Matrix Partitions of Perfect Graphs

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Abstract

Given a symmetric \(m\) by \(m\) matrix \(M\) over \(0, 1, *\), the \(M\)-partition problem asks whether or not an input graph \(G\) can be partitioned into \(m\) parts corresponding to the rows (and columns) of \(M\) so that two distinct vertices from parts \(i\) and \(j\) (possibly with \(i = j\)) are non-adjacent if \(M(i,j) = 0\), and adjacent if \(M(i,j) = 1\). These matrix partition problems generalize graph colourings and homomorphisms, and arise frequently in the study of perfect graphs; example problems include split graphs, clique and skew cutsets, homogeneous sets, and joins.

In this paper we study \(M\)-partitions restricted to perfect graphs, and focus on a natural class of simple matrices \(M\) with \(0, 1\) diagonal. For this class of matrices \(M\), all \(M\)-partition problems restricted to perfect graphs can be solved in polynomial time, and characterized by a finite set of forbidden induced subgraphs. We identify two distinct patterns of such matrices: For the first pattern we are able to characterize partitionability by small forbidden subgraphs. For the second pattern, we can only guarantee a finite set of forbidden subgraphs (which can be quite large). We also discuss matrices for which partitionability of perfect graphs can not be characterized by finite sets of forbidden subgraphs.
1 Introduction

The \textit{M}-partition problem was introduced in [12]: Let $M$ be a fixed symmetric $m \times m$ matrix with entries $M_{i,j} \in \{0, 1, *\}$. An $M$-partition of a graph $G$ is a partition of the vertices of $G$ into $m$ parts, indexed by the rows (and columns) of the matrix $M$, such that for distinct vertices $x$ and $y$ of the graph $G$, placed in parts $i$ and $j$ (possibly with $i = j$) respectively, we have the following:

- if $M(i, j) = 0$, then $xy$ is not an edge of $G$;
- if $M(i, j) = 1$, then $xy$ is an edge of $G$.

(If $M(i, j) = \ast$, then $xy$ may or may not be an edge in $G$.)

An instance of the $M$-partition problem is a graph $G$, and a solution to the instance is an $M$-partition of $G$.

Suppose $H$ is a graph with $m$ vertices and $M$ is obtained from the adjacency matrix of $H$ by replacing each 1 by $\ast$. Then each homomorphism (edge-preserving vertex mapping) $f$ of $G$ to $H$ corresponds to an $M$-partition of $G$, where the parts are $f^{-1}(h), h \in V(H)$. In particular, when $H = K_m$, the matrix $M$ is the matrix with all diagonal entries 0 and all off-diagonal entries $\ast$, and an $M$-partition of $G$ is simply an $m$-colouring of $G$.

With this in mind, we may define a \textit{trigraph} [15] $H$ to consists of a set of vertices, any two of which may either form a non-edge, a weak edge, or a strong edge. The \textit{adjacency matrix} of a trigraph $H$ with $m$ vertices is the symmetric $m \times m$ matrix $M$, with rows (and columns) indexed by the vertices of $H$, which has $M(i, j) = 0$ if $ij$ is not an edge, $M(i, j) = \ast$ if $ij$ is a weak edge, and $M(i, j) = 1$ if $ij$ is a strong edge. A homomorphism of a graph $G$ to a trigraph $H$ is a mapping $f$ of the vertices of $G$ to the vertices of $H$ such that the partition formed by parts $f^{-1}(h)$, over all vertices $h$ of $H$, is an $M$-partition of $G$. This point of view is further explored in [15].

Matrix partitions not only generalize colourings and homomorphisms, but also unify many partition problems arising in the study of perfect graphs. Often these problems are not stated in terms of partitions, but are in fact equivalent to partition problems. For instance, it is evident that $G$ is a \textit{split graph} (admits a partition into a stable set and a clique [16]), if and only if it admits an $M$-partition where $M$ is the matrix

$$
\begin{pmatrix}
0 & * \\
* & 1
\end{pmatrix}
$$
Less obviously, a graph $G$ has a clique cutset [20, 23], if it admits an $M$-partition, into non-empty parts, where $M$ is the matrix

$$
\begin{pmatrix}
1 & * & * \\
* & * & 0 \\
* & 0 & *
\end{pmatrix}
$$

A similar approach allows us to model by $M$-partitions problems such as having an independent cutset [21], a skew cutset [7, 8], a homogeneous set [18], or being a join of various kinds [6]. These connections are explored in more detail in [12], where it is in particular explained how to model restrictions on the size of the parts (for instance requiring parts to be non-empty) by introducing lists.

In [12] we gave polynomial time algorithms for many list $M$-partition problems, and quasi-polynomial ($n^{O(\log n)}$) time algorithms for certain others. In [10] we have shown that all list $M$-partition problems are solvable in quasi-polynomial time, or are NP-complete. Many of our quasi-polynomial time algorithms from [12] were improved to polynomial time algorithms in [5, 8], but it is not known whether all the quasi-polynomial time list $M$-partition problems are polynomial time solvable (even for matrices of size four [5]).

In this paper, we consider the restriction of $M$-partition problems to perfect graphs. A graph $G$ is perfect [1] if it and all of its induced subgraphs have chromatic number equal to its maximum clique size. This also implies that $G$ and all of its induced subgraphs have the chromatic number of their complement equal to maximum size of their independent set.

We shall focus on matrices $M$ without *’s on the diagonal. Thus we may consider $M$ to consist of two diagonal block matrices $A, B$, where $A$ is a $k$ by $k$ matrix with zero diagonal, $B$ is an $\ell$ by $\ell$ matrix with a diagonal of ones, and a $k$ by $\ell$ off-diagonal matrix $C$ (and its transpose). (Note that $m = k + \ell$.)

We have shown in [14] that there exist matrices $M$ of this type which yield NP-complete $M$-partition problems, even when restricted to split (and hence also to perfect) graphs. The matrices constructed in [14] have all off-diagonal entries of $A$ equal to zero, all off-diagonal entries of $B$ equal to *, and have and all entries of $C$ different from 1. For these matrices $M$ we do not expect polynomial time algorithms, and expect that there exist infinitely many minimal forbidden induced subgraphs (something that can sometimes be directly proved.) On the other hand, it follows from [12, 3]
that when $M$ is the matrix

$$
\begin{pmatrix}
0 & 1 & \ast \\
1 & 0 & \ast \\
\ast & \ast & 0
\end{pmatrix}
$$

the $M$-partition problem can be solved in polynomial time, yet there are infinitely many minimal forbidden induced perfect subgraphs: Let $G_t$ be the path with vertices $0, 1, \ldots, 2t$ with the additional edges $0i$ for $i = 2, 3, \ldots, 2t - 1$. Then each $G_t$ is chordal (and hence perfect), and each is easily verified to be a minimal forbidden subgraph.

These examples suggest that we restrict our attention to matrices $M$ in which all off-diagonal entries of $A$ are the same, say $a$, all off-diagonal entries of $B$ are the same, say $b$, and all entries of $C$ are the same, say $c$. Note that we may assume $a \neq 0$ and $b \neq 1$, otherwise we may replace $M$ by a matrix with $k = 1$ or $\ell = 1$ respectively. We shall also consider somewhat more general matrices, where the off-diagonal entries of $A$ or $B$ (or all entries of $C$) are restricted to be either all $\ast$, or all not $\ast$.

In all these situations, we give polynomial time $M$-partition algorithms and forbidden subgraph characterizations. Nevertheless, there is a distinction we observe: If $c \neq \ast$, we give characterizations with forbidden subgraphs all having at most $(k + 1)(\ell + 1)$ vertices. If $c = \ast$, we can only prove an exponential upper bound on the size of the minimal forbidden subgraphs.

Let $R_{k,\ell}$ denote the disjoint union of $\ell + 1$ cliques of size $k + 1$. The result of [17] states that when $a = b = c = \ast$, i.e., for partitions into $k$ cliques and $\ell$ independent sets (first investigated in [4]), a chordal graph is non-partitionable if and only if it contains an induced $R_{k,\ell}$. Thus we have situations in which forbidden subgraphs of size $(k + 1)(\ell + 1)$ are necessary. It is interesting to observe that for perfect graphs, in the case $a = b = c = \ast$ we must admit minimal forbidden subgraphs with more than $(k + 1)(\ell + 1)$ vertices: For instance, the (bipartite, hence perfect) graph $H$, obtained from the three-dimensional cube by deleting one vertex, has seven vertices and is a minimal forbidden subgraph for partitions into two cliques and an independent set. (Here $k = 2$, $\ell = 1$, and $a = b = c = \ast$.) In fact, if $c = \ast$, in the particular case when $k = 1$ and $b = 0$, the largest minimal forbidden perfect subgraphs have on the order of $\ell^2$ vertices [13].

In [13] we shall consider similar problems in the context of chordal graphs. The bounds are generally lower, and the algorithms are more efficient.
2 Small Obstructions

We phrase our results in terms of minimal obstructions. Given a matrix $M$, a graph $G$ is a \textit{minimal obstruction} for $M$-partitionability, if $G$ has no $M$-partition, but each induced subgraph of $G$ has an $M$-partition. If we show, for a matrix $M$, that all minimal obstructions that are perfect have at most a certain number $f(M)$ of vertices, then $M$-partitionability of perfect graphs can be characterized by a finite set of forbidden induced subgraphs.

Recall that we consider matrices $M$ with diagonal blocks $A$ and $B$ with all diagonal entries of $A$ equal to 0, all off-diagonal entries of $A$ equal to $a$, all diagonal entries of $B$ equal to 1, all off-diagonal entries of $B$ equal to $b$, and all other entries of $M$ equal to $c$. Recall also that $A$ is a $k$ by $k$ matrix and $B$ is an $\ell$ by $\ell$ matrix, and that we may assume that $a \neq 0$ and $b \neq 1$ because these cases are covered by $k = 1$ and $\ell = 1$.

In other words, we shall discuss partitions of input graphs $G$ into $k$ independent sets $A_1, A_2, \ldots, A_k$ and $\ell$ cliques $B_1, B_2, \ldots, B_\ell$, with the following restrictions:

1. If $a = 1$ the $k$ independent sets $A_1, A_2, \ldots, A_k$ induce a complete $k$-partite subgraph.
2. If $b = 0$ the $\ell$ cliques $B_1, B_2, \ldots, B_\ell$ are independent (have no edges joining them).
3. If $c = 0$ there is no edge joining a vertex in any $A_i$ to a vertex in any $B_j$.
4. If $c = 1$ there are edges joining each vertex of every $A_i$ to each vertex of every $B_j$.

For each such partition of a graph $G$ we denote by $G_A$ the subgraph induced by the union of the $k$ independent sets $A_i$, and by $G_B$ the subgraph induced by the union of the $\ell$ cliques $B_j$. Note that $G_A$ is $k$-colourable, and that the complement of $G_B$ is $\ell$-colourable.

When $a = b = c = *$, there are no restrictions, and we are talking about partitions into $k$ independent sets and $\ell$ cliques; this is the problem first studied in [4], and solved for chordal graphs in [17], where it is shown that for chordal graphs there is a unique minimal obstruction $R(k, \ell)$ with $(k + 1)(\ell + 1)$ vertices. As we have observed earlier, in this case there are minimal obstructions which are perfect and have more than $(k + 1)(\ell + 1)$ vertices. However, when $c \neq *$, we can prove that no minimal obstruction which is perfect has more than $(k + 1)(\ell + 1)$ vertices:

\[ \text{5} \]
**Theorem 2.1** If \( c \neq \ast \), then every perfect graph which is a minimal obstruction has at most \((k + 1)(\ell + 1)\) vertices.

We have already remarked that this bound does not apply when \( c = \ast \), not even for inputs \( G \) that are trees, cf. the last section.

A careful reading of the proofs below will show that the bound applies more generally. Instead of perfectness of the minimal obstruction \( G \), we only need to assume, if \( a = \ast \), that every induced subgraph of \( G \) which does not have a clique on \( k + 1 \) vertices is \( k \)-colourable, and, if \( b = \ast \), that every induced subgraph of \( G \) which does not have an independent set on \( \ell + 1 \) vertices has a complement which is \( \ell \)-colourable. (Thus if neither \( a \) nor \( b \) is \( \ast \), no restriction is needed.)

From the proof of the theorem, we can extract polynomial time algorithms for the corresponding \( M \)-partition problems on perfect graphs. All our algorithms are certifying algorithms, in that they identify either a partition or an obstruction, in the stated time. Our algorithms actually proceed by detecting the existence of obstructions, along the lines of the proofs. The most computationally expensive part of the algorithms consists of finding cliques of size \( k + 1 \) in the cases with \( a = \ast \), and independent sets of size \( \ell + 1 \) in the cases with \( b = \ast \). Thus if \( c \neq \ast \), the \( M \)-partition problem for perfect graphs can be solved in time \( n^{\max(k, \ell)+O(1)} \), where the dependence of the exponent on \( k \) is only needed if \( a = \ast \), and the dependence on \( \ell \) is only needed if \( b = \ast \). The complexity can be improved to \( O(m + n) \) for fixed \( k, \ell \) in the case of chordal graphs or complements of chordal graphs, since for these graphs cliques and independent sets of a given size

Recall that \( a \in \{1, \ast \} \), \( b \in \{0, \ast \} \). We may also assume that \( c = 0 \); the case \( c = 1 \) is equivalent to the matrix \( M' \) obtained by exchanging 0 and 1 everywhere in \( M \), by complementation of the input graph. There are thus four cases, which are covered by the following four Lemmas.

**Lemma 2.2** Assume that \( a = b = \ast \) and \( c = 0 \).

Every perfect graph which is a minimal obstruction is either isomorphic to \( R_{k, \ell} \) or has strictly fewer than \((k + 1)(\ell + 1)\) vertices.

The \( M \)-partition problem for perfect graphs can in this case be solved by an algorithm of time complexity \( n^{\max(k, \ell)+O(1)} \) by identifying obstructions.

**Proof.** Let \( G \) be a perfect graph. We want to partition \( G \) into a \( k \)-colourable induced subgraph \( G_A \) and an induced subgraph \( G_B \) whose complement is \( \ell \)-colourable, with no edges joining \( G_A \) and \( G_B \). Thus each component of \( G \) must be included entirely either in \( G_A \) or in \( G_B \). All components
of $G$ that contain a clique of size $k + 1$ must be included in $G_B$. Let $D$
be the union of all components $C$ of $G$ that contain a clique of size $k + 1$. Then $G$
is partitionable if and only if $D$ does not have an independent set of size $\ell + 1$. Indeed, if $D$
has $\ell + 1$ independent vertices that its components cannot all be contained in $G_B$. On the other hand, if $D$
does not have $\ell + 1$ independent vertices then its complement can be $\ell$-coloured, since $G$ is perfect. The subgraph $G - D$
can be $k$-coloured for the same reason; thus $G$ is partitionable.

Consider now a perfect minimal obstruction $G$, and its subgraph $D$
defined as above to be the union of all components $C$ of $G$ which contain a clique of size $k + 1$. Let $\ell_C$
denote the independence number of $C$. Since $G$ is minimal, we know that $\sum_C \ell_C = \ell + 1$. We claim that each component $C$
of $D$ has at most $2\ell_C + k - 1$ vertices, and hence $G$ has at most

$$\sum_C (2\ell_C + k - 1) = 2(\ell + 1) + \sum_C (k - 1) \leq 2(\ell + 1) + (k - 1)(\ell + 1) = (k + 1)(\ell + 1),$$

with equality only in the case where all $\ell_C = 1$, corresponding to $G = R_{k,\ell}$.

To prove the claim, let $K$ be a clique with $k + 1$ vertices in $C$, and let $L$
be an independent set with $\ell_C$ vertices in $C$. Note that $K$ and $L$
have at most one vertex in common. The block-cutpoint tree $T$ of $C$ [22] has
as its vertices all the cut-vertices and all the blocks (maximal subgraphs
without cut-vertices) of $C$, with each cut-vertex adjacent to all the blocks
that contain it. We view $T$ as being rooted at the block containing $K$, and
consider, for each vertex $x$ of $C$, the minimum distance $d(x)$ in $T$ of any
block containing $x$ to the root. (A cut-vertex belongs to several blocks and
its distance is determined in its parent block; all other vertices belong to
unique blocks.) It is clear that each leaf of $T$ must be a block; we call such
a block a leaf block of $T$.

Let $S$ denote the set of all the remaining vertices of $C$, i.e., $S = C - K - L$.
Observe that each $x \in S$ is a cut-vertex of $C$, since otherwise $G - x$ would
also not be $M$-partitionable. (The only role of the component $C$ is to contain
a clique of $k + 1$ vertices, and an independent set of $\ell_C$ vertices.) Thus the
tree $T$ contains all vertices of $S$. Moreover, all the leaf blocks of $T$ contain
a vertex of $L$, for the same reason. Some vertices of $L$ may also be cut-
vertices of $T$. We may assume that $L$ has been chosen (from all independent
sets of $C$ with $\ell_C$ vertices) so that $\sum_{x \in L} d(x)$ is minimized. This means, in
particular, that $L$ contains a vertex of $K$, if possible, and that we may not
replace in $L$ any vertex $x$ by a vertex in its parent block.
We shall now define an injection \( f \) of \( S \) to \( L \). We shall arrange \( f \) to take each \( x \in S \) to an \( f(x) \in L \) belonging to a block \( B \) which is a descendant of \( x \) in \( T \).

Suppose first that all children of \( x \in S \) in \( T \) are leaf blocks. Then each of these blocks \( B \) must contain a vertex \( y \in L \), and we may define \( f(x) = y \) for any such vertex \( y \).

In general, for a cut-vertex \( x \) in \( T \), and a block child \( B \) of \( x \), we denote by \( C(B) \) the union of all the blocks that are descendants of \( B \) in \( T \) (including \( B \) itself). We also denote by \( C(x) \) the union of all \( C(B) \) over all children \( B \) of \( x \) in \( T \).

We shall recursively prove the following statement for a cut-vertex \( x \) in \( T \):

There exists an injective mapping \( f_x : S \cap C(x) \to L \cap C(x) \) which takes any \( y \in S \cap C(x) \) to \( f_x(y) \in C(y) - \{y\} \); moreover, if \( x \in S \) and \( B \) is any child block of \( x \) in \( T \) then \( f_x \) can be chosen so that \( f_x(x) \in C(B) \), and if \( B \) contains vertices of \( L \), then \( f_x \) can be chosen so that \( f_x(x) \) is any vertex of \( L \cap B \).

As noted above, this statement holds if all child blocks of \( x \) are leaves.

Thus let \( x \) be a cut-vertex in \( T \) and suppose that the statement holds for all other cut-vertices \( y \) in \( T \) which are descendants of \( x \); in particular, consider the injective mappings \( f_y \) for all \( y \) which are grandchildren of \( x \). Since the domains of these mappings \( f_y \) are disjoint, we may define \( f_x \) by taking the union of all these mappings \( f_y \). This is sufficient if \( x \in L \); if \( x \in S \), we also have to find an image \( f_x(x) \).

Thus suppose that \( x \in S \), and let \( B \) be a child block of \( x \). If \( B \) contains vertices of \( L \), then any such vertex is free to be used as \( f_x(x) \), since it was not used by any of the injections \( f_y \) (as \( B \) is above \( y \) in \( T \)). Otherwise, \( B \) is not a leaf block, and any child \( y \) of \( B \) in \( T \), belongs to \( S \) (since it is a vertex of \( B \)). If \( y \) has two distinct child blocks \( D, D' \), then by our assumption there are two injections \( f_y, f'_y \) of \( S \cap C(y) \) to \( L \cap C(y) \), with \( f_y(y) = b \in C(D), f'_y(y) = b' \in C(D') \). We define an injection \( f_x \) by combining \( f_y \) with \( f(x) = b' \) and all injections \( f_z \) for the other grandchildren \( z \neq y \) of \( x \). On the other hand, if \( y \) has only one child block \( D \) but \( D \) contains at least two vertices of \( L \), then at most one of these can be used as \( f_y(y) \) and hence one is free to be used as \( f_x(x) \). The remaining case has \( y \) with only one child block \( D \), where \( D \) contains at most one vertex \( d \in L \). In \( C \), the vertex \( y \) lies only in the blocks \( B \) and \( D \), and in \( B \cup D \) only \( d \) belongs to \( L \). This contradicts the minimality of \( L \), since \( d \) could be replaced by \( y \).

A similar argument applies if \( D \cap L = \emptyset \), by letting \( d \) be any vertex of \( L \) in \( C(D) \). (Recall that each leaf block must contain an element of \( L \).)
This completes the definition of an injection \( f : S \to L \). It follows that \(|S| \leq |L|\) and so \( C \) has at most \(|K| + 2|L| = 2\ell_C + k + 1\) vertices. To lower the bound to \(2\ell_C + k - 1\) we proceed as follows:

Let \( B \) be the root block of \( T \), containing the clique \( K \). If \( B \) contains at least two elements of \( L \), then these two elements are not in \( f(S) \), and hence \(|S| \leq |L| - 2\), proving that \( C \) has at most \(2\ell_C + k - 1\) vertices. If \( K \) contains a vertex in \( L \) then \(|S| \leq |L| - 1\) and at the same time \( K \) only contributes \( k \) vertices not counted in \( S \cup L \); thus we again have \( C \) with at most \(2\ell_C + k - 1\) vertices.

When \( B \) contains exactly one vertex \( t \) in \( L \), then we may assume that this vertex is not adjacent to all the vertices of \( K \), otherwise we could replace \( K \) by another clique of size \( k + 1 \) which contains \( t \), and the previous case would apply. We may thus choose \( x \) in \( K \) not adjacent to \( t \), and are left with the case where \( x \) is not adjacent to any vertex of \( L \) in the block \( B \).

We may now argue for \( x \) as we did before. If \( x \) has at least two child blocks, or if \( x \) has exactly one child block but the block has two vertices of \( L \), then there exist two vertices \( t \) and \( t' \) in \( L \) that could have been chosen as \( f(x) \), and so there are at most \(|L| - 2\) vertices in \( S \), giving again the bound \(|L| + (|L| - 2) + (k + 1) = 2\ell_C + k - 1\). If \( x \) has exactly one child block \( D \) and \( D \) has at most one vertex in \( L \), then some vertex in \( L \) from \( C(D) \) could have been replaced by \( x \), contrary to our choice of \( L \). In the last case \( x \) is not a cut-vertex of \( C \). Since \( x \) has no neighbour in \( L \) in \( B \), so any vertex in \( L \) could have been replaced by \( x \), contrary to our assumption on \( L \). \( \square \)

**Lemma 2.3** If \( a = * \) and \( b = c = 0 \), then every perfect graph which is a minimal obstruction is either \( R_{k,\ell} \) or has at most \( k + 2 \) vertices if \( k \geq 1 \) (or 3 vertices if \( k = 0 \)).

The \( M \)-partition problem for perfect graphs can be solved in this case in time \( n^{k+O(1)} \) by identifying obstructions.

**Proof.** In this case, a graph is partitionable only if every component is \( k \)-colourable or a clique. If there are more than \( \ell \) components that are cliques and are not \( k \)-colourable, then we have the minimal obstruction \( R_{k,\ell} \). Otherwise, there is a component \( C \) that is neither a clique, nor \( k \)-colourable. For perfect graphs that means that \( C \) contains a clique of size \( k + 1 \) and is not complete. If \( k \geq 1 \), consider a maximum clique \( K \) in \( C \), with at least \( k + 1 \) vertices. There is a vertex \( x \) adjacent to some but not all vertices of \( K \); let \( y \) be a neighbour and \( z \) a non-neighbour of \( x \) in \( K \). Choosing \( x, y, z \), and \( k - 1 \) other vertices in \( K \) gives an obstruction with \( k + 2 \) vertices. Thus all
other minimal obstructions which are perfect consist of a clique with \( k + 1 \) vertices together with an additional vertex adjacent to some but not all of its vertices. If \( k = 0 \) the only additional minimal obstruction is a path with two edges.

In the last section, we provide a finite bound for perfect graphs that are minimal obstructions even for the more general matrices \( M \) with \( a = *, c = 0 \), where \( B \) has arbitrary off-diagonal entries, all different from *.

**Lemma 2.4** If \( a = 1, b = *, \) and \( c = 0 \), then every perfect graph which is a minimal obstruction has at most \((k + 1)(\ell + 1)\) vertices.

The \( M \)-partition problem for perfect graphs can again be solved in time \( n^{\ell + O(1)} \) by identifying obstructions.

**Proof.** Here the goal is to partition the input graph \( G \) into a complete \( k \)-partite subgraph \( G_A \), and a subgraph \( G_B \) whose complement is \( \ell \)-colourable. At most one non-trivial component of \( G \) can be placed in \( G_A \), and that component must be complete \( k \)-partite. Alternately, all single-vertex components of \( G \) can be placed in \( G_A \). In either case, all other components of \( G \) must be placed in \( G_B \), and hence contain altogether at most \( \ell \) independent vertices.

Consider now a perfect graph \( G \) that is a minimal obstruction. We shall bound the size of \( G \) by arguing that each component \( C \) of \( G \) is only required to contribute enough independent vertices.

Indeed, the fact that \( G \) is an obstruction means that

- the graph \( G' \), obtained from \( G \) by deleting all single-vertex components, has independence number greater than \( \ell \), and
- the graph \( G'' \), obtained from \( G \) by deleting any complete \( k \)-partite component, has independence number greater than \( \ell \).

It is now clear that the fact that \( G \) is a minimal obstruction implies that both \( G' \) and \( G'' \) have independence number \( \ell + 1 \). Moreover, those components of \( G \) that can be placed in \( G_A \) may be assumed to be stars, or single-vertex components, since these contribute the same number of independent vertices as more general complete multipartite graphs.

We consider next those components \( C \) of \( G \) that cannot be placed in \( G_A \). The complement of such a \( C \) contains a minimal obstruction to partitionability into \( k \) cliques, identified in the previous proof as the path with two edges, or the graph consisting of \( k + 1 \) isolated vertices. Thus \( C \) contains a
clique with \( k + 1 \) vertices, or the three-vertex graph \( x, y, z \) with the single edge \( xy \). In the latter case, consider a shortest path in \( C \) from \( z \) to \( xy \).

It is clear that \( C \) must contain (up to isomorphism) one of the following two induced subgraphs on four vertices \( x, y, z, t \): The graph \( R_1 \) has edges \( zt, tx, xy \), or the graph \( R_2 \) has edges \( zt, tx, ty, xy \).

Let \( U_1 \) denote the union of all components of \( G \) containing a clique with \( k + 1 \) vertices, and assume that the independence number of \( U_1 \) is \( u_1 \). Then \( U_1 \) is a perfect graph which cannot be partitioned into a \( k \)-colourable graph and the complement of a \((u_1 - 1)\)-colourable graph. By Lemma 2.2, and the minimality of \( U_1 \), we know that \( U_1 \) has at most \((k + 1)u_1 \) vertices.

Let \( U_2 \) denote the union of all components of \( G \) containing induced \( R_1 \) or \( R_2 \), and assume that the independence number of \( U_2 \) is \( u_2 \). By an argument similar to the proof of Lemma 2.2, we can conclude that \( U_2 \) has at most \( 3u_2 \) vertices: There are two independent vertices amongst \( x, y, z, t \), and for each \( \ell_C \geq 3 \) the proof shows at most \( 2\ell_C + 1 < 3\ell_C \) vertices are needed.

Indeed, in that case, we have several possibilities. If no vertex in the \( R_i \) is chosen, then at least three independent vertices in \( L \) were not used as images \( f_x \) of the injective mapping, because if there are only two such independent vertices adjacent to vertices in \( R_i \), then two independent vertices in \( R_i \) could have been included in \( L \) instead: this gives at most \((2\ell_C - 3) + 4 = 2\ell_C + 1 \) vertices. If a single vertex in the \( R_i \) is chosen, then at least two independent vertices in \( L \) outside \( R_i \) were not used as images \( f_x \) of the injective mapping, because if there is only one such independent vertex adjacent to vertices in \( R_i \), then again two independent vertices in \( R_i \) could have been included in \( L \) instead: This gives at most \((2\ell_C - 3) + 3 = 2\ell_C \) vertices. If two vertices in the \( R_i \) are chosen then we have at most \((2\ell_C - 2) + 2 = 2\ell_C \) vertices. We do not need to have just a single independent vertex from a component having \( R_1 \) or \( R_2 \) because in that case just the 3 vertices \( x, y, z \) will give the needed independent vertex.

Since the cases \( k = 0, 1 \) are included in Lemma 2.2, we now assume that \( k \geq 2 \). Therefore \( U_2 \) has at most \((k + 1)u_2 \) vertices.

The graphs \( U_1 \) and \( U_2 \) contain all the components of \( G \) that cannot be placed in \( G_A \). Recall that the components of \( G \) that can be placed in \( G_A \) are either single-vertex components or stars. Let \( U_3 \) denote the union of all single-vertex components of \( G \), and let \( U_4 \) denote the union of all star components of \( G \). Thus \( G \) is the union of \( U_1, U_2, U_3, \) and \( U_4 \). We again denote by \( u_i, i = 3, 4, \) the independence number of \( U_i \). Note that \( u_3 \) is the number of single-vertex components of \( G \), and \( u_4 \) is equal to the number of vertices in \( U_4 \) minus the number \( x \) of stars in \( U_4 \). Finally, we also let \( r \) denote the independence number of the largest star in \( U_4 \).
Recall that we denote by $G'$ the graph obtained from $G$ by deleting $U_3$, and $G''$ the graph obtained from $G$ by deleting the largest star from $U_4$. Since $G$ is a minimal obstruction, the minimum of the independence numbers of $G'$ and $G''$ is $\ell + 1$. The independence number of $G'$ is $u_1 + u_2 + u_4$, the independence number of $G''$ is $u_1 + u_2 + u_3 + (u_4 - r)$; we let $s = \min(u_4, u_3 + u_4 - r)$. Thus $\ell + 1 = u_1 + u_2 + s$.

We now claim that $U_3 \cup U_4$ has at most $3s \leq (k + 1)s$ vertices. This will conclude the proof, since it implies that $G$ has at most $(k + 1)(u_1 + u_2 + s) = (k + 1)(\ell + 1)$ vertices.

To prove the claim, we proceed as follows: The number of vertices of $U_3 \cup U_4$ is $u_3 + u_4 + x$. If $u_4 = u_3 + u_4 - r$, i.e., if $r = u_3$, then $u_3 < u_4$, and since $x \leq u_4$ also, we have $u_3 + u_4 + x < 3u_4 = 3s$. Note that minimality of $G$ implies that we cannot have $u_4 < u_3 + u_4 - r$, since removing a single-vertex component of $G$ would result in the same quantity $s$, and hence still be an obstruction. In the case $u_4 > u_3 + u_4 - r$, a similar contradiction arises unless there are two equal size largest stars in $U_4$, or unless $U_4$ consists of one single star with two vertices. Indeed, in all other cases, we can remove from $G$ a leaf in its largest star component, resulting in the same quantity $s$, (both $r$ and $u_4$ are reduced by one while $u_3$ is unchanged) and hence we would still have an obstruction.

Suppose the two largest stars in $U_4$ have the same number of vertices. Then $U_4$ has $u_4$ independent leaves and at most $u_4 - 2(r - 1)$ other vertices, hence $U_3 \cup U_4$ has at most $u_3 + 2u_4 - 2r + 2$ vertices. If this number is at most $3s = 3u_3 + 3u_4 - 3r$, our claim is proved. For this to hold, it suffices that

- $u_4 \geq r + 2$ (because in this case $2u_4 - 2r + 2 \leq 3u_4 - 3r$), or
- $u_4 = r + 1$ and $u_3 > 0$ (for a similar reason).

If $r \geq 2$, or if $r = 1$ but there are at least three largest stars, the first case occurs; if $r = 1$ and $u_3 > 0$, the second case occurs. Thus it remains to consider the case when $U_4$ consists of two stars with $r = 1$ and $U_3$ is empty. In this case $G$ is not a minimal obstruction, as we can delete one of these four vertices and still have an obstruction. A similar contradiction applies when $U_4$ consists of a single star with two vertices. (In this case we must have $u_3 = 0$ since we are considering the case when $u_4 > u_3 + u_4 - r$, i.e., $1 = r > u_3$.)

In the last section, we provide a finite upper bound on the size of perfect graphs that are minimal obstructions in the more general case of matrices...
$M$ with $b = *, c = 0$, where $A$ has arbitrary off-diagonal entries, all different from $*$. The last case requires no assumption on the graph $G$:

**Lemma 2.5** If $a = 1$ and $b = c = 0$, then every minimal obstruction has at most $(k + 1)(\ell + 1)$ vertices.

The $M$-partition problem can in this case be solved in time $n^{O(1)}$ with the exponent independent of $k$ and $\ell$.

**Proof.** A graph $G$ admits a partition of this kind if and only if it consists of one component that is complete $t$-partite, with $t \leq k$, forming $G_A$, and of at most $\ell$ other components that are cliques, forming $G_B$.

Note that we may now assume that both $k$ and $\ell$ are at least two, as all other cases are covered under previous lemmas.

We begin by making a list of minimal obstructions:

- $R_{k, \ell}$ is a minimal obstruction.
- $R_1$ and $R_2$ from the previous proof are minimal obstructions.
- The graph $P$ consisting of two paths with three vertices each minimal obstruction.
- The graph $K$ consisting of a clique with $k + 1$ vertices and an additional vertex adjacent to all but one of these vertices, is a minimal obstruction.
- The graph $Q$, consisting of $\ell + 2$ isolated vertices, and the graph $Q'$, consisting of $\ell + 1$ isolated vertices and a path of length three are minimal obstructions.

It is easy to check the above claims, For instance $R_1$ and $R_2$ are (minimal) obstructions since they cannot be placed in $G_A$ (they are not complete multipartite), nor in $G_B$ (they are connected and not complete). Similarly, $P$ contains two components that are not cliques and hence is a minimal obstruction. The graph $K$ is a minimal obstruction, since it is connected, not a clique, and not $k$-colourable. Since $k$ and $\ell$ are at least two, these obstructions all have at most $(k + 1)(\ell + 1)$ vertices.

Consider now a graph $G$ which does not contain an induced $R_{k, \ell}$, $R_1$, $R_2$, $P$, $K$, $Q$, or $Q'$. Then $G$ has at most one component $C$ which is not complete (else $G$ contains an induced $P$). If $G$ does have such a $C$ then $C$ is complete.
$t$-partite (else $G$ contains an induced $R_1$ or $R_2$), and $t$ is at most equal to $k$ (else $G$ contains $K$). Thus $C$ can only be placed in $G_A$. Moreover, all the other components of $G$ (which are cliques) can be placed in $G_B$ (since $G$ does not contain $Q'$). If $G$ only contains clique components, then it has at most $\ell + 1$ of them (else $G$ contains $Q$), and at at most $\ell$ of them have more than $k$ vertices (else $G$ contains $R_{k,\ell}$). Therefore $G$ is partitionable.

A generalization of this case (to arbitrary matrices without $*$) is also considered in the next section.

This completes the proof of the four lemmas and the main theorem. The cases of matrices with $c = *$ will be dealt with (in a more general context) in the next section.

3 Large Obstructions

In this section, we consider somewhat more general matrices $M$: We still assume that the diagonal of $M$ has no $*$'s, and view $M$ as consisting of a $k$ by $k$ block matrix $A$ with zero diagonal, an $\ell$ by $\ell$ block matrix $B$ with ones on the diagonal, and an off-diagonal matrix $C$. We still let $a, b, c$ stand for the entries of $A, B, C$ respectively; however, we allow each of $a, b, c$ to denote a set of values. Thus, for instance, $a = \{0, 1\}$, means that the off-diagonal entries of $A$ are 0's and 1's, in arbitrary positions.

We begin with a situation that generalizes that in Lemma 2.5:

**Theorem 3.1** Let $a = b = c = \{0, 1\}$. (In other words, $M$ has no $*$'s.) Then every minimal obstruction has at most $f(k, \ell) = (k + \ell + 1)(\max(k, \ell) + 1)$ vertices.

A corresponding certifying algorithm solves the $M$-partition problem for general graphs in time $O((m + n + f(k, \ell)^{k+\ell})(k + \ell))$.

**Proof.** Given an instance $G$, partition the vertices of $G$ into $r$ sets $S_1, S_2, \ldots, S_r$ so that two vertices $x, y$ are in the same $S_i$ if and only if $x$ and $y$ have the same neighbors in $G$, other than $x$ and $y$.

If $r \geq k + \ell + 1$, then $G$ has no $M$-partition, since vertices that are placed in the same part in $M$ must have the same neighbors. In that case, choose $k + \ell + 1$ vertices that pairwise do not have the same neighbors, and for every pair $x, y$, of two such vertices, choose a vertex $z$ that is a neighbor of exactly one of $x, y$. This gives an obstruction with $k + \ell + 1 + \binom{k+\ell+1}{2} \leq (k + \ell + 1)(\max(k, \ell) + 1)$ vertices.
Otherwise \( r \leq k + \ell + 1 \), and \( r = k' + \ell' \), where there are \( k' \) independent sets \( S_i \) and \( \ell' \) of the sets \( S_i \) are cliques. If \( G \) has an \( M \)-partition in which at least one vertex in an independent set \( S_i \) is placed in \( G_A \), i.e., to an independent part \( p \) of the \( M \)-partition), then all vertices in \( S_i \) can be placed to the part \( p \). One can guarantee that some vertex in \( S_i \) must be placed in an independent part \( p \) if \( |S_i| \geq \ell + 1 \), so if \( G \) does not have an \( M \)-partition, then there is still no \( M \)-partition when we remove vertices from \( S_i \) until \( |S_i| \leq \ell + 1 \). Similarly, we may remove vertices from a clique \( S_j \) until \( |S_j| \leq k + 1 \) and still not have an \( M \)-partition. The number of vertices in the resulting obstruction is at most \( k' (\ell + 1) + \ell' (k + 1) \leq r (\max(k, \ell) + 1) \leq (k + \ell + 1)(\max(k, \ell) + 1) \).

The algorithm takes time \( O((m + n)(k + \ell)) \) to identify up to \( k + \ell + 1 \) sets \( S_i \) and other vertices that pairwise differentiate them, and time \( O(f(k, \ell)^{k+\ell}(k + \ell)) \) once the number of vertices has been reduced to \( f(k, \ell) \) to test the possible assignments of these vertices to the \( k + \ell \) parts.

We now deal with a situation that generalizes the context of Lemma 2.3:

**Theorem 3.2** Let \( a = *, b = \{0, 1\}, c = 0 \).

Then each minimal obstruction which is perfect has at most \( (\ell + 1)^2 (k + 2\ell) \) vertices.

The \( M \)-partition problem for perfect graphs can be solved in time \( n^{k+O(1)} \). On chordal graphs and complements of chordal graphs, time \( O((m + n + (\ell + 1)^{2\ell})(\ell + 1)) \) is sufficient.

**Proof.** Each connected component that does not have a clique of size \( k + 1 \) can be placed in \( G_A \). The remaining components, which must be placed in \( G_B \), have an obstruction for \( G_B \) from Theorem 3.1 of size at most \( (\ell + 1)^2 \). These vertices are each joined to a clique \( K \) of size \( k + 1 \) by a path \( P \), and \( P \) can be taken of length at most \( 2\ell - 1 \). Otherwise \( P \) gives \( \ell + 1 \) independent vertices with length \( 2\ell \) and thus an obstruction of size \( k + 1 + 2\ell \). We thus have at most \( k + 2\ell \) vertices per vertex in the obstruction for \( G_B \), giving at most \( (\ell + 1)^2 (k + 2\ell) \) vertices in the final obstruction. The running time on chordal graphs and complements of chordal graphs follows from being able to find the largest cliques in linear time [16], and the bound in Theorem 3.1 for \( k = 0 \). An additional \( n^{k+O(1)} \) time to find large cliques is used on perfect graphs.

Finally, here is the context generalizing that of Lemma 2.4:
Theorem 3.3 Let \( a = \{0, 1\}, b = *, c = 0 \).

Then each perfect graph which is a minimal obstruction has at most 
\((k + \ell)(k + 1)^2 + 2(\ell + 1)^2\) vertices.

The \( M \)-partition problem for perfect graphs can be solved in time \( n^{\ell + O(1)} \). For chordal graphs and complements of chordal graphs, time \( O((m + n)k + 2^{k+\ell}(k + 1)^{2k+1}) \) is sufficient.

**Proof.** We may assume that an instance \( G \) has at most \( \ell + 1 \) isolated vertices, since when an isolated vertex is placed to a part in \( G_A \), all isolated vertices can be placed in that part as well. A component of \( G \) with at least one edge that is placed in \( G_A \) uses at least two parts in \( G_A \), so there are at most \( \lfloor k/2 \rfloor \) such components, and at most \( \ell \) components are placed in \( G_B \), for a total of at most \( \lfloor k/2 \rfloor + \ell \) components with at least one edge, otherwise an obstruction consisting of \( \lfloor k/2 \rfloor + \ell + 1 \) independent edges is obtained.

A component that cannot be placed in a part in \( G_A \) gives an obstruction for \( G_A \) by the proof of Theorem 3.1, that is connected, and has at most \((k + 1)^2\) vertices. If this component \( C \) has \( \ell_C \) independent vertices, then we may assume \( \ell_C \leq \ell + 1 \), since \( \ell + 1 \) independent vertices suffice for an obstruction for \( G_B \). They can be joined with paths of length at most \( 2\ell C \), so only \((k + 1)^2 + 2\ell C \leq (k + 1)^2 + 2(\ell + 1)^2\) vertices are needed from such a component. If a component can be placed in a part in \( G_A \), then it has at most \( k \) sets \( S_i \) from Theorem 3.1, may have at most \( k \) elements in sets \( S_i \) that are cliques, and the size of sets \( S_i \) that are independent sets can be reduced to \( \ell + 1 \) without affecting the existence of a partition. This is so, since at least one of \( \ell + 1 \) vertices in \( S_i \) must be placed in \( G_A \), and they can all be placed in \( G_A \) if is. Thus only \( k \max(k, \ell + 1) \) vertices are needed from such a component. Adding the \( \ell + 1 \) isolated vertices to the at most \( \lfloor k/2 \rfloor + \ell \) components with at most \((k + 1)^2 + 2(\ell + 1)^2\) vertices in each component gives the stated bound on obstruction size.

On chordal graphs and complements of chordal graphs, the algorithm takes time \( O((m + n)k) \) to find the obstruction as in Theorem 3.1, and then tests it by considering the at most \( 2^{k+\ell} \) partitions of the components and the set of isolated vertices to be placed in \( G_A \) or in \( G_B \), using \( O((k + 1)^{2k+1}) \) time as in Theorem 3.1 to test such a partition. An additional \( n^{\ell + O(1)} \) time to find large independent sets is used on perfect graphs. \( \square \)

We now consider the cases with \( c = * \). In these cases we can only prove the existence of large obstructions, even for the case of basic matrices where \( a, b, c \) are individual values (and not sets as in the above generalizations). This should not be surprising, as we have already mentioned an example
with \(a = b = c = \ast\), where there are minimal obstructions with strictly more than \((k + 1)(\ell + 1)\) vertices. (More examples are given in the next section.)

All our results in these cases are applications of the following general theorem.

Let \(\mathcal{A}\) and \(\mathcal{B}\) be two classes of graphs, each closed under taking induced subgraphs, and such that the largest graph that is both in \(\mathcal{A}\) and in \(\mathcal{B}\) has at most \(r\) vertices. An \(\mathcal{A},\mathcal{B}\)-partition of a graph \(G\) is a partition of the vertices of \(G\) into two sets \(A\) and \(B\), such that \(A\) induces a subgraph \(G_A\) of \(G\) that is in \(\mathcal{A}\), and \(B\) induces a subgraph \(G_B\) of \(G\) that is in \(\mathcal{B}\). In [12] we studied these partitions (called sparse-dense partitions), and proved that if \(\mathcal{A},\mathcal{B}\) have polynomial time recognition algorithms, then so do \(\mathcal{A},\mathcal{B}\)-partitionable graphs. Here we prove that, moreover, if both classes \(\mathcal{A}\) and \(\mathcal{B}\) have bounded minimal obstruction size, then so does the class of \(\mathcal{A},\mathcal{B}\)-partitionable graphs.

To be specific, a minimal \(\mathcal{A}\)-obstruction is a graph \(G\) that is not in \(\mathcal{A}\), but every induced subgraph of \(G\) is in \(\mathcal{A}\). Minimal \(\mathcal{B}\)-obstructions are defined analogously. Finally, a minimal \(\mathcal{A},\mathcal{B}\)-obstruction is a graph \(G\) that is not \(\mathcal{A},\mathcal{B}\)-partitionable, but every induced subgraph of \(G\) is \(\mathcal{A},\mathcal{B}\)-partitionable.

**Theorem 3.4** If each minimal \(\mathcal{A}\)-obstruction has at most \(s\) vertices, and each minimal \(\mathcal{B}\)-obstruction has at most \(t\) vertices, then each minimal \(\mathcal{A},\mathcal{B}\)-obstruction has at most \(1 + 2\max(s,t)^{2r+1}\) vertices.

**Proof.** Let \(G\) be a minimal \(\mathcal{A},\mathcal{B}\)-obstruction, and let \(v_0\) be any vertex of \(G\). The graph \(G - v_0\) has an \(\mathcal{A},\mathcal{B}\)-partition into subgraphs \(G_A \in \mathcal{A}\) and \(G_B \in \mathcal{B}\). Imagine trying to certify that \(G\) is not \(\mathcal{A},\mathcal{B}\)-partitionable. In any \(\mathcal{A},\mathcal{B}\)-partition of \(G\) into \(A', B'\), we would have \(|A \cap B'| \leq r\) and \(|A' \cap B| \leq r\), by the definition of \(r\). Thus \(G\) is certified as non-partitionable by any set of \(2r + 1\) distinct vertices that are guaranteed to be in \((A \cap B') \cup (A' \cap B)\), for any \(\mathcal{A},\mathcal{B}\)-partition \(A', B'\) of \(G\). We now consider how much of the graph \(G\) is needed to obtain such a certificate of \(2r + 1\) vertices. Assume that \(A', B'\) is any \(\mathcal{A},\mathcal{B}\)-partition of \(G\), and suppose that \(v_0 \in A'\).

Consider the partition of \(G\) into sets \(A \cup v_0\) and \(B\). The set \(A \cup v_0\) is not in \(\mathcal{A}\), since \(G\) is not \(\mathcal{A},\mathcal{B}\)-partitionable. Thus \(G_{A \cup v_0}\) contains an induced subgraph \(G'\) with at most \(s\) vertices that is not in \(\mathcal{A}\). Then some vertex \(v_1\) in \(G'\) must be in \(A \cap B'\), since \(A'\) is assumed to be in \(\mathcal{A}\). Now we have a new partition of \(G\) into sets \(A \cup v_0 - v_1\) and \(B \cup v_1\); thus again \(A \cup v_0 - v_1\) is not in \(\mathcal{A}\) or \(B \cup v_1\) is not in \(\mathcal{B}\). This yields a subset of the first set with at most \(s\) vertices that is not in \(\mathcal{A}\), or a subset of the second set with at most \(t\) vertices that is not in \(\mathcal{B}\). In each case, one of these vertices must be
moved from the one set to the other, giving us a choice of \( v_2 \). Continuing this way, we obtain a sequence of vertices \( v_1, \ldots, v_{2r+1} \), all distinct, and all in \( (A \cap B') \cup (A' \cap B) \).

The vertex \( v_0 \) has two choices, going to \( A \) or going to \( B \). At each iteration we select at most \( \max(s, t) \) possible choices for \( v_i \) to be moved from \( A \) to \( B \) or from \( B \) to \( A \), and the number of iterations is \( 2r + 1 \), thus identifying an induced subgraph \( H \) of \( G \) that is not partitionable with at most \( 1 + 2\max(s, t)^{2r+1} \).

We now apply the theorem to our \( M \)-partition problems.

**Corollary 3.5** Assume \( c = * \), and \( a = * \) or \( a = \{0, 1\} \), and \( b = * \) or \( b = \{0, 1\} \). Let \( w \) be the maximum of \( (k+1)^2 \) (or \( k+1 \) if \( a = * \)) and \( (\ell+1)^2 \) (or \( \ell+1 \) if \( b = * \)).

Then each perfect graph which is a minimal obstruction to \( M \)-partitionability has at most \( 1 + 2w^2k\ell+1 \) vertices.

**Proof.** We take as \( A \) the class of perfect graphs that admit an \( A \)-partition (\( A \) is the diagonal \( k \times k \) block of \( M \)), and as \( B \) the class of perfect graphs that admit a \( B \)-partition. Since \( A \)-partitionable graphs are \( k \)-colourable, and \( B \)-partitionable graphs have \( \ell \)-colourable complements, the size of graphs \( G \) in \( A \cap B \) is bounded by \( r = k\ell \). Indeed, such a graph \( G \) is \( k \)-colourable and has all independent sets (and hence all colour-classes) of size at most \( \ell \). It remains to bound the size of perfect minimal \( A \)- and \( B \)-obstructions: When \( a = * \), there is only one minimal obstruction – the clique with \( s = k + 1 \) vertices. When \( a = \{0, 1\} \), we apply Theorem 3.1 to deduce that each obstruction has at most \( s \leq (k+1)^2 \) vertices. Similarly, \( b = * \) yields \( t = \ell + 1 \), and \( b = \{0, 1\} \) yields \( t \leq (\ell+1)^2 \). Since \( r = k\ell \), we derive the claimed bound from Theorem 3.4.

We note that we can again weaken the assumption of perfectness to the assumption that the absence of clique of size \( k+1 \) implies \( k \)-colourability, needed only when \( a = * \), and the assumption that the absence of an independent set of size \( \ell+1 \) implies \( \ell \)-colourability of the complement, needed only when \( b = * \). In all cases, the bound \( r \) on \( A \cap B \) can be taken to be the Ramsey number \( r(k+1, \ell+1) \), since a graph with more than \( r \) vertices has a clique of size \( k+1 \) or an independent set of size \( \ell+1 \).

Note that the corollary covers all the four cases with \( c = * \) and \( a = *, b = *; a = *, b = 0; a = 1, b = *; \) and \( a = 1, b = 0 \). This completes the study of the simple class of matrices introduced in the previous section.
4 Conclusions

Recall that when $c = \ast$, the bound of $(k + 1)(\ell + 1)$ on the size of minimal obstructions in perfect graphs no longer applies.

We show, more generally, that the upper bound is not of the order $O(k\ell)$, not even for trees. The matrices $M$ we shall consider have $k = 1, b = 0$, and $c = \ast$. Thus an $M$-partition of a graph $G$ has one independent set and at most $\ell$ cliques, with no edges between the vertices of different cliques.

We shall show that (for infinitely many values $\ell$) there are perfect graphs (in fact, trees) which are minimal obstructions and have at least $(\ell/3)^2$ vertices.

Let $p$ be a positive integer. For each $t \geq 2^p$, we shall construct a tree obstruction $R$ with $v = 3 \cdot 2^p - 3 + p \cdot 2^p + t \cdot 2^p + t$ vertices and no $M$-partition with $\ell < r = 2^{p+1} - 2 + t$, but such that every proper induced subgraph of $R$ has such an $M$-partition. Thus $R$ is a minimal obstruction for $\ell = r - 1$. If $t = 2^p$ and $\ell = r - 1$ then $\ell \leq 3t$ and $v \geq t^2 \geq (\ell/3)^2$.

Let $R'$ be an auxiliary tree in which the root (at depth 0) has two children, and in general every vertex at depth $2i < 2p$ has two children, while each vertex at depth $2i - 1 < 2p - 1$ has just one child. Each vertex at level $2p - 1$ is a leaf. It is easy to see that the tree $R'$ has $3 \cdot 2^p - 3$ vertices, has an $M$-partition with $\ell = 2^p - 1$, in which all leaves are in cliques and the root is in the independent set, and an $M$-partition with $\ell = 2^{p+1} - 2$, in which all leaves are in the independent set and the root is in a clique. It can also be seen that $R'$ has no $M$-partition with $\ell < 2^p - 1$ or $\ell > 2^{p+1} - 2$. Furthermore, for every leaf $x$ of $R'$, there is an $M$-partition with $\ell = 2^{p+1} - 2 - p$ that puts $x$ in the independent set and all other leaves in cliques, and puts the root in a clique. Let $X$ denote the set consisting of the root of $R'$ and all the leaves of $R'$. Note that for any $x \in X$ there is a partition that puts $x$ in the independent set and all other $y \in X$ is cliques. Finally, we note that if we remove any vertex from $R'$, then there is an $M$-partition that puts all remaining vertices of $X$ in cliques.

We now define the tree $R$ by letting $t \geq 2^p$ and attaching $t$ new vertices to the root, and $t + p$ vertices at each of the leaves of $R'$. The tree $R$ has $v = 3 \cdot 2^p - 3 + p \cdot 2^p + t \cdot 2^p + t$ vertices. The partitions of $R'$ that put $x \in X$ in the independent set and all $y \in X$ in cliques, add $t$ vertices in cliques if $x$ is the root, and add $t + p$ vertices in cliques if $x$ is a leaf; thus these $M$-partitions have $\ell = r = 2^{p+1} - 2 + t$ cliques. Any other partition of $R$ has at least $2^p - 1$ cliques in $R'$, and at least $2t + p$ cliques at the new leaves; thus it has $\ell \geq 2^p - 1 + 2t + p > r$ cliques. If we remove a vertex of $R'$, then there is a partition of $R$ where the remaining vertices out of $X$ are
in cliques, with $\ell \leq 2p+1 - 2 < r$.

Thus there is no $M$-partition of $R$ with $\ell < r$, yet $R$ minus any vertex has an $M$-partition with $\ell < r$. Therefore $R$ is a minimal obstruction for $\ell = r - 1$. Finally, we note that when $t = 2^p$ the tree $R$ has $v \geq (\ell/3)^2$ vertices.

We know from Corollary 3.5 that the above $M$-partition problems still have only finitely many perfect minimal obstructions. For chordal graphs we have established the correct order of the largest minimal obstruction to be $O(\ell^2)$ [13].

The bounds given here for perfect graphs are decreased for chordal graphs in [13] (and the algorithms are generally more efficient). If $a = 1$, the bound on the size of chordal minimal obstructions in [13] is $t^k$, where $t = O(\ell)$ if $b = \ast$, and $t = O(\ell^2)$ if $b = 0$. The case $a = 0$ is symmetric, by complementation. The case $a = b = c = \ast$ is solved in [17], with the bound of $(k+1)(\ell+1)$. Finally, if $a = \ast, b = 0$, [13] gives the upper bound of $2(k+1)^{(k+2)\ell+1}$.

References


[3] Brandstadt and Lozin

[4] Brandstadt old


