Remarks on Transient Amplitude Analysis of MOS Cross-Coupled Oscillators

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SUMMARY In this paper, closed-form analytical equations for the time-domain amplitude of the MOS cross-coupled oscillators are derived. The procedure of the paper is based on estimating an accurate equation for describing the behavior of the cross-coupled MOS configurations and finding a reasonable solution for the nonlinear differential equation governing the circuit. The solution method is presented for a general equation and is valid for all possible second-order oscillators. Both of the long channel and short channel transistor topologies have been investigated. The resulted equations are in good agreement with simulation results for a wide range of the circuit parameters and enable us to analyze and synthesize the oscillators with the desired transient behavior.

key words: cross-coupled oscillator, MOS VDP oscillator, nonlinear circuits, oscillator, transient response

1. Introduction

The growing demand for higher data transfer rates and lower power consumption has had a major impact on the design of RF communication systems. In both wireless and wireline applications, this has been achieved using more spectrally efficient modulations and/or wider channel bandwidth in combination with engineering techniques to lower power and fabrication costs. Oscillators are essential building blocks of radio transceivers, where they are used for frequency up/down conversion of the information bearing signal. The two most important specifications for a local oscillator operating at RF are phase noise and oscillation amplitude. At present, there are a great number of papers devoted to voltage-controlled oscillators (VCOs) [1]–[15], dealing mostly with the problem of predicting the phase noise, which critically affects the stringent specifications of modern wireless systems of which the VCOs are a crucial component [7], [9]. The amplitude of the local oscillator signal is also very important to the RF designer. Insufficient LO amplitude leads to the mixer circuit not being switched fully on or off during each half cycle. As a result the mixer noise figure is considerably degraded. In some systems, the oscillator signal is amplified first before being applied to the mixer [14]. On the other hand, in many applications, it is also necessary to predict the transient response of oscillators [3], [4] as in phase-locked loops (PLLs) [5] and, when the precision is of some interest, the frequency deviation due to nonlinear effects, as in quartz crystal oscillators [8]. Also having the VCOs with faster transient response enables us to design the high performance blocks with faster responses. The limitations of the linear analysis of oscillators based on the instability of the equilibrium state are well known, as this analysis allows us to determine only the condition for a reliable oscillation build-up and the oscillation frequency in first approximation. To overcome these limitations, a simplified nonlinear analysis based on the describing function method is usually performed [7], [13].

Cross-coupled topologies of oscillators are popular among CMOS designers for their relaxed startup conditions. These topologies as in the Van Der Pol (VDP) case give a useful low-voltage structure with an improved intrinsic phase noise performance. The differential configuration of these designs helps limit even-order harmonics in the output and reduces the amount of noise (from the rails, substrate, and current source) appearing as a differential disturbance across the tank. This, and the benefit of a convenient biasing scheme, marks the extent of the advantages available to the cross-coupled VCO topologies.

This paper is dedicated to finding analytical equations for the time-domain oscillation amplitude in the MOS cross-coupled oscillators. This is performed by analyzing nonlinear differential equations of these circuits that are achieved using an accurate estimation for the nonlinear characteristics of cross-coupled MOS configurations with standard models of MOS devices. As the device dimension continue to scale down, reaching below 0.2μm by the year 2000, higher order effects necessitate more complex models so as to attain more accuracy in designs and simulations [17] therefore, both of the topologies with long channel and short channel transistors are investigated. The channel length of \( L = 0.2 \mu m \) is chosen for modeling of the short channel topology and \( L = 1 \mu m \) is chosen for modeling of the long channel topology also the validity of these equations is investigated through three interesting examples. Although the oscillation can be rigorously predicted with the acceptable accuracy, in this work, we focus on the expression for the first approximation (fundamental component) only, which is accurate enough for practical purposes. The proposed simple closed-form equations gives us a design insight that we can simply design circuits with desired features while using simulations only is somehow a trial and error that may be bothering.
2. General Procedure of this Paper (Averaging Method)

As is known, every nonlinear Van Der Pol type oscillator can be modeled as the second-order oscillator shown in Fig. 1, which is composed of a locally nonlinear active one-port connected to a LC-tank whose losses are attributed to a parallel resistance $R$. Using the differential operator $\overline{D}$ with respect to the time $t$ the circuit describing equation is written in the form

$$\left(\overline{D}^2 + \frac{1}{RC}\overline{D} + \frac{1}{LC}\right)V = -\frac{1}{C}\overline{D}i(V)$$

(1)

Equation (1) takes the form (3) by introducing the parameter $\varepsilon$ as the inverse of the circuit quality factor $Q$, that is

$$\varepsilon = \frac{1}{Q} = \frac{1}{\omega_0 RC}$$

(2)

Where, $\omega_0^2 = 1/\LC$. Using the variable $t = \omega_0 t$, (1) can be rewritten as

$$\left[D^2 + \varepsilon F(V, DV) \cdot D + 1\right]V = 0$$

(3)

Where $D$ is the derivative with respect to the time $t$ and $F(V, DV)$ represents the effect of nonlinear active element and parallel resistance $R$.

Choosing high quality factor $Q$ for the $LC$-tank guarantees that $\varepsilon \ll 1$ is a very small parameter. Therefore, (3) is in a sense very close to the equation $(D^2 + 1)V = 0$ whose phase diagram consists of circles on the origin. It should be possible to take advantage of this fact to construct approximate solutions. The phase paths will be nearly circular for significantly small $\varepsilon$ [18]. But the nonlinear part makes the path far from an ideal circle. Supposing that at least one periodic time solution corresponding to the closed path exists [19], let any phase path of (3) be represented parametrically by time-dependent polar coordinates $a(t), \theta(t)$ as shown in Fig. 2. Where $V = a \cos \theta$, $y = DV = a \sin \theta$ so that (3) can be written as (4).

$$y = a \sin \theta, D_{y} = -\varepsilon \cdot F(a \cos \theta, a \sin \theta) - a \cos \theta$$

(4)

And

$$a^2 = V^2 + y^2, \tan \theta = \frac{y}{V}$$

(5)

Differentiating these equations with respect to the time results in

$$2a Da + 2DV y + 2y Dy, \sec^2 \theta D\theta = \frac{V Dy - y DV}{V^2}$$

(6)

Equations of (6) are simply reduced to (7).

$$Da = \frac{V Dy + y Dy}{a}, \ D\theta = \frac{V Dy - y DV}{a^2}$$

(7)

Now, substituting $V = a \cos \theta$, $DV = y = a \sin \theta$ in (7) leaves beneficial equations of (8) and (9).

$$Da = -\varepsilon F \sin \theta$$

(8)

$$D\theta = -1 - \varepsilon a^{-1} F \cos \theta$$

(9)

And the differential equation of the phase path will be obtained from division of (8) and (9).

$$\frac{da}{d\theta} = -\varepsilon F \sin \theta \sin \theta$$

(10)

In (8)–(10), for brevity $F$ stands for $F(a \cos \theta, a \sin \theta)$. It is assumed that a representative point with $a(t), \theta(t)$ moves round the origin repeatedly in phase plane at the angular speed $D\theta \approx -1(D\theta \approx -\omega = -1/\sqrt{LC}$ in (1)), where $a(t)$ from the origin changes slowly. We should obtain an approximate differential equation for the phase paths. After extending Eq. (10) for paths, it can be written as

$$\frac{da}{d\theta} = \varepsilon F(a \theta, \sin \theta) - a \sin \theta + G(\varepsilon^2)$$

(11)

Where $G(\varepsilon^2)$ represents sum of the components that contain $\varepsilon^n, n > 1$, which are negligible for $\varepsilon \ll 1$. Note that the function $\varepsilon F(a \theta, \sin \theta)$ is not in general periodic in $\theta$ because $a(\theta)$ is not periodic. Nevertheless $F(a \theta, \sin \theta)$ can be approximated by a pseudo-Fourier series for all values of $\theta$ if $a$ is treated as if it were an arbitrary constant parameter. In this condition $F(a \theta, \sin \theta)$ is periodic. So, it can be shown by an ordinary Fourier series. This is valid for all $\theta$ and a fixed value of $a$.

$$F(a \cos \theta, a \sin \theta) \sin \theta = N_0(a) + \sum_{n=1}^{\infty} \left\{N_n(a) \cos n \theta + M_n(a) \sin n \theta \right\}$$

(12)

In (12) the coefficients are as
\[
N_0(a) = \frac{1}{2\pi} \int_0^{2\pi} F(a \cos z, a \sin z) \sin z \, dz
\] (13)

\[
\left( \frac{N_n(a)}{M_n(a)} \right) = \frac{1}{2\pi} \int_0^{2\pi} F(a \cos z, a \sin z) \sin z \frac{\cos nz}{\sin nz} \, dz
\] (14)

For \( n \geq 1 \), Eqs. (12)–(14) still hold good if we put \( a(\theta) \) in place of \( a \) wherever it appears [20]. Therefore, the differential equation of (11) can be rewritten as

\[
\frac{da}{d\theta} = \varepsilon N_0(a) + \varepsilon \sum_{n=1}^{\infty} \left[ N_n(a) \cos n\theta + M_n(a) \sin n\theta \right] + G(\varepsilon^2) \] (15)

In which \( a \) is a function of \( \theta \). However it will be shown that for significantly small \( \varepsilon \) increment in \( a(\theta) \) over any complete loop \( \theta_0 < \theta < \theta_0 + 2\pi \) depends only on the \( eN_0(a) \). Suppose that on one loop of phase path \( \theta_0 < \theta < \theta_0 + 2\pi \).

\[ a(\theta_0) = a_1, \quad a(\theta_0 + 2\pi) = a_2 \] (16)

Now the increment \( a_2 - a_1 \) should be calculated. From (11):

\[
\frac{da}{d\theta} = \varepsilon F(a(\theta) \cos \theta, a(\theta) \sin \theta) \sin \theta + G(\varepsilon^2).
\]

Integrating (11) with respect to \( \theta \), starting at \( \theta = \theta_0 \), it will be obtained that

\[ a(\theta) - a(\theta_0) = G(\varepsilon) \Rightarrow a(\theta) = a_1 + G(\varepsilon) \] (17)

In which \( a(\theta_0) = a_1 \). Substituting (17) in to the terms under summation sign in (15), obtaining, \( N_n(a) = N_n(a_1) + g(\varepsilon), M_n(a) = M_n(a_1) + g(\varepsilon) \) then (15) - becomes

\[
\frac{da}{d\theta} = \varepsilon N_0(a) + \varepsilon \sum_{n=1}^{\infty} \left[ N_n(a_1) \cos n\theta + M_n(a_1) \sin n\theta \right] + O(\varepsilon^2)
\] (18)

By integrating (18) over the complete loop \( \theta_0 < \theta < \theta_0 + 2\pi \), the integral over each term under the summation will be zero, and we are left with

\[ a_2 - a_1 = \varepsilon \int_{\theta_0}^{\theta_0+2\pi} N_0(a(\theta))d\theta + O(\varepsilon^2) \] (19)

Therefore the increment over a complete loop depends only on \( N_0(a) \) to this order of accuracy [20], which is suitable for practical proposes. So the differential equation that is left from (15) is as (20).

\[
\frac{da}{d\theta} = \varepsilon N_0(a),
\]

\[
\frac{da}{d\theta} = \frac{\varepsilon}{2\pi} \int_0^{2\pi} F(a(\theta) \cos z, a(\theta) \sin z) \sin z \, dz
\] (20)

Approximated equations for the time variation of \( a(t) \) can be derived from (8).

\[
\frac{da}{dt} = \frac{da}{d\theta} \frac{d\theta}{dt} = \frac{da}{d\theta} + G(\varepsilon^2)
\] (21)

And from (20)

\[
\frac{da}{dt} = -\varepsilon N_0(a)
\] (22)

Equation (22) is the principal beneficial equation of the used method that leads to straightforward analysis for finding the time-domain amplitude variation of the MOS cross-coupled oscillators.

3. Applying This procedure for the MOS Cross-coupled Van Der Pol Oscillator

MOS cross-coupled Van Der Pol oscillator shown in Fig. 3 is one of the most important building blocks for many of RF applications [7]. As is shown in Fig. 3, this oscillator is composed of two major parts: 1. Amplifier part that is implemented using cross-coupled MOS tank. The behavior of this circuit can be efficiently described considering the nonlinear I-V characteristic of the amplifier that is given by: \( i_{df} = F(V_d) \). An example of the simulated differential I-V characteristics of the amplifier is shown in Fig. 4. This plot consists of five main parts, which denotes the operating point of the transistors [7]. When both of the transistors are operating in the saturation region near to the quiescent bias point \( i_{df} \) decreases with \( V_d \) (region 3). As \( V_d \) continues to increase one of the transistors slips into triode region (region 2, 4). Increasing \( V_d \) further, one of the devices falls into the off region whereby another falls deeper
into the triode region (regions 1, 5). So Fig. 4 gives the evidence that the differential I-V characteristics of the amplifier can be estimated by a cubic function like (23).

\[ i_{df} = AV_d^3 + BV_d \]  

(23)

3.1 Approximating the I-V Characteristics for the Configurations with Long Channel MOS Devices

As is known [19], the slope at \( V_d = 0 \) is equal to \(-\frac{\omega_0}{2} \), therefore, \( B = -\frac{\omega_0}{2} \). Where \( g_m \) is equal to: \( \mu C_{ox} W/L(V_{CM} - V_t) \) in long channel devices. \( \mu \) is the carrier mobility, \( C_{ox} \) is the transistor capacitance per unit area, \( W, L \) are the channel width and length and \( V_{CM} \) is the common mode voltage at output nodes (in this topology \( V_{CM} = V_{DD} \)) and \( V_t \) is the threshold voltage of transistors.

In order to calculate \( A \), we should have two symmetric points. For calculating these two points let us assume that one of the devices is going to be off (Fig. 4). At this point, another transistor is exactly in the triode region (for example M1 is going to be off and M2 is in triode). So, \( V_{O1} = V_t \) and knowing that the common mode voltage at \( V_{O1}, V_{O2} \) is \( V_{CM} = V_{DD} \), it will be obtained that

\[ V_{O2} = V_{DD} + (V_{DD} - V_t) = 2V_{DD} - V_t \quad (24) \]

Consequently,

\[ i_{df} = i_{D2} - i_{D1} = K(V_{GS2} - V_t)V_{DS2} - K(V_{GS1} - V_t)^2 = 2KV_t(V_{DD} - V_t) \]  

(25)

Substituting this point in (23), \( A \) can be easily calculated as follow.

\[ A = \frac{2KV_t + g_m}{8(V_{DD} - V_t)^3} \]  

(26)

3.2 Approximating the I-V characteristics for the Topologies with Short Channel MOS Devices

As it is presented in [18], the drain current of the short channel MOS transistor in saturation region can be described as (27).

\[ i_d = \frac{1}{2} \cdot \frac{\mu C_{ox}}{1 + \theta_{on}} \cdot \frac{W}{L} \cdot v_{on}^2 \]  

(27)

Where, the on-voltage \( v_{on} = V_{gs} - V_t \) is the difference between the gate-source voltage and the transistor’s threshold voltage and \( \theta \) is the mobility degradation term. Similar to the former case, the differential I-V characteristics of the amplifier can be estimated by a third order function like (23). And the slope at \( V_d = 0 \) is equal to \(-\frac{\omega_0}{2} \) [7], therefore, \( B = -\frac{\omega_0}{2} \) where \( g_m \) is [18].

\[ g_m = \frac{\mu C_{ox}(V_{CM} - V_t)}{2(1 + \theta (V_{CM} - V_t))} \cdot \frac{W}{L} \]  

(28)

As in the former case \( V_{CM} = V_{DD} \). Similar to the procedure that has been used in the long channel devices, the \( A \) coefficient in this case can be calculated as the following relation [7].

\[ A = \frac{g_m}{6(V_{DD} + 0.5V_t)^2} \]  

(29)

3.3 Calculating the Transient Amplitude Behavior for the MOS VDP Oscillators

Now look at the cross-coupled Van Der Pole oscillator in Fig. 3. Writing KCL equations at nodes \( V_{O1}, V_{O2} \) the differential equation of the circuit is written as

\[
\begin{align*}
\frac{d^2V_{O1}}{dt^2} + \frac{1}{RC} \frac{dV_{O1}}{dt} + \frac{3}{C} AV_{O1} - \frac{g_m}{2C} + \frac{V_{O1} - V_{DD}}{L} &= 0 \\
\frac{d^2V_{O2}}{dt^2} + \frac{1}{RC} \frac{dV_{O2}}{dt} + \frac{3}{C} AV_{O2} - \frac{g_m}{2C} + \frac{V_{O2} - V_{DD}}{L} &= 0
\end{align*}
\]

(30)

Subtracting (30.a) and (30.b) and using the estimated cubic characteristics of (23) for \( i_{df} \) and the method that was introduced in the former section can be used to achieve the transient response for the oscillator amplitude \( a(t) \). As is mentioned before, it can be assumed

\[ V_d(t) = a(t) \cos \omega t \]  

(32)

That is accurate enough for practical purposes [1]. In (32), \( \omega \) is assumed to be \( 1/\sqrt{LC} \) because of high quality factor (Q) assumption for the LC tank. From (32)

\[ eF(a \cos \omega t, a\omega \sin \omega t) = a\omega \sin \omega t \left( \frac{1}{RC} - \frac{g_m}{2C} + \frac{3}{C} \frac{A^2}{a} \cos^2 \omega t \right) \]  

(33)

Now \( eN_0(a) \) can be calculated from (13) as follows.

\[ eN_0(a) = \frac{\omega}{2\pi} \int_0^\infty 3 \frac{A^2}{a} \cos^2 \omega t - \left( \frac{g_m}{2C} - \frac{1}{RC} \right) a \sin \omega t dt \]  

(34)

In order to calculate \( a(t) \), (22) can be used as

\[ \frac{da}{dt} = - \left( \frac{3\lambda a}{8C} a^3 - \frac{\omega}{2} \left( \frac{g_m}{2C} - \frac{1}{RC} \right) a \right) \]  

(35)

Solving the above differential equation and assuming that \( a(0) = a_0 \), \( a(t) \) will be calculated as (36).

\[ a(t) = \left( \frac{3\lambda a_0}{8C} \left( 1 - \frac{1}{2\pi} (\frac{g_m}{2C} - \frac{1}{RC}) \right) e^{-(\frac{\pi}{2\sqrt{\lambda}})^2} \right)^{\frac{1}{3}} \]  

(36)
Equation (36) is a useful closed-form equation for the time-domain amplitude variation of this type of oscillator in terms of the circuit parameter. Also, (36) allows us to analyze the dynamic response of amplitude under supply voltage variations through the \( g_m \) parameter, which is a function of supply voltage (28).

4. Applying This Procedure for the MOS Cross-coupled Oscillator with Tail-Current

The present section is devoted to derivation of analytical equations for the time domain amplitude behavior of the well-known tail-current biased MOS cross-coupled LC oscillator shown in Fig. 5(a). All the published papers for estimating the amplitude of cross-coupled oscillators have not represented the acceptable accuracy for the amplitude transient of this type of oscillator because the cubic approximation for the \( I-V \) characteristic of the cross-coupled pair is used in these works and as is seen in Fig. 6, this approximation is not accurate while this paper presents an exact closed-form equation for the time-domain amplitude variation of this oscillator by using the \( \tan^{-1} \) approximation for the \( I-V \) characteristics.

The differential \( I-V \) characteristic of the composite one-port active part of this oscillator formed by the two cross-coupled MOS transistors is shown in Fig. 5(b). The behavior of this circuit can be efficiently described considering the nonlinear \( i_{doff}(V_d) \) characteristic in Fig. 5(b). As in the former case, the differential equation governing the circuit can be obtained by writing \( KCL \) at output nodes (\( \text{Vo}_1, \text{Vo}_2 \)). Now, writing \( KCL \) equations at outputs, the circuit describing equation can be written as (30).

As in the prior section, the differential oscillator under investigation is essentially equivalent to the second-order oscillator shown in Fig. 1. Using the standard model for the MOS transistor the differential \( I-V \) characteristics of the active part formed by the cross-coupled MOS transistors can be simply obtained [1] as (37) at the bottom of the page.

Where \( V_{eff} = \sqrt{2I_b|K_N|} \), \( K_N = \mu_NC_{OX}W/L \). Analyzing the circuit using the differential characteristics of (37) is very difficult and time consuming. Therefore, the differential characteristics of the active part should be approximated through a more simple function. In most cases like [1] and [12] this differential characteristic is approximated through a cubic function as (23) that leads to a simple analysis procedure but this approximation does not give an acceptable accuracy. Therefore, in this paper this differential characteristic is modeled through the \( \tan^{-1} \) function that leads to a simple and straightforward analysis along with excellent results. Therefore the differential \( I-V \) characteristic is estimated as (38)

\[
\begin{align*}
i_{doff}(V_d) = \begin{cases} 
-\frac{I_b}{V_{eff}} (V_d - V_i)^2 - I_b \frac{V_{eff}}{V_d} \sqrt{\left(1 - \frac{V_d^2}{V_{eff}^2}\right) \left(1 + \frac{2V_dV_i}{V_{eff}^2}\right)} & V_d < -V_i \\
-I_b \sqrt{\frac{V_d - V_i}{V_{eff}}} \left(1 - \frac{V_d^2}{V_{eff}^2}\right) & -V_i < V_d < V_i \\
\frac{I_b}{V_{eff}} (V_d - V_i)^2 - I_b \frac{V_{eff}}{V_d} \sqrt{\left(1 - \frac{V_d^2}{V_{eff}^2}\right) \left(1 + \frac{2V_dV_i}{V_{eff}^2}\right)} & V_d > V_i 
\end{cases}
\end{align*}
\]  

\[(37)\]

\[
i_{doff}(V_d) = b \tan^{-1}(cV_d)
\]  

\[(38)\]

**Fig. 5**  (a) Tail-current biased MOS cross-coupled LC oscillator (b) The differential \( I-V \) characteristics of the composite one port active part of this oscillator.

**Fig. 6** Comparison between the characteristic curve \( i_{doff}(V_d) \), obtained from simulation, the \( \tan^{-1} \) approximation and the best fitted cubic approximation.
Whose parameters are identified imposing the equality of the asymptotic values of current, which are ±Ib, and that the derivative at the origin be equal to −gm/2, we get

\[
b = -\frac{2I_b}{\pi}, \quad c = \frac{\pi g_m}{2I_b}
\] (39)

In (39), \(g_m = \sqrt{K_N I_b}\) for the configurations with long channel devices and \(g_m\) is defined as (28) for the configurations with short channel devices. Note that, (39) is true for both of topologies with long channel and short channel transistors and as is mentioned earlier, for short channel devices \(g_m\) is as (28). In Fig. 6, \(I_{ddiff}(V_d)\) obtained from simulation, cubic and tan−1 approximations are reported for a visual comparison.

Now, using (38) in differential equation of the circuit that has been achieved by writing KCL at output nodes, the following estimated describing differential equation governing this circuit will be obtained that enables us to have an accurate analysis on the circuit for achieving the time domain amplitude behavior.

\[
ds^2 V_d\ \frac{dt^2}{dt} + 1 \\frac{dV_d}{dt} \left[ \frac{1 - bc}{R + \frac{b}{1 + (ca \cos \omega t)^2}} \right] + V_d \frac{dV_d}{LC} = 0
\] (40)

Assuming \(V_d(t) = a(t) \cos \omega t\) as in the former case and assuming \(\omega = 1/\sqrt{LC}\) because of high quality factor (Q) assumption for the LC tank.

\[
eF(a \cos \omega t, \omega \sin \omega t) = \frac{a \omega \sin \omega t}{C \left[ \frac{1 - bc}{R + \frac{b}{1 + (ca \cos \omega t)^2}} \right]}
\] (41)

And \(eN_0(a)\) can be calculated using (13) as follows.

\[
eN_0(a) = \frac{\omega}{2\pi} \int_0^{\pi} \frac{a \omega \sin \omega t}{C \left[ \frac{1 - bc}{R + \frac{b}{1 + (ca \cos \omega t)^2}} \right]} dt
\]

\[
= \frac{\omega}{2\pi} \int_0^{\pi} \frac{a \omega \sin \omega t}{RC} \frac{2\pi}{\omega} \int_0^{\pi} \frac{ca \omega \sin \omega t}{1 + (ca \cos \omega t)^2} dt
\] (42)

The second term on the right hand side of (42) can be calculated by substituting \(\tan \omega t = u\).

\[
4b\omega \int_0^{\pi/2} \frac{ca \omega \sin^2 \omega t}{1 + (ca \cos \omega t)^2} dt = 4b\omega \int_0^{\infty} \frac{ca \omega du}{(1 + u^2)(1 + c^2 a^2 u^2)} = \frac{b \omega}{caC} \left( \sqrt{1 + (ca)^2} - 1 \right)
\] (43)

The first term on the right hand side of (42), as can be easily calculated, is equal to \(\omega a/2RC\). Therefore, in order to calculate \(a(t)\) it can be written

\[
\frac{da}{dt} = \frac{\omega a}{2RC} + \frac{b \omega}{caC} \left( \sqrt{1 + (ca)^2} - 1 \right)
\] (44)

Solving (44) is too time consuming and on the other hand it does not give a closed-form equation for transient amplitude of this type of oscillator but it can be used for calculating the steady state oscillation amplitude of the oscillator by choosing \(da/dt = 0\) and applying (39) values for \(b, c\) as follows.

\[
\frac{da}{dt} = 0 \Rightarrow a(\infty) = \frac{4RI_b}{\pi} \sqrt{R \left( \frac{1}{g_m} + R \right)}
\] (45)

Also, the estimate solution can be used for (44). As a good estimation, the solution of (44) can be estimated by the following function

\[
a(t) = a(\infty) - (1 - n)[a(\infty) - a(0)]e^{-\frac{\pi}{\tau}} - n[a(\infty) - a(0)]e^{-\frac{\pi}{\tau}}
\] (46)

The unknown parameters in (46) are \(n, \tau_1, \tau_2\) and \(\tau_2\) that can be achieved by substituting (46) into (44) and equating both sides in critical times of \(\tau_1, \tau_2\) and \(\tau_2\) that leaves the three beneficial equations in (47). There are just three unknowns in algebraic Eqs. (47.a), (47.b) and (47.c).

\[
a) \frac{0.368(1-n)[a(\infty) - a(0)]}{g_m + 2RC} e^{-\frac{\tau_1}{\tau_2}}
\]

\[
= \frac{\omega a(\tau_1)}{2RC} + \frac{b \omega a(cN_c \tau_1)}{caC} \left( \sqrt{1 + (ca(\tau_1)^2} - 1 \right)
\]

\[
b) \frac{1-n[a(\infty) - a(0)]}{g_m + 2RC} e^{-\frac{\tau_1}{\tau_2}} + 0.3682[a(\infty) - a(0)]
\]

\[
c) \frac{1-n[a(\infty) - a(0)]}{g_m + 2RC} e^{-\frac{\tau_1}{\tau_2}} + \frac{b \omega}{caC} \left( \sqrt{1 + (ca(\tau_2)^2} - 1 \right)
\]

Hence, \(\tau_1, \tau_2\) and \(n\) are obtained using a simple numerical solution method as follows.

\[
\frac{\tau_2}{2} = 3 \tau_1
\]

\[
1 = \left( \frac{g_m}{C} - \frac{2}{RC} \right)
\]

\[
n = 0.25
\]

Therefore the time-domain amplitude can be represented by the useful estimated closed-form equation of (51).

\[
a(t) = a(\infty) - 0.75[a(\infty) - a(0)]e^{-\frac{\pi a}{\tau_1}} - 0.25[a(\infty) - a(0)]e^{-\frac{\pi a}{\tau_2}}
\] (51)

As seen, (51) represents a very simple closed-form equation for the transient amplitude that is very useful for quick hand calculation of the amplitude variation.

It should be mentioned that because the tail current is in series with the supply voltage in this structure, the dynamic of the oscillator is somehow independent of \(V_{DD}\) but the value of the tail current is a function of \(V_{DD}\) itself in practical cases and from this we can investigate the dynamic response of amplitude under supply voltage variations through \(g_m\) parameter.

5. Simulations and Examples

To determine the validity of the amplitude equations derived here, (36) and (51), a test bench was created with CMOS differential cross-coupled oscillators using Advanced Design
System software (Agilent ADS). Three interesting examples are carefully expressed and simulation is used to compare the results from theoretical analysis and simulation of actual circuits. Example1, Example2 provide examples of long channel and short channel topology for the Van Der Pol oscillator respectively and Example3 provides an example of the oscillator with the tail current and short channel device (note that, as mentioned before, the procedure of the amplitude calculation for both of the long channel and short channel configurations is the same in this case). The simulations and design tests were done in a 1.8 V, 0.18-μm CMOS process.

EXAMPLE 1: Suppose a cross-coupled VDP oscillator has to be designed with $V_{DD} = 1.8$ V, $C = 2$ nF, $L = 2$ nH, $R = 1$ kΩ and a pair of NMOS transistors with $W/L = 100 \mu m/1 \mu m$ in 0.18-μm CMOS technology with $\mu n_{Cox} = 263 \times 10^{-6} A/V^2$, $V_t = 0.482$ V. For this circuit calculate the equation of transient amplitude.

As is mentioned before, the channel length ($L = 1 \mu m$), therefore we can use long-channel relation and

$$g_m = \mu n_{Cox} \frac{W}{L} (V_{CM} - V_t) = 96.092 \times 10^{-4} \text{ mho}$$

$$\Rightarrow B = -48.046 \times 10^{-4}$$

$$A = \frac{2K V_t + g_m}{8(V_{DD} - V_t)} = 5.323 \times 10^{-4}$$

And from (36), the equation for the transient amplitude can be written as follow

$$a(t) = \frac{30.046}{1.99612 - (1.99612 - \frac{60.098}{a_0}) \exp(-60.098 \times 10^5 t)}^{0.5}$$

(54)

Figure 7 shows the comparison between transient amplitude obtained from (54) and simulation of the actual circuit for different initial conditions ($a_0$). The calculated peak (steady-state) value of amplitude is 3.91 V that is very close to the simulated value that is 3.96 V.

EXAMPLE 2: Suppose a cross-coupled VDP oscillator has to be designed with $V_{DD} = 1.8$ V, $C = 1$ nF, $L = 1$ nH, $R = 1$ kΩ and a pair of NMOS transistors with $W/L = 20 \mu m/0.2 \mu m$ in 0.18-μm CMOS technology with $g_m = 90.548 \times 10^{-4} A/V^2$, $V_t = 0.482$ V. For this circuit calculate the equation of transient amplitude.

$$g_m = 90.548 \times 10^{-4} \text{ mho} \Rightarrow B = -45.274 \times 10^{-4}$$

$$A = \frac{2K V_t + g_m}{8(V_{DD} - V_t)} = 5.323 \times 10^{-4}$$

And from (36) we can write the equation for the transient amplitude as follows

$$a(t) = \frac{24.247}{1.358 - (1.358 - \frac{24.247}{a_0}) \exp(-45.274 \times 10^5 t)}^{0.5}$$

(57)

Figure 8 shows the comparison between transient amplitude obtained from (57) and simulation of the actual circuit for different initial conditions ($a_0$). The calculated peak (steady-state) value of amplitude is 4.12 V that is in a very good agreement with the simulated value that is 4.08 V.

EXAMPLE 3: Suppose a cross-coupled oscillator with the tail current has to be designed with $V_{DD} = 1.8$ V, $C = 2$ nF, $L = 2$ nH, $R = 3$ kΩ, $I_b = 2$ mA and a pair of NMOS transistors with $W/L = 20 \mu m/0.2 \mu m$ in 0.18-μm CMOS technology with $g_m = 6954 \times 10^{-6} A/V^2$, $V_t = 0.511$ V.

From (38) and (39) the estimated I-V characteristics of the cross-coupled MOS configuration will be calculated as

$$I_{diff}(V_d) = -1.273 \times 10^{-3} \tan^{-1}(5.46 V_d)$$

(58)

Also the calculated steady-state oscillation amplitude from (45) is 7.82 V that is in a very good agreement with the simulated value that is 7.79 V. Therefore the time-domain variation of the output oscillation amplitude can be written from (51) as
Fig. 8  Comparison between transient amplitude obtained from simulation and analysis result for the circuit in example 2. (a) $a_0 = 1$. (b) $a_0 = 1.5$.  (c) $a_0 = 2$.

\[
a(t) = 7.82 - 0.75(7.82 - a_0) \exp(-10.43 \times 10^3 t) \\
- 0.25(7.82 - a_0) \exp(-3.477 \times 10^3 t)
\]  

(59)

Figure 9 shows the comparison between transient amplitude obtained from (59) and simulation of the actual circuit for different initial conditions ($a_0$).

As is seen from simulations and comparisons, the proposed Eqs. (36) and (51) are in excellent agreement with simulation results.

6. Conclusion

MOS cross-coupled differential oscillators have been analyzed using a nonlinear method and a hand calculation technique that is useful as a preliminary design aid and also as a behavioral model in a top/down design methodology is proposed. The simple closed-form equations for the transient oscillation amplitude have been derived that the time-domain amplitude can be easily calculated from them. Also, these equations enable the designer to estimate the time needed for the circuit to reach the steady state and this helps to design the oscillators with faster response. The analysis is done using standard model of transistors and both of the configurations with short-channel devices or with long-channel devices were studied. The nonlinear characteristics of cross-coupled configuration are estimated by useful equations that made the modeling of the circuits more traceable and the validity of all derived equations have been investigated in some simulations and examples. The obtained equations are very useful to link between circuit parameters and circuit specifications and can be used for design problems to achieve the desired transient behavior.

References


