these items are displayed in large amounts in retail stores. Moreover, from a financial standpoint, an inventory represents a capital investment and must compete with other assets for a firm’s limited capital funds. The effects of inflation are not usually considered when an inventory system is analysed because most people think that inflation would not influence the inventory policy to any significant degree. Owing to high inflation, the financial situation has changed in many developing countries. As a result, while determining the optimal inventory policy, the effect of inflation cannot be ignored especially in case of deteriorating items having stock-dependent consumption rate, for a retailer. Further, in classical inventory analysis, it was tacitly assumed that the supplier would not allow a delay period to settle the account with the retailer. But in most business transactions, it is more common to see that the retailer is allowed some delay period to settle the account with the supplier. This provides an advantage to the retailer, due to the fact that he does not have to pay the supplier immediately after receiving the order, but instead, can defer
Supply chain model

the payment until the end of allowed period. The retailer pays no interest during the fixed period they are supposed to settle the account, but if the payment is delayed beyond that period, interest is charged. Hence, paying later indirectly reduces the cost of holding stock. Thus, the retailer invests more on inventory so as to survive in the competitive market of stock-dependent consumption rate, and at the same time he does not have to pay cash immediately.

In the present paper, an inventory model for deteriorating items with stock-dependent consumption rate and shortages under the condition of permissible delay of payment has been formulated. In addition, the effects of inflation and time value of money on optimal replenishment policy are also considered over a finite planning horizon. An optimisation solution procedure is presented to find the optimal replenishment policy and the convexity of the total cost function is discussed. Finally, a numerical example is solved to illustrate the results, and comprehensive sensitivity analysis has been performed on key model parameters of the system. The proposed model can be very helpful for the managers of large retail outlets or big supermarkets.

2 Literature survey

Many researchers have given a considerable attention to the situation where the demand rate is dependent on the on-hand inventory level, as it is usually observed in the supermarket that display consumer goods in large quantities to attract new customers which in turn give rise to the demand. This idea was reported by Levin et al. (1972) and Silver and Peterson (1985). Gupta and Vrat (1986) developed an inventory model with stock-dependent consumption rate to minimise the cost. Padmanabhan and Vrat (1988) defined stock-dependent consumption rate as a function of inventory level at any instant of time and developed model for perishable items. Padmanabhan and Vrat (1995) further presented inventory models for perishable items with stock-dependent selling rate, where the selling rate is assumed to be a function of current inventory level. Other related analyses on inventory systems with stock-dependent consumption rate have been performed by Sarker et al. (1997), Datta and Paul (2001), Goh (1992), Pal et al. (1993), Lev et al. (1994), Urban (1995), Mandal and Maji (1997, 1999) and Giri and Chaudhuri (1998). Balkhi and Benkherouf (2004) presented an inventory model for deteriorating items with stock-dependent and time varying demand rates over a finite planning horizon. Recently, Wu et al. (2006) determined an optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging.

Moreover, deterioration is a well-established fact in literature, due to which utility of an item does not remain same over the period of time. Ghare and Schrader (1963) presented an economic order quantity (EOQ) model for deteriorating items assuming exponential decay. Covert and Phillip (1973) extended the model with the assumption of Weibull distribution deterioration. Thereafter, several interesting papers have appeared in different Journals viz., Dave and Patel (1981), Sachan (1984), Chung and Ting (1994), Hargia and Benkherouf (1994), Hargia (1995), Chakrabarti and Chaudhuri (1997). Recently Samanta and Roy (2004) developed a production inventory model where distribution of the time to deterioration of an item follows exponential distribution.

In all the above models, the time value of money and inflation were not considered because of the belief that the time value of money and inflation would not affect significantly the decisions regarding inventory management. But in real life the impact of
time value of money and inflation cannot be ignored while deciding the optimal inventory policies. The fundamental result in the development of EOQ model with inflation is that of Buzacott (1975) who discussed EOQ model with inflation subject to different types of pricing policies. Several other interesting and relevant papers in this direction are Datta and Pal (1991), Hargia and Ben-Daya (1996). Later, Ray and Chaudhuri (1997), Chen (1998), Chung and Lin (2001), Wee and Law (2001), Moon et al. (2005) all have investigated the effects of inflation, time value of money and deterioration on inventory models. Jaggi et al. (2006) developed optimal order policy for deteriorating items with inflation induced demand. Recently, Hou (2006) formulated an inventory model for deteriorating items with stock-dependent consumption rate and shortages under inflation and time discounting.

Apart from all this, the trade credit also plays a vital role in today’s competitive market, as the permissible delay in payments offered by the supplier to the retailer has become a very powerful promotional tool to attract new customers and a good incentive policy for the buyers. Owing to this fact, during the past few years, a lot of research work has been done on inventory models with permissible delay in payment. Haley and Higgins (1973) introduced the first model to determine EOQ under condition of permissible delay in payment with deterministic demand, no shortages, and zero lead-time. Aggarwal et al. (1997) investigated the economic ordering policies in the presence of trade credit with inflation for non-deteriorating items. Other inventory models for deteriorating items under a permissible delay in payment have been developed by Jaggi and Aggarwal (1994), Aggarwal and Jaggi (1995), Jamal et al. (1997), Sarker et al. (2000a,b), Liao et al. (2000) and Jaggi and Goel (2005). Dye (2002) in his paper considered the stock-dependent demand for deteriorating items with partial backlogging and condition of permissible delay in payment. Chang et al. (2004) considered an inventory model for deteriorating items with instantaneous stock-dependent demand and time value of money when credit period is provided. Chung et al. (2005) formulated the optimal inventory policies under permissible delay in payments depending on the order quantity. Chung and Liao (2006) extended the model incorporating DCF analysis. Arcelus et al. (2007) developed the ordering and pricing policies of a retailer, confronted with a price-dependent demand and a vendor offering a discount on the regular price. Liao (2007) derived a production model for the lot-size inventory system with finite production rate, taking into consideration the effect of decay and the condition of permissible delay in payments. Sana (2008) derived an EOQ/EPQ model with a varying demand followed by advertising expenditure and selling price under permissible delay in payments, for a retailer. Tsao (2009) developed an EOQ model under advance sales discount and trade credits. Huang and Hsu (2008) have developed an inventory model under two-level trade credit policy by incorporating partial trade credit option at the customers of the retailer. Recently, Jaggi and Khanna (2009) formulated retailer’s ordering policy for deteriorating items with inflation induced demand under trade credit policy ‘d/D, Net D’.

Unfortunately, none of these models have incorporated all these realistic features of an inventory system collectively viz. stock-dependent consumption rate, deterioration, permissible delay in payments, inflation and time value of money. Thus, in the present paper an attempt has been made to incorporate all these features and develop an inventory model which can assist retailer in making important replenishment decisions.
3 Assumptions and notation

Single-supplier single-retailer supply chain model has been developed under the following assumptions and notations:

1. The consumption rate is stock dependent.
2. The replenishment rate is infinite and lead-time is zero.
3. Shortages are allowed and fully backlogged.
4. A constant fraction of the on-hand inventory deteriorates per unit time. The item value switches to zero upon deterioration.
5. There is no repair or replenishment of the deteriorated items during the inventory cycle.
6. Inflation rate is constant over the planning horizon.
7. A discounted cash flow (DCF) approach is used to consider the various costs at various times.
8. The planning horizon is finite.
9. Supplier offers a fixed credit period to the retailer to settle the accounts.

Notations adopted in this paper are as below:

\( D(t) = \alpha + \beta I(t) \), stock-dependent consumption rate
\( \alpha \) = a positive constant
\( \beta \) = stock-dependent consumption rate parameter, \( 0 \leq \beta \leq 1 \)
\( I(t) \) = inventory level at time \( t \)
\( r \) = discount rate, representing the time value of money
\( i \) = inflation rate
\( R = r - i \), representing the net discount rate of inflation is constant
\( H \) = planning horizon
\( T \) = replenishment cycle
\( n \) = the number of replenishment during the planning horizon, \( n = H/T \)
\( T_j \) = the total time that is elapsed up to and including the \( j \)th replenishment cycle \((j = 1, 2, \ldots, n)\) where \( T_0 = 0\), \( T_1 = T \) and \( T_n = H \)
\( t_j \) = the time at which the inventory level in the \( j \)th replenishment cycle drops to zero \((j = 1, 2, \ldots, n)\)
\( T_j - t_j \) = time period when shortage occurs \((j = 1, 2, \ldots, n)\)
\( Q \) = the 2nd, 3rd, \ldots, \( n \)th replenishment lot size
\( Im \) = maximum inventory level
\( Ip \) = the interest paid per $ per unit time
496  C.K. Jaggi and A. Khanna

$I_e$ = the interest earned per $ per unit time

$M$ = permissible delay in settling the account

$\theta$ = deterioration rate, units per unit time

$A$ = cost per replenishment, $ per order

$c$ = per unit cost of the item, $ per unit

$p$ = per unit selling price of the item, $ per unit

$c_1$ = per unit inventory holding cost per unit time, $ per unit per unit time

$c_2$ = per unit shortage cost per unit time, $ per unit per unit time

4 The supply chain model formulation under inflation

The planning horizon $H$ is divided into $n$ equal parts of length $T = H/n$. Hence, the reorder times over the planning horizon $H$ are $T_j = jT$ ($j = 0, 1, 2, \ldots, n$). When the inventory is positive, consumption rate is dependent on stock levels, whereas for negative inventory, the demand (backlogging) rate is constant. Therefore, the period for which there is no-shortage in each interval $[jT, (j+1)T]$ is a fraction of the scheduling period $T$ and is equal to $kT$ ($0 < k < 1$). Shortages occur at time $t_j = (k + j - 1)T$ ($j = 1, 2, \ldots, n$) and are accumulated until time $t = jT$ ($j = 1, 2, \ldots, n$) before they are backordered. This model is illustrated in Figure 1.

As we have assumed stock dependent consumption rate, that is, $D(t)$, therefore, demand at any time $t$ is given by

$$D(t) = \alpha + \beta I(t)$$

where $\alpha$ is a positive constant, $\beta$ is the stock-dependent consumption rate parameter, $0 \leq \beta \leq 1$ and $I(t)$ is the inventory level at time $t$.

Figure 1  The graphical representations of the inventory cycles for case 1($M \leq t_1 < T$)
Supply chain model

4.1 General decaying function

At $t = 0$, the first replenishment lot size of $I_m$ is received. Now, during the time interval $[0, t_1]$, the inventory level decreases due to stock dependent consumption rate and deterioration until it depletes to zero at $t = t_1$. Further, during the time interval $[t_1, T]$, we accumulate the backorders that are fulfilled from the next replenishment. Therefore, the inventory system at any time $t$ can be represented by the following differential equations:

$$\frac{dI(t)}{dt} + \theta I(t) = -\left[\alpha + \beta I(t)\right], \quad 0 \leq t \leq t_1$$

$$\frac{dI(t)}{dt} = -\alpha, \quad t_1 \leq t \leq T$$

The solutions of the above differential equations after applying the boundary condition $I(t_1) = 0$, are

$$I(t) = \frac{\alpha}{\beta + \theta} \left[e^{(eta + \theta)k(t-t_1)} - 1\right], \quad 0 \leq t \leq t_1$$

$$I(t) = -\alpha(t-t_1), \quad t_1 \leq t \leq T$$

Therefore, the maximum inventory level and maximum shortage quantity during the first replenishment cycle are

$$I_m = \frac{\alpha}{\beta + \theta} \left[e^{(eta + \theta)kH/n} - 1\right]$$

and

$$I_b = \alpha(T-t_1) \alpha(1-k)H/n$$

respectively.

4.2 Inventory scenarios

Since the shortages are allowed and are fully backlogged, retailer is being offered a fixed credit period $M$ to settle the accounts with the supplier; therefore, two distinct cases have been considered.

1. payment at or before the total depletion of inventory ($M \leq t_1 < T$)
2. after-depletion payment ($t_1 < M$).

4.2.1 Case 1: $M \leq t_1 < T$ (i.e. payment at or before the total depletion of inventory)

This situation indicates that the delay period expires on or before the inventory depletes to zero. As a result, the variable cost is comprised of the sum of the ordering cost, purchase cost, holding cost, backorder cost, and the interest payable minus the interest earned.
Moreover, the present model has been developed under inflationary conditions. Thus, by using continuous compounding of inflation and discount rate, the present worth of the various costs for the first replenishment cycle is calculated as follows:

1. Present value of ordering cost during the first replenishment cycle is
   \[ C_r = A \]  
   (since ordering is made at time \( t = 0 \), the inflation does not affect the ordering cost).

2. Present value of purchase cost during the first replenishment cycle is
   \[ C_p = cI_m + ce^{-RT} \int_0^{T-t_1} \alpha dt \]
   \[ = \frac{ca}{\beta + \theta} \left[ e^{(\beta + \theta)kH/n} - 1 \right] + c\alpha e^{RH/n} \left[ \frac{H}{n} - \frac{kH}{n} \right]. \]  
   (8)

3. Present value of holding cost during the first replenishment cycle is
   \[ C_h = c_1 \int_0^{t_1} I(t) e^{-Rt} dt \]
   \[ = \frac{c_1 \alpha}{(\beta + \theta)(\beta + \theta + R)} \left[ e^{(\beta + \theta)kH/n} - e^{-RkH/n} \right] + \frac{c_1 \alpha}{(\beta + \theta) R} e^{-RkH/n} - 1. \]  
   (9)

4. The maximum shortage level is \( I_s = a(T-t_1) \), all shortages during the interval \([t_1, T]\) will be completely backordered at \( T \), therefore, present value of shortage cost during the first replenishment cycle is
   \[ C_s = c_2 \int_{t_1}^{T_1} \alpha (t-t_1) e^{-Rt} dt \]
   \[ = \frac{c_2 \alpha}{R^2} \left[ R(kH/n - H/n) + e^{-RkH/n} - 1 \right] e^{-RkH/n}. \]  
   (10)

5. The present value of interest payable for the inventory not being sold after the due date \( M \) during the first replenishment cycle is
   \[ CI_p = cI_p \int_M^{t_1} I(t) e^{-Rt} dt \]
   \[ = \frac{cI_p \alpha}{(\beta + \theta)} \left[ e^{(\beta + \theta)(kH/n-M)} e^{-RM} - e^{-RM} \right] + \frac{e^{-RkH/n} - e^{-RM}}{R}. \]  
   (11)
6 The present value of interest earned during the first replenishment cycle is

\[ CL_e = pL \int_0^h \frac{I(t)}{e^{Rt}} \, dt \]

\[ = \frac{pL \alpha}{(\beta + \theta)} \left( e^{(\beta + \theta)kH/n} - e^{-RkH/n} \right) + \frac{e^{-RkH/n} - 1}{R} \cdot \left(1 - e^{(\beta + \theta)kH/n} \right) \cdot \left(1 - e^{-RkH/n} \right). \] (12)

Consequently, the present value of total cost of system during the first replenishment cycle can be formulated as

\[ TRC_1 = C_r + C_p + C_h + C_s + CI_p - CI_e \] (13)

There are \( n \) cycles during the planning horizon. Since inventory is assumed to start and end at zero, an extra replenishment at \( T_n = H \) is required to satisfy the backorders of the last cycle in the planning horizon. Therefore, there are \( n + 1 \) replenishments in the entire planning horizon \( H \). The first replenishment lot size is \( I_m \), the 2nd, 3rd, …, \( n \)th replenishment lot size is

\[ I_m = \int_0^{T-n} \alpha \, dt \]

\[ = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)kH/n} - 1 \right] + \alpha(H/n - kH/n) \] (14)

and the last or \((n + 1)\)th replenishment lot size is

\[ I_b = \alpha(H/n - kH/n) \] (15)

Now, the present value of total cost of system over a finite planning horizon \( H \) is

\[ TC_1(n,k) = \sum_{j=0}^{n-1} TRC_j e^{-RjT} + Ae^{-RH} = TRC_1 \left( \frac{1-e^{-RH}}{1-e^{-RkH/n}} \right) + Ae^{-RH} \] (16)

where \( T = H/n \) and \( TRC \) are derived by substituting Equations (7)–(12) into Equation (13).
On simplification and summation, we get

\[
TC_J(n,k) = \left( A + \frac{c_i \alpha}{(\beta + \theta)(\beta + \theta + R)} \right) \left[ e^{(\beta + \theta)kH / n} - e^{-RH / n} \right] \\
+ \frac{c_i \alpha}{(\beta + \theta) R} \left[ e^{-RH / n} - 1 \right] E + A e^{-RH} \\
+ \left[ \frac{c_i \alpha}{R} \left( R(kH / n - H / n) + e^{-R(kH / n - H / n)} - 1 \right) e^{-RH / n} \right] \\
F + \frac{c \alpha E}{R} \left( e^{(\beta + \theta)kH / n} - 1 \right) + c \alpha (1-k) FH / n \\
+ \left[ \frac{c_i \alpha}{(\beta + \theta)} \left( \frac{e^{(\beta + \theta)(kH / n-M)} - e^{-(kH / n) / R}}{R} \right)^2 \right] E \\
- \left[ \frac{c \alpha}{(\beta + \theta)} \left( \frac{e^{(\beta + \theta)kH / n} - e^{-R(kH / n)} / R}{R} + e^{-R(kH / n)} / R \right) \right] E
\]

(17)

where \( E = \frac{1-e^{-RH / n}}{1-e^{-RH / n}} \) and \( F = \frac{1-e^{-RH / n}}{e^{-RH / n}} \).

4.2.2 Case 2: \( t_1 < M \) (i.e. after depletion payment)

In this case, the retailer makes the payment after depletion of the inventory, as illustrated in Figure 2.

The ordering cost \( C_o \), purchase cost \( C_p \), holding cost \( C_h \) and shortage cost \( C_s \) for the first replenishment cycle are the same as in Equations (7), (8), (9) and (10), respectively. The interest payable for the cycle, \( C_l_p \) in this case is equal to zero because the supplier can be paid in full at the end of the permissible delay, \( M \).

The interest earned for the cycle is the interest earned during the positive inventory plus the interest earned from the cash invested during the time period \([t_1, M]\) after the inventory is exhausted at \( t_1 \).

**Figure 2**  The graphical representations of the inventory cycles for case 2 \((t_1 < M)\)
Thus, the present value of interest earned during the first replenishment cycle is

\[
CI_e = pl_e \left\{ t_i \int_0^t I(t)e^{-Rt} dt \right\} + pl_e \left\{ t_i \int_0^t I(t)dt \right\} (M - t_i) e^{-RM}
\]

\[
= \frac{pl_e \alpha}{(\beta + \theta)} \left[ e^{(\beta + \theta)kH/n} - e^{-RkH/n} \right] + \frac{e^{-RkH/n} - 1}{R} \]

\[
+ \frac{pl_e \alpha}{(\beta + \theta)} \left( M - \frac{kH}{n} \right) e^{-RM} \left[ e^{(\beta + \theta)kH/n} - 1 - \frac{kH}{n} \right]
\]

(18)

Consequently, the present value of total cost of system during the first replenishment cycle in this case is given by

\[
TRC_e = C_r + C_p + C_h + C_s + CI_e
\]

(19)

Now, the present value of total cost of system over a finite planning horizon \(H\) is

\[
TC_e(n,k) = \sum_{j=0}^{n-1} TRC e^{-RjT} + Ae^{-RH} = TRC e^{\left( 1 - e^{-RH} \right)} + Ae^{-RH}
\]

(20)

Where, \(T = H/n\) and \(TRC\) are derived by substituting Equations (7)–(10) and (18) into Equation (19).

On simplification and summation, we get

\[
TC_e(n,k) = \left\{ A + \frac{C_1 \alpha}{(\beta + \theta)(\beta + \theta + R)} \left[ e^{(\beta + \theta)kH/n} - e^{-RkH/n} \right] \right\} E
\]

\[
+ \frac{C_2 \alpha}{(\beta + \theta)R} \left[ e^{-RkH/n} - 1 \right] E
\]

\[
+ Ae^{-RH} + \left( \frac{C_3 \alpha}{R^2} \left( R(kH/n - H/n) + e^{-R(kH/n-H/n)} - 1 \right) e^{-RH/n} \right) F
\]

\[
+ \frac{C\alpha E}{\beta + \theta} \left( e^{(\beta + \theta)kH/n} - 1 \right) + C\alpha(1-k)FH/n
\]

\[
- \frac{pl_e \alpha}{(\beta + \theta)} \left[ e^{(\beta + \theta)kH/n} - e^{-RkH/n} \right] + \frac{e^{-RkH/n} - 1}{R} \right\} E
\]

\[
- \frac{pl_e \alpha}{(\beta + \theta)} \left( M - \frac{kH}{n} \right) e^{-RM} \left[ e^{(\beta + \theta)kH/n} - 1 - \frac{kH}{n} \right]
\]

(21)

Where, \(E = \left( \frac{1 - e^{-RH}}{1 - e^{-RH/n}} \right)\) and \(F = \frac{1 - e^{-RH}}{e^{RH/n} - 1}\)
The present value of total cost $TC(n, k)$ can be given by

$$TC(n, k) = \begin{cases} 
TC_1(n, k) & \text{if } M < t_1 < T, \\
TC_2(n, k) & \text{if } t_1 < M.
\end{cases}$$  \hspace{1cm} (22)

Further, replacing $t_1 (= kH/n) = M$ in Equations (17) and (21), we get

$$TC_1(n, k)_M = TC_2(n, k)_M$$

### 5 Optimal solution and solution procedure

#### 5.1 Optimal solution

The total cost $TC(n, k)$ is a function of two variables $n$ and $k$, where $n$ is a discrete variable and $k$ is a continuous variable. Our problem is to determine the optimum value of $k$ and $n$ which minimises $TC(n, k)$.

The necessary condition for $TC_1(n, k)$ and $TC_2(n, k)$ to be minimised with respect to $k$ for a given value of $n$, is

$$\frac{dTC_1(n, k)}{dk} = 0 \quad \text{and} \quad \frac{dTC_2(n, k)}{dk} = 0,$$

respectively, which gives

$$\frac{dTC_1(n, k)}{dk} = \left\{ \begin{array}{c}
c_1 \alpha H \left( \frac{(\beta + \theta)(\beta + \theta + R)n - e^{-RkH/n}}{n} \right) \\
- \frac{c_1 \alpha H}{(\beta + \theta)n} e^{-RkH/n} \end{array} \right\} E + \frac{c_2 \alpha}{R^2} \left[ RH/n - re^{-R(kH/n - H/n)} H/n \right] F
+ c\alpha E \left( \frac{e^{(\beta + \theta)kH/n} H/n - c\alpha F H/n}{n} \right) E
+ \left\{ \begin{array}{c}
cl_{1} \alpha H \left( \frac{(\beta + \theta)e^{(\beta + \theta)kH/n - M} n}{(\beta + \theta + R)} e^{-RM/n} + e^{-RkH/n} \right) \\
- \frac{pl_{1} \alpha H}{(\beta + \theta)n} \left( \frac{e^{(\beta + \theta)kH/n} n}{(\beta + \theta + R)} - e^{-RkH/n} \right) \end{array} \right\} E = 0$$  \hspace{1cm} (23)

and

$$\frac{dTC_2(n, k)}{dk} = \left\{ \begin{array}{c}
c_1 \alpha H \left( \frac{(\beta + \theta)(\beta + \theta + R)n - e^{-RkH/n}}{n} \right) \\
- \frac{c_1 \alpha H}{(\beta + \theta)n} e^{-RkH/n} \end{array} \right\} E + \frac{c_2 \alpha}{R^2} \left[ RH/n - re^{-R(kH/n - H/n)} H/n \right] F
+ c\alpha E \left( \frac{e^{(\beta + \theta)kH/n} H/n - c\alpha F H/n}{n} \right) E
+ \left\{ \begin{array}{c}
pl_{1} \alpha H \left( \frac{(\beta + \theta)e^{(\beta + \theta)kH/n - M} n}{(\beta + \theta + R)} e^{-RM/n} \right) \\
- \frac{pl_{1} \alpha H}{(\beta + \theta)n} \left( \frac{e^{(\beta + \theta)kH/n} n}{(\beta + \theta + R)} - e^{-RkH/n} \right) \end{array} \right\} E = 0$$  \hspace{1cm} (24)
Further sufficient conditions are
\[ d^2TC_1(n, k)/dk^2 > 0, \] which holds good provided
\[ c_1 \geq pl_e \]
and \((\beta + \theta) \geq R.\]
and \(d^2TC_2(n, k)/dk^2 > 0,\) which holds good provided
\[ c_1 \geq pl_e \]
and \( \frac{kH}{n} e^{(\beta+\theta)kH/n} \geq M.\]
respectively (Appendix).

Since, it is very difficult to get the close form solution from Equations (23) and (24), therefore, we make use of software What’s Best in order to determine the optimal value of \(k,\) for a given positive integer \(n.\)

Let \(t_{11}\) be the value of \(t_1\) in case 1 and \(t_{12}\) be the value of \(t_1\) in case 2. The optimal value of \(t_1\) and \(T\) can be obtained using the relation \(t_1^* = kH/n^*\) and \(T^* = H/n^*,\) while the optimal value of \(TC_1(n, k)\) or \(TC_2(n, k)\) can be obtained by Equations (17) or (21), respectively.

5.2 Solution procedure

Now, to find the optimal value of \(k\) and \(n,\) which minimises the retailer’s total cost, we make use of the following algorithm:

**Step 1** Put \(n = 1.\)

**Step 2** Determine optimal value of \(k\) say \(k_1\) and \(k_2\) from Equations (23) and (24), respectively. Let \(t_{11}\) be the value of \(t_1\) in case 1, and \(t_{12}\) be the value of \(t_1\) in case 2. Compute \(t_{11}^* = k_1H/n^*\) and \(t_{12}^* = k_2H/n\) and determine \(TC_1(n, k)\) and \(TC_2(n, k)\) from Equations (17) and (21), respectively.

**Step 2.1** If \(t_{11}^* > M\) and \(t_{12}^* < M,\) then \(TC = \min\{TC_1(n, k), \ TC_2(n, k)\}\) and the corresponding value of \(t_1\) would be optimal.

**Step 2.2** If \(t_{11}^* > M\) and \(t_{12}^* < M,\) then \(TC = TC_1\) is the optimum cost and \(t_1^*\) is the optimum value of \(t_1.\)

**Step 2.3** If \(t_{11}^* > M\) and \(t_{12}^* < M,\) then \(TC = TC_2\) is the optimum cost and \(t_1^*\) is the optimum value of \(t_1.\)

**Step 2.4** If \(t_{11}^* > M\) and \(t_{12}^* < M,\) then the optimum \(t_1\) is \(M\) and the optimum \(TC = TC_1\) or \(TC_2.\)

**Step 3** Determine \(TC(n + 1, k^*)\) using Step 2.

**Step 4** If \(TC(n, k^*) > TC(n + 1, k^*),\) then increment the value of \(n\) by 1 and go to Step 3, else current value of \(TC(n, k)\) is the optimal total cost and the corresponding \(n\) and \(k\) are optimal. The optimal value of \(T\) can be obtained using the relation \(T^* = H/n^*.\)
The first replenishment lot size, \( I_1^* \); the 2nd, 3rd, …, \( n \)th replenishment lot size \( Q^* \); and the last (i.e. \((n+1)\)th) replenishment lot size \( I_n^* \) can be obtained from Equations (5), (14) and (15), respectively.

6 Numerical example and sensitivity analysis

6.1 Numerical example

The situation of optimal ordering policies for deteriorating items with stock-dependent consumption rate and shortages under inflationary conditions with permissible delay in payment has been viewed as two scenarios: payment before total depletion (Case 1) and payment after total depletion (Case 2) of inventory. An example is devised to illustrate the effect of the general model developed in this paper.

The inventory parametric values \( \alpha = 600 \text{units/year}, \quad \beta = 0.05, \quad \theta = 0.10, \quad A = \$250.00/\text{order}, \quad c = \$5/\text{unit}, \quad p = \$6/\text{unit/year}, \quad c_1 = \$1/\text{unit/year}, \quad c_2 = \$3/\text{unit/year}, \quad I_p = 0.15, \quad I_e = 0.10, \quad R = 0.08, \quad H = 10 \text{ year} \) and \( M = 60 \). Using the solution procedure described above, the results are presented in Table 1.

From this table, we see that for the number of replenishments \( n = 12 \) the total cost \( TC \) becomes minimum. Hence, the optimal values of \( n \) and \( k \) are \( n^* = 12, \quad k^* = 0.546 \), respectively, and the minimum total cost \( TC(n^*, k^*) = \$24566 \). Subsequently, we get \( T^* = H/n^* = 0.833 \text{ year} = 304.2 \text{ days}, \quad t_1^* = k^*T^* = 0.455 \text{ year} = 166.1 \text{ days} \) and \( Q^* = 509.5 \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( k^*(n) )</th>
<th>( T_1^* )</th>
<th>( T^* )</th>
<th>( Q^* )</th>
<th>( TC^* )</th>
</tr>
</thead>
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<td>2</td>
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<td>33,941</td>
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<td>2123.1</td>
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<tr>
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<td>0.494</td>
<td>451.0</td>
<td>912.5</td>
<td>1573.2</td>
<td>27,524</td>
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<tr>
<td>5</td>
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<td>369.5</td>
<td>730.0</td>
<td>1248.5</td>
<td>26,375</td>
</tr>
<tr>
<td>6</td>
<td>0.515</td>
<td>313.3</td>
<td>608.3</td>
<td>1034.2</td>
<td>25,674</td>
</tr>
<tr>
<td>7</td>
<td>0.522</td>
<td>272.2</td>
<td>521.4</td>
<td>883.1</td>
<td>25,228</td>
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<tr>
<td>8</td>
<td>0.528</td>
<td>240.9</td>
<td>456.3</td>
<td>770.3</td>
<td>24,939</td>
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<td>9</td>
<td>0.533</td>
<td>216.2</td>
<td>405.6</td>
<td>682.9</td>
<td>24,755</td>
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<td>365.0</td>
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<td>24,643</td>
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<td>179.9</td>
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<tr>
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<td>0.546*</td>
<td>166.1*</td>
<td>304.2*</td>
<td>509.5*</td>
<td>24,566*</td>
</tr>
<tr>
<td>13</td>
<td>0.550</td>
<td>154.4</td>
<td>280.8</td>
<td>469.8</td>
<td>24,577</td>
</tr>
<tr>
<td>14</td>
<td>0.553</td>
<td>144.3</td>
<td>260.7</td>
<td>435.7</td>
<td>24,611</td>
</tr>
<tr>
<td>15</td>
<td>0.557</td>
<td>135.5</td>
<td>243.3</td>
<td>406.3</td>
<td>24,665</td>
</tr>
<tr>
<td>16</td>
<td>0.560</td>
<td>127.8</td>
<td>228.1</td>
<td>380.6</td>
<td>24,734</td>
</tr>
<tr>
<td>17</td>
<td>0.563</td>
<td>120.9</td>
<td>214.7</td>
<td>358.0</td>
<td>24,816</td>
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<tr>
<td>18</td>
<td>0.566</td>
<td>114.8</td>
<td>202.8</td>
<td>337.9</td>
<td>24,908</td>
</tr>
</tbody>
</table>
6.2 Special cases

Here, we explore some special cases that influence the replenishment policy of the system.

Table 2 shows the comparative results for the special cases. Based on the comparative results, we infer the following:

1. When the permissible delay in payments is not considered, that is, $M = 0$, the present value of total cost is higher than the case permissible delay in payments is considered. Thus, trade credit plays a vital role in controlling the cost of the inventory system.

2. In case of the stock-dependent consumption rate, the order quantity is higher than the case of deterministic and constant demand (i.e. $\beta = 0$). As the stock-dependent consumption rate influences the demand, hence it suggests the retailer to order more quantity.

3. When the deterioration of items is not considered, that is, $\theta = 0$, the total order quantity is lower than the case deterioration of items is considered. Since when the items are deteriorating, retailer has to order more quantity than the requirement, so as to satisfy the demand.

4. In the presence of inflation and time value of money, the present value of total cost is substantially lower than the case inflation and time value of money are considered (i.e. $R = 0$); because DCF approach helps in proper recognition of financial implication of opportunity cost in inventory analysis.

Table 2  Comparison of the results for the above special conditions

<table>
<thead>
<tr>
<th>Special conditions</th>
<th>$M$</th>
<th>$k$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$Q_{total}$ (over $H$)</th>
<th>$TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our model</td>
<td>12</td>
<td>0.546</td>
<td>166.1</td>
<td>304.2</td>
<td>509.50</td>
<td>6114</td>
<td>24,566</td>
</tr>
<tr>
<td>$M = 0$</td>
<td>12</td>
<td>0.514</td>
<td>156.4</td>
<td>304.2</td>
<td>508.44</td>
<td>6101</td>
<td>24,798</td>
</tr>
<tr>
<td>$\beta = 0$</td>
<td>12</td>
<td>0.579</td>
<td>176.0</td>
<td>304.2</td>
<td>507.00</td>
<td>6084</td>
<td>24,417</td>
</tr>
<tr>
<td>$\theta = 0$</td>
<td>11</td>
<td>0.611</td>
<td>202.8</td>
<td>331.8</td>
<td>550.13</td>
<td>6051</td>
<td>24,244</td>
</tr>
<tr>
<td>$R = 0$</td>
<td>12</td>
<td>0.633</td>
<td>192.7</td>
<td>304.2</td>
<td>512.80</td>
<td>6154</td>
<td>35,136</td>
</tr>
</tbody>
</table>

6.3 Sensitivity analysis

In this section, we perform the sensitivity analysis on the key parameters $M$, $\beta$, $\theta$ and $R$ of the model, in order to study their effect on the optimal replenishment policy. The results are summarised in Tables 3–6.

Table 3  Effects of changing the permissible delay time $M$ on the optimal replenishment policy

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
<th>$k$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$Q_{total}$ (over $H$)</th>
<th>$TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12</td>
<td>0.514</td>
<td>156.4</td>
<td>304.2</td>
<td>508.4</td>
<td>24,798</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>12</td>
<td>0.533</td>
<td>161.3</td>
<td>304.2</td>
<td>509.0</td>
<td>24,670</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>12</td>
<td>0.546</td>
<td>166.1</td>
<td>304.2</td>
<td>509.5</td>
<td>24,566</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>12</td>
<td>0.562</td>
<td>170.8</td>
<td>304.2</td>
<td>510.1</td>
<td>24,484</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>12</td>
<td>0.577</td>
<td>175.5</td>
<td>304.2</td>
<td>510.7</td>
<td>24,426</td>
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</table>
Table 4: Effects of changing the stock-dependent consumption rate $\beta$ on the optimal replenishment policy

<table>
<thead>
<tr>
<th>$B$</th>
<th>$M$</th>
<th>$k$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
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<td>508.63</td>
<td>24,508</td>
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<td>166.1</td>
<td>304.2</td>
<td>509.53</td>
<td>24,566</td>
</tr>
<tr>
<td>0.08</td>
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<td>0.528</td>
<td>160.6</td>
<td>304.2</td>
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<td>24,648</td>
</tr>
<tr>
<td>0.10</td>
<td>12</td>
<td>0.517</td>
<td>157.1</td>
<td>304.2</td>
<td>511.45</td>
<td>24,700</td>
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</table>

Table 5: Effects of changing the deterioration rate $\theta$ on the optimal replenishment policy

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$N$</th>
<th>$k$</th>
<th>$T_1$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>11</td>
<td>0.611</td>
<td>202.8</td>
<td>331.8</td>
<td>550.1</td>
<td>24,244</td>
</tr>
<tr>
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<td>12</td>
<td>0.546</td>
<td>166.1</td>
<td>304.2</td>
<td>509.5</td>
<td>24,566</td>
</tr>
<tr>
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<td>13</td>
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<td>138.6</td>
<td>280.8</td>
<td>472.7</td>
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</tr>
<tr>
<td>0.30</td>
<td>13</td>
<td>0.448</td>
<td>125.7</td>
<td>280.8</td>
<td>474.5</td>
<td>25,011</td>
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</table>

Table 6: Effects of changing the net discount rate of inflation $R$ on the optimal replenishment policy

<table>
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<tr>
<th>$R$</th>
<th>$M$</th>
<th>$n$</th>
<th>$k$</th>
<th>$T_1$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$TC$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>12</td>
<td>0.559</td>
<td>169.9</td>
<td>304.2</td>
<td>509.80</td>
<td>29580.0</td>
</tr>
<tr>
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<td>12</td>
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<td>174.8</td>
<td>304.2</td>
<td>510.57</td>
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<tr>
<td>60</td>
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<td>511.16</td>
<td>29276.9</td>
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</tr>
<tr>
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<td>12</td>
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<td>184.3</td>
<td>304.2</td>
<td>511.76</td>
<td>29166.4</td>
<td></td>
</tr>
<tr>
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<td>12</td>
<td>0.621</td>
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<td>512.37</td>
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</tr>
<tr>
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<td>0.536</td>
<td>163.2</td>
<td>304.2</td>
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<td>27043.9</td>
</tr>
<tr>
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<tr>
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<td>24669.7</td>
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<td>24565.5</td>
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<td>175.5</td>
<td>304.2</td>
<td>510.65</td>
<td>24425.9</td>
<td></td>
</tr>
</tbody>
</table>

6.4 Observations

As the permissible delay time $M$ increases, $t_1$ increases whereas total cost decreases, because large delay period helps the retailer to prolong the payments to the supplier without penalty, which subsequently encourages the retailer to order more quantity.
2 When the stock-dependent consumption rate $\beta$ increases, which signifies that the slope of the demand function moves slightly upwards, resultant is higher order quantity which is very much on the expected lines, as our demand depends on the stock on display.

3 As the deterioration rate $\theta$ increases, then the total cost increases whereas the cycle length decreases. Since due to deterioration the value of the goods decreases, hence it is optimal for the retailer to order for a shorter period of time so as to manage the loss due to deterioration.

4 When the net discount rate of inflation $R$ is decreasing (i.e. inflation rate is increasing), then total cost and cycle length are increasing. The managerial implication of this result is quite obvious, since due to inflation the cost of goods increases, therefore, the retailer would like to order large quantity for a longer period of time.

7 Conclusions

In this article, an inventory model has been presented by incorporating some of the realistic phenomenon viz. deterioration, stock-dependent consumption rate, shortages, permissible delay in payments, inflation and time value of money; that are associated with inventory systems. Firstly, deterioration of goods over time is a natural feature. Secondly, it has been observed in supermarkets that the demand is influenced by the amount of stock displayed on the shelves. Thirdly, occurrence of shortages in inventory is inevitable in reality. Next, the permissible delay in payment has become a very powerful promotional tool to attract new customers and a good incentive policy for the buyers. Further, inflation suggests one to procure more that means more investment in inventory, which is highly correlated with the return on investment. Moreover, the paper considers the time value of money in developing the model, as it permits the proper recognition of financial implication of opportunity cost in inventory analysis. Thus incorporating the above features, an inventory model has been developed, and the convexity of the total cost function has been established.

An optimisation solution procedure is presented to find the retailer’s optimal replenishment policy. Finally, a numerical example has been solved to validate the results obtained, and sensitivity analysis has been carried out on different parameters of the system. The study shows that the total cost $TC$ increases when the consumption rate ($\beta$) and the deterioration rate ($\theta$) increases, where as the total cost $TC$ decreases when the net discount rate of inflation ($R$) and permissible delay time ($M$) increases. Thus, the model can be very useful in retail business and can provide insight in managerial decision-making.

Acknowledgements

The authors are thankful to anonymous referees for their constructive comments and valuable suggestions that have led to the improvement of the paper. The first author would like to acknowledge the support of Research Grant No.: Dean(R)/R&D/2009/487, provided by University of Delhi, Delhi, India for conducting this research.
References


Appendix

Theorem 1: $TC_1(n, k)$ is convex with respect to $k$.

The present value of total cost $TC_1(n, k)$ for the planning time horizon $H$ is given by

$$
TC_1(n, k) = \left\{ \begin{array}{l}
A + \frac{c_1 \alpha}{(\beta + \theta)(\beta + \theta + R)} \left[ e^{\beta kH/n} + e^{RkH/n} + e^{-BkH/n} - 1 \right] \\
+ \frac{c_1 \alpha}{(\beta + \theta)R} \left[ e^{-RkH/n} - 1 \right] E + A e^{-RH} \\
+ \left\{ \frac{c_2 \alpha}{R^2} \left[ R(kH/n - H/n) + e^{-R(kH/n - H/n)} - 1 \right] e^{-RkH/n} \right\} E \\
+ \frac{c \alpha E}{(\beta + \theta)R^2} \left[ e^{(\beta + \theta)kH/n} - 1 \right] + c \alpha (1 - k) FH/n \\
+ \left\{ \frac{c_1 \alpha}{(\beta + \theta)} \left[ e^{(\beta + \theta)kH/n - 1} + c \alpha (1 - k) FH/n \right] \\
+ \left\{ \frac{c_1 \alpha}{(\beta + \theta)} \left[ e^{(\beta + \theta)kH/n - 1} + c \alpha (1 - k) FH/n \right] \right\} E \\
\end{array} \right. \quad (A.1)
$$

Therefore, we have

$$
\frac{dTC_1(n, k)}{dk} = \frac{c_1 \alpha H}{(\beta + \theta)(\beta + \theta + R)n} \left[ (\beta + \theta)e^{(\beta + \theta)kH/n} + Re^{-RkH/n} \right] \\
- \frac{c_1 \alpha H}{(\beta + \theta)n} e^{-RkH/n} E + \frac{c_1 \alpha}{(\beta + \theta)R} \left[ RH/n - Re^{-R(kH/n - H/n)H/n} \right] E \\
+ \frac{c \alpha E}{(\beta + \theta)R} \left[ e^{(\beta + \theta)kH/n} - 1 \right] + c \alpha FH/n \\
+ \left\{ \frac{c_1 \alpha H}{(\beta + \theta)n} \left[ e^{(\beta + \theta)kH/n - 1} + c \alpha (1 - k) FH/n \right] \right\} E \\
- \left\{ \frac{c_1 \alpha H}{(\beta + \theta)n} \left[ e^{(\beta + \theta)kH/n} + Re^{-RkH/n} \right] \right\} E \\
\right. \quad (A.2)
$$
and
\[
\frac{d^2TC_1(n,k)}{dk^2} = \left\{ \frac{c_1\alpha}{(\beta + \theta)} \left[ \frac{(\beta + \theta)^2 e^{(\beta + \theta)kH/n} - R^2 e^{-RkH/n}}{(\beta + \theta + R)} + Re^{-RkH/n} \right] \right\} E \frac{H^2}{n^2}
\]
\[+ \frac{c_2\alpha FH^2}{n^2} e^{-R(kH/n - H/n)} + \frac{c_3\alpha E H^2}{n^2} (\beta + \theta) e^{(\beta + \theta)kH/n}
\]
\[+ \left\{ \frac{cl_\alpha}{(\beta + \theta)} \left[ \frac{(\beta + \theta)^2 e^{(\beta + \theta)kH/n - M} e^{-RM} - R^2 e^{-RkH/n}}{(\beta + \theta + R)} + Re^{-RkH/n} \right] \right\}
\]
(A.3)
\[E \frac{H^2}{n^2} - \left\{ \frac{pl_\alpha}{(\beta + \theta)} \left[ \frac{(\beta + \theta)^2 e^{(\beta + \theta)kH/n} - R^2 e^{-RkH/n}}{(\beta + \theta + R)} + Re^{-RkH/n} \right] \right\}
\]
\[E \frac{H^2}{n^2} \]
where \( E = \left( \frac{1 - e^{-R\theta}}{1 - e^{-R\theta/n}} \right) \) and \( F = \left( \frac{1 - e^{-R\theta}}{e^{R\theta/n} - 1} \right) \)

Now \( d^2TC_1(n,k)/dt^2 > 0 \) if
\[
\left\{ \frac{c_1\alpha}{(\beta + \theta)} \left[ \frac{(\beta + \theta)^2 e^{(\beta + \theta)kH/n} - R^2 e^{-RkH/n}}{(\beta + \theta + R)} + Re^{-RkH/n} \right] \right\} E \frac{H^2}{n^2}
\]
\[\left\{ \frac{pl_\alpha}{(\beta + \theta)} \left[ \frac{(\beta + \theta)^2 e^{(\beta + \theta)kH/n} - R^2 e^{-RkH/n}}{(\beta + \theta + R)} + Re^{-RkH/n} \right] \right\} E \frac{H^2}{n^2} \geq 0
\]
and \( (\beta + \theta)^2 e^{(\beta + \theta)(kH/n - M)} e^{-RM} - R^2 e^{-RkH/n} \geq 0 \)
i.e. if \( c_1 \geq pl_\alpha \)
and if \( (\beta + \theta) \geq R \), since \( (kH/n - M) \geq 0 \)

Thus, for a fixed value of \( n \), \( TC_1(n,k) \) is convex on \( k > 0 \), provided the following conditions hold
\[ c_1 \geq pl_\alpha \]
and \( (\beta + \theta) \geq R \).

Theorem 2: \( TC_2(n,k) \) is convex with respect to \( k \).
Supply chain model

The present value of total cost $TC_2(n, k)$ for the planning time horizon $H$ is given by

\[
TC_2(n, k) = \left\{ A + \frac{c_1\alpha}{(\beta + \theta)(\beta + \theta + R)} e^{(\beta + \theta)kH/n} - e^{-R^2H/n} \right\} + \frac{c_1\alpha}{(\beta + \theta)R} e^{-R^2H/n} - 1 \right\} E + Ae^{-RH} + \frac{C_2\alpha R}{R^2} \left[ R(kH/n - H/n) + e^{-R(kH/n - H/n)} - 1 \right] e^{-RH/n} F + \frac{c\alpha E}{\beta + \theta} \left( e^{(\beta + \theta)kH/n} - 1 \right) + c\alpha(1 - k)FH/n - \frac{pl\alpha}{(\beta + \theta)} \left[ (\beta + \theta)e^{(\beta + \theta)kH/n} + Re^{-R^2H/n} \right] E - \frac{pl\alpha}{(\beta + \theta)} \left[ (\beta + \theta) e^{(\beta + \theta)kH/n} - e^{-R^2H/n} \right] E - \frac{pl\alpha}{(\beta + \theta)} \left[ M - kH/n \right] e^{-RM} \left[ (\beta + \theta) e^{(\beta + \theta)kH/n} - 1 \right] E \right) \] (A.4)

\[
\frac{dTC_2(n, k)}{dk} = \left\{ \frac{c_1\alpha H}{(\beta + \theta)(\beta + \theta + R)n} \left[ (\beta + \theta)e^{(\beta + \theta)kH/n} + Re^{-R^2H/n} \right] \right\} - \frac{c_1\alpha H}{(\beta + \theta)n} e^{-R^2H/n} \right\} E + \frac{C_2\alpha R}{R^2} \left[ RH/n - Re^{-R(kH/n - H/n)} H/n \right] F + \frac{c\alpha E e^{(\beta + \theta)kH/n} H/n - c\alpha FH/n - \frac{pl\alpha}{(\beta + \theta)} \left[ (\beta + \theta)e^{(\beta + \theta)kH/n} + Re^{-R^2H/n} \right] E - \frac{pl\alpha}{(\beta + \theta)} \left[ (\beta + \theta) e^{(\beta + \theta)kH/n} - e^{-R^2H/n} \right] E - \frac{pl\alpha}{(\beta + \theta)} \left[ M - kH/n \right] e^{-RM} \left[ (\beta + \theta) e^{(\beta + \theta)kH/n} - 1 \right] E \right) \right) \] (A.5)

and

\[
\frac{d^2TC_2(n, k)}{dk^2} = \left\{ \frac{c_1\alpha}{(\beta + \theta)} \left[ (\beta + \theta)^2 e^{(\beta + \theta)kH/n} - R^2 e^{-R^2H/n} \right] + Re^{-R^2H/n} \right\} E \frac{H^2}{n^2} + c_2\alpha e^{-R(kH/n - H/n)} F \frac{H^2}{n^2} + c\alpha(\beta + \theta) E e^{(\beta + \theta)kH/n} \frac{H^2}{n^2} - \frac{pl\alpha}{(\beta + \theta)} \left[ (\beta + \theta)^2 e^{(\beta + \theta)kH/n} - R^2 e^{-R^2H/n} \right] + Re^{-R^2H/n} \right\} E \frac{H^2}{n^2} + \frac{pl\alpha}{(\beta + \theta)} 2e^{-RM} \left[ e^{(\beta + \theta)kH/n} - 1 \right] E \frac{H^2}{n^2} - \left[ pl\alpha \left[ M - kH/n \right] e^{-RM} \left[ e^{(\beta + \theta)kH/n} - 1 \right] E \frac{H^2}{n^2} \right) \right) \] (A.6)
where \( E = \left( \frac{1 - e^{-RH}}{1 - e^{-RH/n}} \right) \) and \( F = \left( \frac{1 - e^{-RH}}{e^{RH/n} - 1} \right) \).

Now \( d^2TC_2(n, k)/dt^2 > 0 \) if

\[
\left\{ \frac{c_i \alpha}{(\beta + \theta)} \left[ (\beta + \theta)^2 e^{(\beta + \theta)\delta H/n} - R^2 e^{-RkH/n} + Re^{-RkH/n} \right] \right\} E \frac{H^2}{n^2}
\]

\[
- \left\{ \frac{p L \alpha}{(\beta + \theta)} \left[ (\beta + \theta)^2 e^{(\beta + \theta)\delta H/n} - R^2 e^{-RkH/n} + Re^{-RkH/n} \right] \right\} E \frac{H^2}{n^2} \geq 0
\]

i.e. if \( c_i \geq p L \alpha \)

and \( \left\{ \frac{p L \alpha}{(\beta + \theta)} 2e^{-\delta M} \left[ e^{(\beta + \theta)\delta H/n} - 1 \right] \right\} E \frac{H^2}{n^2} \)

\[
- \left\{ \frac{p L \alpha}{(\beta + \theta)} \left( M - \frac{kH}{n} \right) e^{-\delta M} e^{(\beta + \theta)\delta H/n} \right\} E \frac{H^2}{n^2} \geq 0
\]

or \( \frac{p L \alpha}{(\beta + \theta)} 2e^{-\delta M} e^{(\beta + \theta)\delta H/n} \geq p L \alpha Me^{-\delta M} e^{(\beta + \theta)\delta H/n} \)

and \( p L \alpha \left( M - \frac{kH}{n} \right) e^{-\delta M} e^{(\beta + \theta)\delta H/n} \geq \frac{p L \alpha}{(\beta + \theta)} 2e^{-\delta M} \)

i.e. \( \frac{2}{(\beta + \theta)} \geq M \)

and \( \frac{kH}{n} e^{(\beta + \theta)\delta H/n} \geq \frac{2}{(\beta + \theta)} \)

\[
= \frac{kH}{n} e^{(\beta + \theta)\delta H/n} \geq M
\]

Thus, for a fixed value of \( n \), \( TC_2(n, k) \) is convex on \( k > 0 \), provided the following conditions hold

\( c_i \geq p L \alpha \)

and \( \frac{kH}{n} e^{(\beta + \theta)\delta H/n} \geq M \).