Performance Analysis of Non-Orthogonal AF Relaying in Cognitive Radio Networks

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Abstract—In this letter, we address the problem of maximizing the throughput of underlay cognitive networks, through optimal power allocation of non-orthogonal amplify-and-forward relays. The optimization problem is formulated and transformed to a quadratically constrained quadratic problem (QCQP). The optimal power allocation is obtained through an eigen-solution of a channel-dependent matrix where the corresponding signal-to-noise ratio (SNR) is shown to be the dominant eigenvalue of this matrix. Our optimal power allocation is shown to transform the transmission over the non-orthogonal relays into parallel channels, resulting in the received SNR to be the sum of the SNRs over the relaying channels. While closed-form expressions for statistics of the received SNR are mathematically intractable, we propose an approximation for the probability density function of the received SNR based on Gamma random distribution. The outage probability of the cognitive network is analyzed where the Gamma approximation is shown to be accurate and insightful.

I. INTRODUCTION

The increase of wireless applications requiring high data rates is vast, and is not bearable with the current spectrum allocation strategies. It has been shown that the fixed spectrum allocation (licensing) is under-utilizing the radio spectrum and the need for new strategies and technologies is necessary. In that quest, cognitive radio is a promising solution that can help increasing the efficiency of the current spectrum allocation policies [1]. In cognitive radio design, the overlay and the underlay access paradigms are proposed in the literature to enable spectrum sharing [2]. In the overlay access, the secondary users assume higher priority for the primary transmissions and hence employ spectrum sensing. On the other hand, in the underlay access paradigm, secondary users are allowed to share the spectrum used by primary users conditioned on a guaranteed Quality-of-Service (QoS) for primary users. Such requirement is guaranteed by limiting the secondary-to-primary interference below a predefined threshold referred to as the interference temperature [3].

It is known that node cooperation can reduce fading effects and improve the channel reliability or the transmission capacity through diversity gain [4]. Several cooperative techniques have been proposed in the literature to leverage such diversity [5]. Furthermore, the availability of multiple relaying nodes can enhance the performance through relay selection. While single relay selection is common, the general case of multiple relay selection is addressed in [6] where multiple AF relays transmit over the same frequency band simultaneously. With the advantage offered by cooperative communication, cognitive users can enhance their throughput and coverage using cognitive relays. Recently, several studies have been conducted to analyze the performance of cooperation among cognitive nodes. Multiple-relay cognitive networks are studied in [7], where an algorithm is proposed to select multiple relays to maximize the secondary network capacity while preserving the QoS requirements of the primary network. In [8], a study of the secondary network throughput scaling with the number of relays is conducted while relays either cooperate with full transmission power or do not cooperate at all. Apart from this binary power allocation problem in [7] and [8], beamforming is exploited to enhance the performance of cognitive users. In [9], a simplified suboptimal power allocation for multiple amplify-and-forward (AF) relays with individual maximum power constraint is developed. In [10], the performance of zero-forcing beamforming at the cognitive relays is studied for AF and DF relays. In the aforementioned work, the proposed power allocation schemes are suboptimal, and to the best of our knowledge there is no work studying the performance of optimal power allocation in a multiple-relay scenario.

In this paper, we study the optimal power allocation of an underlay cognitive relay network in order to maximize the received SNR. The system under consideration consists of multiple AF relays that use the same frequency band simultaneously (i.e., non-orthogonal transmissions). Different from previous works, we formulate the optimal power allocation using a quadratically constrained quadratic optimization problem (QCQP). Using this formulation, the optimal power allocation is obtained through a simple eigenvector calculation where the maximum received SNR is shown to be the dominant eigenvalue of a well-defined matrix. We prove that the optimal SNR is the sum of independent SNR of each relay, suggesting that the obtained power allocation transforms the non-orthogonal relaying channel into parallel channels. The statistics of the resulting SNR of each relay are developed in terms of the probability density function (PDF) and the cumulative distribution function (CDF). It is noted that, closed-from expressions for such statistics for the total received SNR appeared are intractable. Therefore, we propose approximating the individual SNRs as exponential random variables using moment matching method. Hence, the received SNR is approximated using a Gamma distribution.

II. SYSTEM MODEL

We consider an underlay secondary relay network with one source $S$ and one destination $D$ and $K \geq 2$ relays denoted $R_1, R_2, ..., R_K$. All nodes are assumed to have single antenna. The nearby primary system consists of a single transmitter-receiver link. The interference on the primary receiver from
secondary transmissions must be kept below a pre-defined threshold referred to as the **interference temperature**. The interference from the primary network to the relays as well as the destination $D$ is treated as Gaussian noise.

A block-fading model is assumed, and all channel coefficients are assumed to be independent. Assuming no direct link between $S$ and $D$, the transmission from $S$ to $D$ occurs through the AF relays over two time slots. In the first time slot, $S$ selects the proper transmission power $P_S$ to transmit a data symbol to all relays. The received signal at the $i$th relay, $y_{R_i}$, is then given by

$$y_{R_i} = \sqrt{P_S} h_{S,R_i} x_S + w_{R_i},$$

where $x_S$ is the transmitted data symbol (with $\mathbb{E}[|x_S|^2] = 1$, where $\mathbb{E}[\cdot]$ denote expectation), and $h_{S,R_i}$ is the channel coefficient from $S$ to the $i$th relay, $i \in \{1,2,...,K\}$, modeled as zero mean and unit variance circular symmetric complex Gaussian random variable (CSCGVR), denoted by $\mathcal{CN}(0,1)$. Here $w_{R_i}$ is a complex random variable capturing the effect of the thermal noise and the interference due to the primary activities at $D$, simultaneously, the received signal at $D$ is given by

$$y_D = \sum_{i=1}^{K} \left( \sqrt{P_{R_i}} \frac{|h_{R_i,D}|}{\sqrt{P_S |h_{S,R_i}|^2 + \sigma_{R_i}^2}} \right) x_S + \sum_{i=1}^{K} \left( \sqrt{P_{R_i}} \frac{|h_{R_i,D}|}{\sqrt{P_S |h_{S,R_i}|^2 + \sigma_{R_i}^2}} \right) + w_D,$$

where $w_D$ is CSCGVR with zero mean and variance $\sigma_D^2$ capturing the effect of the thermal noise and the interference due to the primary activities at $D$. The instantaneous SNR, denoted by $\gamma$, is then given by

$$\gamma = P_S \times \left( \frac{\sum_{i=1}^{K} \frac{|h_{R_i,D}|}{\sqrt{P_S |h_{S,R_i}|^2 + \sigma_{R_i}^2}}^2}{\sum_{i=1}^{K} \frac{1}{\sqrt{P_S |h_{S,R_i}|^2 + \sigma_{R_i}^2}} + \sigma_D^2} \right).$$

Hence, the transmission rate of the secondary link for a unit bandwidth is given by $R = \frac{1}{2} \log_2(1 + \gamma)$, where the $\frac{1}{2}$ factor accounts for the dual hop transmission.

Let $g_{S,P}$ be the channel coefficient from $S$ to the primary receiver. The interference generated by $S$ in the first time slot is given by $I_S = P_S |g_{S,P}|^2$. In the second time slot, with all relays transmitting, the total interference generated at the primary receiver is given by $I = \sum_{i=1}^{K} P_{R_i} |f_{R_i,P}|^2$, where $f_{R_i,P}$ is the channel coefficient from the $i$th relay to the primary receiver. Hence, $I_S$ and $I$ should be kept below the interference temperature, $I_{max}$, during the first and the second time slots, respectively.

### III. Optimal Power Allocation

#### A. Problem Formulation

The secondary transmission rate mainly depends on the transmission power of secondary relays and source node. Thus, the objective is to find the set of transmission power $\{P_{R_1},P_{R_2},...,P_{R_K}\}$ that maximizes the SNR (or equivalently the secondary transmission rate) for a given interference constraint at the primary receiver. Given that the secondary source determines its transmission power $P_S^*$ independently from the transmission powers of the relays $P_{R_i}$, the power allocation problem can then be formulated as,

$$P_R^* = \text{arg max}_{P_R} \left\{ \frac{\left( \sum_{i=1}^{K} \sqrt{P_S} \alpha_i |h_i| \sqrt{P_{R_i}} \right)^2}{\sum_{i=1}^{K} \left( \beta_i \sigma_{R_i}^2 + \sigma_D^2 \right)} \right\},$$

subject to $\sum_{i=1}^{K} \left( \frac{P_{R_i}}{\gamma} \right) \leq I_{max},$ (4)

where $\alpha_i = |h_{S,R_i}|^2$, $\beta_i = |h_{R_i,D}|^2$, $\gamma = |f_{R_i,P}|^2$. This problem can be formulated as QCQP to obtain an optimal solution in a simple form as follows. Let $x = [x_1,x_2,...,x_K]^T$ be the new optimization variable, where $x_i = \sqrt{P_{R_i}}$ and $|.|^T$ denotes transpose operation. The optimization problem in (4) can be reformulated as

$$x_{opt} = \text{arg max}_{x} \left\{ \frac{x^T A x}{x^T B x + \sigma_D^2} \right\},$$

subject to $x^T C x \leq I_{max},$ (6)

where $A$ is a $K \times K$ matrix with elements $a_{i,j} = P_S \sqrt{\alpha_i \beta_j}$, $B$ and $C$ are diagonal matrices of size $K \times K$ with the diagonal elements $b_{i,i} = \theta_i$ and $c_{i,j} = \zeta_i$, respectively. Now, the optimal solution, $x_{opt}$, has to satisfy the constraint in (6) with equality, which transforms (5) into a Rayleigh quotient. A Rayleigh quotient is maximized using the dominant eigenvector of the numerator matrix $M$, and the corresponding maximum value is the dominant eigenvalue (for more details, the reader is referred to [11]). Hence, the optimal power allocation is given as

$$P_R^* = \mu^2 \text{diag} \left\{ (D^{-\frac{1}{2}})^T v \right\},$$

where $D = B + \sigma_D^2 C$, $M = D^{-\frac{1}{2}} A (D^{-\frac{1}{2}})^T$, and $\text{diag}\{\cdot\}$ is a vector whose elements are the diagonal elements of a matrix. The vector $v$ is the dominant eigenvector of $M$, and

$$\mu^2 = \frac{I_{max}}{v^T D^{-\frac{1}{2}} C (D^{-\frac{1}{2}})^T v}$$

is a scaling factor.
With the optimal power allocation deployed at the cognitive relays, the corresponding maximum received SNR, \( \gamma^* = \lambda_{\text{max}} \), where \( \lambda_{\text{max}} \) is the dominant eigenvalue of the matrix \( M \). The eigenvalues of the matrix \( M \) are given by,

\[
\text{Eigen}\{M\} = \text{Eigen}\{AD^{-1}\}.
\]

(8)

Noting that matrix \( A \) is of rank one, the product of the two matrices \( A \) and \( D^{-1} \) is also of rank one. This leads to the resulting SNR being the only nonzero eigenvalue of the matrix \( AD^{-1} \). After some mathematical manipulations, the maximum received SNR can be expressed as,

\[
\gamma^* = \gamma_s \sum_{i=1}^{K} \gamma_i,
\]

(9)

\( \gamma_s = \frac{\lambda}{\sigma^2} \), and \( \gamma_i \) is given by

\[
\gamma_i = \frac{[h_{s,R_i}]^2 [h_{R_i,r}]^2}{[h_{R_i,D}]^2 + \kappa_P f_{R_i,p}^2 [h_{s,R_i}]^2 + \kappa_n |f_{R_i,p}|^2},
\]

(10)

where \( \kappa_P = \frac{P_m}{\lambda_{\text{max}}} \), \( \kappa_n = \frac{\sigma^2}{m} \), and assuming the noise variance is the same at all secondary nodes. Note that (9) suggests that the optimal power allocation in (7) transforms the nonorthogonal transmission of the secondary relays into parallel channels, resulting in the received SNR to be the sum of the SNRs generated independently by each relay.

IV. SNR STATISTICS

A. Statistics of the received SNR \( \gamma^* \)

Given the channel coefficients \( h_{s,R_i}, h_{R_i,D} \) and \( f_{R_i,p} \) are zero mean and unit variance CSCGRV, \( \mathcal{CN}(0,1) \), the power gains \( |h_{s,R_i}|^2, |h_{R_i,D}|^2 \) and \( |f_{R_i,p}|^2 \) are exponential random variables. Hence, the PDF and the CDF of the SNR at each relay, \( \gamma_i \) \((i \in \{1,2,\ldots,K\})\), are given by (see Appendix),

\[
F_{\gamma_i}(t) = 1 - \frac{e^{-t}}{(\kappa_p + 1)} \left[ 1 + \frac{t(\kappa_p t + \kappa_n)}{(\kappa_p t + 1)} \right] e^{\frac{t(\kappa_p t + \kappa_n)}{\kappa_p t + 1}}, t \geq 0,
\]

(11)

\[
f_{\gamma_i}(t) = \frac{e^{-t}}{(\kappa_p + 1)^2} \left[ \kappa_n t^2 + \kappa_P (1 - \kappa_n + 2 \kappa_p) t + \frac{t(\kappa_p t + \kappa_n)}{(\kappa_p t + 1)} - \frac{t(\kappa_p t + \kappa_n)}{(\kappa_p t + 1)} \right],
\]

(12)

where \( E(t) \) is the exponential integral function defined by \( E(t) = -\int_x^\infty \frac{e^{-t}}{t} dt \), \( m_1 = \kappa_n (2 - \kappa_n) - \kappa_P (1 - \kappa_n) \) and \( m_2 = 2 \kappa_P - \kappa_n (1 - \kappa_n) \). Noting that a closed-form expression for CDF of the received SNR, \( \gamma^* \), \( F_{\gamma^*}(t) \), is intractable, in what follows we approximate such a CDF using the Gamma distribution.

B. Approximating the statistics of the received SNR \( \gamma^* \)

As noted earlier, obtaining a closed form expression for the statistics of \( \gamma^* \) appeared intractable. While evaluating a numerical solution for these statistics might be useful, obtaining expressions for the overall system performance (outage probability, bit error rate and achievable throughput) is not possible with this solution. Therefore, we suggest approximating the statistics of \( \gamma^* \). To obtain such approximation, we propose approximating the PDF of \( \gamma_i \) in (12) using an exponential distribution given by,

\[
f_{\gamma_i}(x) \approx \frac{1}{\bar{\mu}_{\gamma_i}} e^{\frac{-x}{\bar{\mu}_{\gamma_i}}}, \quad x \geq 0,
\]

(13)

where \( \bar{\mu}_{\gamma_i} \) is the mean of exponential distribution obtained by matching the mean of the original distribution in (12), and is given by,

\[
\bar{\mu}_{\gamma_i} = -e^{\frac{-\kappa_n}{\kappa_p}} \frac{1}{Ei[-1]} - \frac{\kappa_n}{\kappa_p} \Gamma_{3,0}^1 \left( 0, 2 \right) \left[ 1, 1, 1 \right]
\]

(14)

where \( \Gamma(a, x) \) is the upper incomplete Gamma function [12]. Accordingly, the approximate PDF of \( \gamma^* \), is then given by,

\[
f_{\gamma^*}(x) \approx \frac{1}{\Gamma(K)} x^{K-1} e^{\frac{-x}{\bar{\mu}_{\gamma^*}}}, \quad x \geq 0.
\]

(16)

V. OUTAGE PROBABILITY ANALYSIS

Using the proposed power allocation, the transmission rate of the secondary network \( R_s = \frac{1}{2} \log_2 (1 + \gamma^*) \). An outage event occurs when \( R_s \) drops below a predefined target rate \( R_{th} \). Let \( P_{out} \) be the outage probability, then,

\[
P_{out} = P_R(R_s \leq R_{th}) = P_R(\gamma^* \leq \gamma_{th}),
\]

(17)

where \( \gamma_{th} = 2 R_{th} \). The outage probability can then be simply obtained using the approximate CDF of \( \gamma^* \) in (15) as,

\[
P_{out}(\gamma_{th}) = \frac{1}{\Gamma(K)} \gamma \left( K, \frac{\gamma_{th}}{\bar{\mu}_{\gamma^*}} \right).
\]

(18)

where \( \gamma(a, x) \) is the lower incomplete Gamma function [12].

VI. NUMERICAL RESULTS AND SIMULATION

In this section, we present the performance results of the secondary network through the derived expressions and simulation. In Fig. 1, we simulate the outage probability \( P_{out} \) of the cognitive network versus \( \gamma_s \), the secondary source power \( P_S \) measured relative to the noise power at the cognitive nodes \( \sigma^2 \) for a target SNR \( \gamma_{th} = 2 \) dB. The interference temperature \( I_{max} \) is set at 13 dB and the noise power is normalized to 1. We compare the performance of the optimal power allocation to the equal power allocation where \( P_{R_i} = \frac{I_{max}}{\sum_i |f_{R_i,p}|^2} \). It
maximum SNR statistics based on the Gamma distribution are provided.

**APPENDIX**

Let $X = |h_{R_i,D}|^2$, $Y = |h_{S,R_i}|^2$, $Z = |f_{R_i,P}|^2$, and define $U = X + Y + Z$. Noting that for the Rayleigh fading scenario, the random variable $X, Y$ and $Z$ are exponential random variables with unit mean. To find the CDF of $\gamma_i$, we start by finding the CDF of $U$ given by,

$$F_U(u) = \int_0^\infty (1 - e^{-uz}) e^{-z} dz = \frac{u}{u + 1}, \quad u \geq 0.$$  \hspace{1cm} (18)

Then it is easy to show that,

$$F_{\gamma_i}(t) = 1 - \int_t^\infty F_U \left( \frac{\kappa_p + \kappa_n}{x - t} \right) e^{-x} dx.$$ \hspace{1cm} (19)

Substituting (18) into (19), with some manipulation and using variable change $x = u - t$,

$$F_{\gamma_i}(t) = 1 - e^{-t} \int_0^\infty \frac{\kappa_p t x e^{-x}}{(\kappa_p + 1)x + (\kappa_p + \kappa_n)t} dx \hspace{1cm} (20)$$

$$- e^{-t} \int_0^\infty \frac{\kappa_p t^2 + \kappa_n t}{(\kappa_p + 1)x + (\kappa_p + \kappa_n)t} e^{-x} dx$$

Using [12, eq. 3.353.5] and [12, eq. 3.352.4], a closed-form expression for $I_1$ and $I_2$ can be obtained. Substituting those results into (20) with term arrangement resulting in the CDF (11). Differentiating the CDF in (11) with respect to the variable $t$, one can easily arrive at the PDF of $\gamma_i$ in (12).

**REFERENCES**


