An elementary characterization of the Gini index

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Abstract

The Gini index is one of the most used indicators of social and economic inequality. In this paper we characterize the Gini index as the unique function that satisfies the properties of scale independence, symmetry, standardization and separability. Furthermore, we propose a simpler way to compute it.

Keywords: Gini index; income inequality; axiomatization.

JEL Classification: D31, D63, I31.

1 Introduction

In this article we offer a novel characterization of the Gini index\(^1\), exploiting the geometry of the contour lines of this index restricted to the unit simplex. In the case of three agents, these contour lines are hexagons whose edges are line segments on distribution sets that maintain a similar ordering (see Foster and Sen [4]). There are six permutations of the agents, which lead to the six edges of any contour line of the Gini index. The basic idea is to find appropriate axioms that take advantage of this fact. The treatment of extreme

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\(^1\)The coefficient attributed to Gini [5] has been presented and analyzed by Dalton [2], Atkinson [1] and Sen [6], among others. The discrete formula of the Gini coefficient as a standardized average of income differences is presented in several equivalent versions, a good explanation is available in the classic paper of Sen [6] or in the survey of Dutta [3]. Atkinson [1] provides the basis to discuss the relation between inequality and social welfare orders.
points of subsets of distributions that maintain the same ordering of incomes is essential. If we impose the condition that the index has a certain value at these extreme points, we can use it as a stepping stone to extend the index to the rest of the simplex by making use of symmetry and linearity arguments.

Our characterization is substantially different from others proposed in the literature and throughout this introduction we will discuss the main differences. The main idea of the characterization is the following. An inequality index is a real function whose domain is the set of vectors with nonnegative real entries. We impose four properties or axioms directly on functions that constitute potential inequality indices. Our first two axioms are well known. The first one, scale independence, allows us to restrict the domain of the index to the unit simplex in $\mathbb{R}^n$.2

A second property, symmetry or anonymity, is used to restrict our domain even more; it will be sufficient to define the index on sets of income distribution whose elements are non-decreasingly arranged. We say that two income distributions are comonotone if their elements are ordered in the same way—we give the precise definition in the next section. In other words, symmetry implies that the inequality index of any distribution is independent of the ordering of its entries.

The third axiom, standardization, allows us to define the index at the extreme points of these special sets. The coordinates of these extreme points may only take two possible values: zero or identical values, such that the coordinates add up to one. For the case of three individuals and distributions with $y_1 \leq y_2 \leq y_3$, the extreme points are $(1/3, 1/3, 1/3), (0, 1/2, 1/2)$ and $(0, 0, 1)$. From the standardization axiom we define the inequality index at extreme points, as the ratio of the number of individuals with no income divided by the total number of individuals.

Finally, the fourth axiom, separability3, extends linearly the inequality index, from the extreme points to the rest of the corresponding special set.

One of the main differences between the characterization offered in this work and alternative axiomatizations in the literature lies in the fact that we want to characterize a cardinal inequality index, whereas many authors look instead for a representation of an inequality ordering: the axiom of standardization is crucial in this respect. Among the advantages of our characterization we note the following: it works very well for the discrete case and it is simple; the axioms are imposed directly on indices and not on welfare orderings; there is no need to appeal to standard axioms of rationality (completeness and transitivity) and continuity to obtain numerical representations of welfare orderings.

A related work, closer to the approach followed in the paper, is the one by Thon [8]. This author proposes an axiomatization of the Gini coefficient in a general case. He works in a framework with income distributions, whose aggregate incomes and populations are not necessarily the same. Its axioms are imposed directly over the numerical function representing the inequality index, as in our case. But the axioms used by Thon are different from the ones used in this paper; although some of those axioms refer to the pure distribution case and have a clear interpretation from the welfare point of view, he has also taken into account considerations which transcend the pure distribution case, when comparing distributions with different populations and comparability of income distributions with the same population but different total incomes. Basically, those axioms show us other properties of the Gini index, more concerned with asymptotic properties.

2We restrict our attention to a fixed population of $n$ agents ($n \geq 2$).
3The precise definitions will be provided in Section 3.
when the size of the population increases. In our framework the population size is fixed and we explore algebraic distribution properties instead of asymptotic characteristics of the Gini coefficient.

The rest of the paper is organized as follows. In Section 2 we introduce our notation and recall some definitions, particularly that of comonotone distribution. The four axioms that we employ are presented in Section 3. We state in Section 4 our main result and proofs, plus some remarks and examples.

2 Framework and notation

Let \( N = \{1, 2, ..., n\} \) be a set of individuals. When the income distribution function is discrete, such incomes takes \( n \) values that can be denoted by a vector \( x \in \mathbb{R}_+^n \setminus \{0\},^4 \) where \( x_i \) denotes the income of the agent \( i \).

For a given \( x \in \mathbb{R}_+^n \), let \( x^* \) denote the vector obtained from \( x \) by rearranging its elements in a non-decreasing way; i.e., \( x_1^* \leq x_2^* \leq \cdots \leq x_n^* \). For example, if \( x = (9, 7, 0, 11, 3, 5, 9, 0) \), then \( x^* = (0, 0, 3, 5, 7, 9, 9, 11) \).

Now, the group of permutations of \( N \), \( S_n = \{ \theta : N \rightarrow N \mid \theta \text{ is bijective} \} \) acts on the space of income vectors \( \mathbb{R}_+^n \setminus \{0\} \) in a natural way; i.e., for \( \theta \in S_n \) and \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n \setminus \{0\} \), we define

\[
x^\theta := (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})^5
\]

Notice that for an arbitrary \( x \in \mathbb{R}_+^n \), there exists at least one permutation \( \theta \) such that \( x^\theta = x^* \).

Let \( x, y \in \mathbb{R}_+^n \setminus \{0\} \) be two income distributions. We say that \( x \) and \( y \) are comonotone if \( (x_i - x_j)(y_i - y_j) \geq 0 \), \( \forall i, j \in N \). In other words, two income distributions are comonotone if the income positions for every agent is the same in both distributions \( x^* \) and \( y^* \). For example, the income distributions \( x = (6, 0, 3, 18, 1, 24) \) and \( y = (11, 5, 10, 19, 7, 33) \) are comonotone; since for both distributions, agent 2 is the poorest one, agent 5 is the second poorest, agent 6 is the richest one, and so on.

An inequality index is a function \( I : \mathbb{R}_+^n \setminus \{0\} \rightarrow \mathbb{R} \) that assigns to each income vector a real number, which represents the society’s inequality level.

The attractiveness of the Gini index to many economists is that it has an intuitive geometric interpretation. That is, the Gini index can be defined geometrically as the ratio of the area that lies between the line of perfect equality and the Lorenz curve, over the total area under the line of equality.

Sen [6] defined the Gini index as a function \( G : \mathbb{R}_+^n \setminus \{0\} \rightarrow \mathbb{R} \) such that

\[
G(x) = \frac{1}{n} \left[ n + 1 \sum_{i=1}^{n} (n + 1 - i) x_i^* \right] \sum_{i=1}^{n} x_i^*
\]

(1)

This formulation illustrates the fact that the income-rank-based weights are inversely related to the incomes sizes. That is, the highest incomes get lower weights while the incomes of the poor get higher weights. Notice that for this discrete version, \( 0 \leq G(x) \leq \frac{n-1}{n} \) for every \( x \in \mathbb{R}_+^n \setminus \{0\} \).

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^4Here, we discard null distributions \( \mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n \) in the space of admissible income vectors.

^5This permutation implies a change of position of the individuals within the income distribution.
Example 1 Let the income distribution be \( x = (30, 20, 60, 10) \), then \( x^* = (10, 20, 30, 60) \) and

\[
G(x) = \frac{1}{4} \left[ 5 - 2 \cdot \frac{4(10) + 3(20) + 2(30) + 1(60)}{120} \right] = \frac{1}{3}
\]

This paper will show that the Gini index as it is defined above is uniquely determined by a certain set of properties, described in the next section.

3 The axioms

Next, we introduce a set of axioms which an inequality index is required to satisfy in this work.

The first axiom deals with scales; that is, if every person’s income \( x_i \) in an economy is multiplied by any positive constant, then inequality should not change (it is independent of the aggregate level of income). So, if the income is measured in other currencies, then the index remains the same.

Axiom 1 (Scale independence) The inequality index \( I \) is said to be scale-independent if

\[ I(\lambda x) = I(x) \]

for all \( x \in \mathbb{R}_+^n \setminus \{0\} \) and \( \lambda > 0 \).

The next axiom requires that the inequality index be independent of any characteristic, other than their incomes, of the individuals. More precisely,

Axiom 2 (Symmetry) The inequality index \( I \) is symmetric if and only if

\[ I(x^\theta) = I(x) \]

for every \( \theta \in S_n \) and \( x \in \mathbb{R}_+^n \setminus \{0\} \).

The previous two axioms are common and standard in the literature, they are satisfied by most of the well-known inequality indices. However, the next two axioms are new and crucial for our characterization of the Gini index.

Consider a society where there are only two types of individuals: rich people that share equally 1 unit of income and the poor with no income. The next property requires that in this case, the inequality index is equal to the proportion of individuals with no income. Let \( v^{k+1} = \left(0, \ldots, 0, \frac{1}{n-k}, \ldots, \frac{1}{n-k}, \frac{1}{n-k} \right) \), \( k = 0, ..., n-1 \), be the income vector of one such society.

Axiom 3 (Standardization) The inequality index \( I \) is said to be standardized if

\[ I(v^{k+1}) = \frac{k}{n} \]

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The reader may wonder about why we used the term *standardization*. The intuition behind the previous axiom is the following. Suppose we start with an income vector where 1 unit of income is equally divided among all individuals. Next, one agent is deprived of income and the unit is equally divided among the remaining agents. We are interested in the difference of inequality:

\[ I \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) - I \left( 0, \frac{1}{n-1}, \ldots, \frac{1}{n-1} \right) \]

Again in a second step, one more individual is deprived of income and we focus in the difference:

\[ I \left( 0, \frac{1}{n-1}, \ldots, \frac{1}{n-1} \right) - I \left( 0, 0, \frac{1}{n-2}, \ldots, \frac{1}{n-2} \right) \]

We repeat the process until we reach the distribution where exactly one agent have the entire income:

\[ \vdots \]

\[ I \left( 0, \ldots, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) - I \left( 0, \ldots, 0, \frac{1}{2}, \frac{1}{2} \right) \]

\[ I \left( 0, \ldots, 0, \frac{1}{2}, \frac{1}{2} \right) - I \left( 0, \ldots, 0, 1 \right) \]

What we required is that all such differences (of inequality) are equal to a common constant, such that \( I \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) = 0 \) and \( I \left( 0, 0, 0, 1 \right) = \frac{n-1}{n} \). Those conditions capture the idea of inequality in a very particular way by restricting attention to special distributions. To some extent, it is reminiscent of the idea of deprivation in cases where the extent of deprivation (or poverty) is assimilated with the percentage of the poor.

Finally, the next property represents an independence or separability condition\(^7\) that requires a specific treatment for comonotone income distributions. It enables an overall index value to be computed from subaggregates. These subaggregates share a common characteristic: they are comonotone income vectors with the same total income. In particular, the index applied to a mixture of two comonotone income vectors can be expressed as a weighted sum of the index value calculated for their components.

**Axiom 4 (Separability)** The inequality index \( I \) is separable if

\[ I [\beta x + (1 - \beta)z] = \beta I(x) + (1 - \beta)I(z) \]

for every \( \beta \in [0, 1] \), every \( x, z \in \mathbb{R}_+^n \setminus \{0\} \) that are comonotone and \( \sum_{i=1}^n x_i = \sum_{i=1}^n z_i \).

It is important to notice that an index satisfying the last axiom, has a separability property only on sets of comonotone distributions with the same total income\(^8\). For

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\(^6\)Thus, one can think that such axiom has a component of normalisation and a component of constant inequality changes.

\(^7\)Weaker than the Strong Separability Condition used in the Expected Utility model (Yaari [10,11]), since it applies to a restricted set of distributions: precisely, to those that are comonotone.

\(^8\)However, we should note that there are some implications based on this axiom for comonotone distributions with different total incomes. For example, if \( x \) and \( y \) are comonotone distributions with total income \( a \) and \( b \), respectively; then from axioms of scale independence and separability, we have \( I(x + y) = I \left( \frac{a}{a+b} x + \frac{b}{a+b} y \right) = \frac{a}{a+b} I(x) + \frac{b}{a+b} I(y) \), which could have an intuitive interpretation. Such an index of the sum of two distributions are affected each by a factor of the corresponding proportion of the sum of total incomes \( a + b \).
example, suppose that $I$ is an index satisfying the above axioms and that we have a situation where there are four individuals with income distributions (with same total income) $x = (2, 2, 0, 0)$ and $z = (0, 0, 2, 2)$. Then we have that $I(x) = I(z) = \frac{1}{2}$ and $I\left(\frac{1}{2}x + \frac{1}{2}z\right) = I(1, 1, 1, 1) = 0$. Thus $I\left(\frac{1}{2}x + \frac{1}{2}z\right) \neq \frac{1}{2}I(x) + \frac{1}{2}I(z)$, which is compatible with Axiom 4 since $x$ and $z$ are not comonotone.

On the other hand, take two comonotone income vectors (for two individuals) $x = (0, 10)$ and $z = (20, 20)$. In this case we get $I(x) = \frac{1}{2}$, $I(z) = 0$ and $I\left(\frac{1}{2}x + \frac{1}{2}z\right) = I(4, 12) = I\left(\frac{1}{2}(8, 8) + \frac{1}{2}(0, 16)\right) = \frac{1}{4}$. Then $I\left(\frac{1}{2}x + \frac{1}{2}z\right) \neq \frac{1}{2}I(x) + \frac{1}{2}I(z)$, which is again compatible with Axiom 4 since $x$ and $z$ have different total income.

4 The main result

This section is devoted to the characterization of the Gini index by means of the previous axioms, which constitutes the main result of this work. First of all, we shall present some preliminary results that are used in the proof of our main result.

The key idea is to show that axioms 3 and 4 characterize the Gini index on the convex subset

$$K = \left\{ y \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} y_i = 1 \text{ and } y_1 \leq y_2 \leq \cdots \leq y_n \right\}.$$  

Then axioms 1 and 2, allow us to extend the Gini index on the whole space of income vectors $\mathbb{R}^n_+ \setminus \{0\}$. To wit, let $I : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}$ be a symmetric and scale-independent index. For $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists $\theta \in S_n$ such that $x_{\theta(1)} \leq x_{\theta(2)} \leq \cdots \leq x_{\theta(n)}$ and hence $\frac{1}{\sum_{i=1}^{n} x_i} \left(x^\theta\right) \in K$. Therefore,

$$I(x) = I(x^\theta) = I\left[ \frac{1}{\sum_{i=1}^{n} x_i} \left(x^\theta\right) \right].$$

For subsequent developments, we need to express each element of $K$ as a convex combination of the vectors $\{v^j\}_{j=1}^{n}$, which are the extreme points of $K$ and are given by

$$v^j_i = \begin{cases} \frac{1}{n-i+1} & \text{if } j \leq i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

We know that such decomposition always exists by Carathéodory’s Theorem, in convex analysis. The precise values of the components in the decomposition are indicated in the following lemma.

**Lemma 1** For every $y \in K$, there exist unique non-negative scalars $\{\alpha_j\}_{j=1}^{n}$ with $\sum_{j=1}^{n} \alpha_j = 1$, such that $y$ decomposes

$$y = \sum_{j=1}^{n} \alpha_j v^j \quad (3)$$

Moreover, the scalars are given by

$$\alpha_j = (n-j+1)(y_j - y_{j-1})$$

for $j = 1, 2, \ldots, n$, where $y_0 = 0$. The $\alpha_j$’s are called the varycentric coordinates of $y$ with respect to $\{v^j\}$.

*Note that if $y \in K$, then $y^* = y$.  

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Proof. Let \( V \) be a \( n \times n \) matrix with entries \( v^j_i \), and let \( \alpha \) be the \( 1 \times n \) matrix \( \alpha^T = [\alpha_1 \ldots \alpha_n]^T \). Notice that \( y = \sum_{j=1}^n \alpha_j v^j_i \) can be represented by the system of equations \( y = V \alpha \). Since \( V \) is non-singular, then the scalars \( \{\alpha_j\}_{j=1}^n \) are unique.

Now, for arbitrary \( y \in K \):

\[
y_i - y_{i-1} = \sum_{j=1}^n \alpha_j v^j_i - \sum_{j=1}^{i-1} \alpha_j v^j_{i-1} = \sum_{j=1}^i \frac{\alpha_j}{n-j+1} - \sum_{j=1}^{i-1} \frac{\alpha_j}{n-j+1} = \alpha_i \frac{n-i+1}{n-i+1}
\]

Hence,

\[
\alpha_i = (n-i+1)(y_i - y_{i-1})
\]

and they are not negative since \( y \in K \).

For the other direction:

\[
\sum_{j=1}^n \alpha_j v^j_i = \sum_{j=1}^i (n-j+1)(y_j - y_{j-1}) \frac{1}{n-j+1} = \sum_{j=1}^i y_j - \sum_{j=1}^{i-1} y_{j-1} = y_i
\]

Finally, those scalars add up 1:

\[
\sum_{j=1}^n \alpha_j = \sum_{j=1}^n (n-j+1)(y_j - y_{j-1}) = \sum_{j=1}^n (n-j+1)y_j - \sum_{j=0}^{n-1} (n-j)y_j = \sum_{j=1}^n y_j = 1
\]

Lemma 2 Let \( w^j \in K, j = 1, \ldots, m \) and \( \beta_j \geq 0 \) be such that \( \sum_{j=1}^m \beta_j = 1 \). If \( I : \mathbb{R}_+^n \setminus \{0\} \to \mathbb{R} \) verifies separability, then

\[
I \left( \sum_{j=1}^m \beta_j w^j \right) = \sum_{j=1}^m \beta_j I(w^j)
\]

Proof. First, it is clear that every pair of elements in \( K \) are comonotone, since by definition they all are ordered increasingly.

The proof is done by induction on \( m \). Assume that the statement holds for \( k < m \):

\[
I \left( \sum_{j=1}^k \beta_j w^j \right) = \sum_{j=1}^k \beta_j I(w^j)
\]
for \( w^j \in K, j = 1, \ldots, k \) and \( \beta_j \geq 0 \) are such that \( \sum_{j=1}^k \beta_j = 1 \).

Now, let \( w^j \in K, j = 1, \ldots, k+1 \) and let \( \gamma_j \geq 0 \) be such that \( \sum_{j=1}^{k+1} \gamma_j = 1 \) (assume \( \gamma_{k+1} < 1 \)). Since \( I \) is separable:

\[
I \left( \sum_{j=1}^{k+1} \gamma_j w^j \right) = I \left( \sum_{j=1}^k \gamma_j w^j + \gamma_{k+1} w^{k+1} \right) = (1 - \gamma_{k+1})I \left( \sum_{j=1}^k \frac{\gamma_j}{1 - \gamma_{k+1}} w^j \right) + \gamma_{k+1} I(w^{k+1})
\]

Notice that \( \frac{\gamma_j}{1 - \gamma_{k+1}} \geq 0 \) and \( \sum_{j=1}^k \frac{\gamma_j}{1 - \gamma_{k+1}} = 1 \). Thus, by the induction hypothesis:

\[
I \left( \sum_{j=1}^{k+1} \gamma_j w^j \right) = (1 - \gamma_{k+1}) \sum_{j=1}^k \frac{\gamma_j}{1 - \gamma_{k+1}} I(w^j) + \gamma_{k+1} I(w^{k+1}) = \sum_{j=1}^{k+1} \gamma_j I(w^j)
\]

Note that the scale independence and standardization axioms determine the inequality index on the extreme points of \( K \), whereas separability, extends the index on the rest of \( K \). Finally, symmetry extends \( I \) on all the simplex from its values on \( K \).

We are now ready to state our main result.

**Theorem 1**  Let \( I : \mathbb{R}^n_+ \setminus \{0\} \rightarrow \mathbb{R} \). Then, \( I \) equals the Gini index, given by (1), if and only if it satisfies the axioms of scale independence, symmetry, standardization and separability.

**Proof.** It is straightforward to prove that the Gini index given by (1) satisfies the four properties.

For the converse, let \( I : \mathbb{R}^n_+ \setminus \{0\} \rightarrow \mathbb{R} \) be any index satisfying the four axioms and let \( z \in \mathbb{R}^n_+ \setminus \{0\} \) be an income vector. Define \( y_i = \frac{z_i^2}{\sum_{j=1}^n z_j} \). It is clear that \( y \in K \) and so, it can be decomposed as \( y = \sum_{j=1}^n \alpha_j v^j \) for unique real numbers \( \{\alpha_j \geq 0\} \) with \( \sum_{j=1}^n \alpha_j = 1 \).

Thus, using the symmetry and scale independence axioms, and appealing to Carathéodory’s Theorem, we have:

\[
I(z) = I(z^*) = I(y) = I \left( \sum_{i=1}^n \alpha_i v^i \right)
\]

Next, invoking Lemmas 2 and 1, we obtain

\[
I(z) = \sum_{i=1}^n \alpha_i I(v^i) = \sum_{i=1}^n (n - i + 1)(y_i - y_{i-1})I(v^i)
\]
By the axiom of standardization, we get:

\[
I(z) = \sum_{i=1}^{n} (n - i + 1)(y_i - y_{i-1}) \frac{i-1}{n} \\
= \sum_{i=1}^{n} \frac{(n - i + 1)(i - 1)}{n} y_i - \sum_{i=2}^{n} \frac{(n - i + 1)(i - 1)}{n} y_{i-1} \\
= \sum_{i=1}^{n} \frac{(n - i + 1)(i - 1)}{n} y_i - \sum_{i=1}^{n-1} \frac{(n - i)}{n} y_i \\
= \frac{n - 1}{n} y_n + \sum_{i=1}^{n-1} \left[ \frac{(n - i + 1)(i - 1)}{n} - \frac{(n - i)}{n} \right] y_i \\
= \frac{n - 1}{n} y_n + \sum_{i=1}^{n-1} \frac{2i - n - 1}{n} y_i = \sum_{i=1}^{n} \frac{2i - n - 1}{n} y_i \\
= \frac{1}{n} \left[ -n - 1 + 2 \sum_{i=1}^{n} iy_i \right] = \frac{1}{n} \left[ -n - 1 + 2 \sum_{i=1}^{n} i \left( \frac{z_i^*}{\sum_{j=1}^{n} z_j} \right) \right] \\
= \frac{1}{n} \left[ n + 1 - 2 \sum_{i=1}^{n} \frac{z_i^*}{\sum_{j=1}^{n} z_j} \right] \\
= G(z)
\]

which is precisely (1). ■

Note that, an index that verifies Axioms 1, 2 and 4, can be written as \( I(z) = \sum_{i=1}^{n} \alpha_i I(v^i) \). Thus, different ways of assigning the value of \( I(v^i) \) can lead to different indices. We can redefine the Standardization Axiom and obtain other possible indices. For example, we could give a measure of “poverty” by setting \( I(v^{k+1}) = [k - (n - k)] \) (# of poor people – # of rich people). Such an index is given by \( I(z) = \sum_{i=1}^{n} (-3n - 2 + 4i)y_i \).

**Example 2** As an illustration of the above derivation, we compute the Gini index of Example 1.

For the income distribution \( z = (30, 20, 60, 10) \), we have \( y = \left( \frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2} \right) \). Furthermore,

\[
y = \left( \frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2} \right) = \frac{1}{12} v^1 + \frac{1}{6} v^2 + \frac{1}{4} v^3 + \frac{1}{2} v^4 \\
= \frac{1}{3} \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 1 \right) + \frac{1}{4} \left( 0, \frac{1}{3}, \frac{1}{3}, 1 \right) + \frac{1}{6} \left( 0, 0, \frac{1}{2}, 1 \right) + \frac{1}{2} (0, 0, 0, 1)
\]

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10 This index reflects in some way an effect of extreme poverty and takes values in \([-n, n - 2]\). When there are an even number of individuals, half with no income and half with the total income equally distributed, the index is equal to 0. We are in a situation of latent conflict because both groups are large and antagonistic, a kind of maximum polarization. The greatest value of the index is \( n - 2 \), occurs when all individuals have zero income, except for one individual. It is the maximum effect of poverty; however, we do not necessarily associate this case with conflict as it would require the organization and coordination of the dispossessed to generate conflict. The lowest value in the index is \(-n\), where the distribution of income is in an egalitarian way, there is no extreme poverty, it is the smallest effect of it.
and hence,
\[ G(z) = G(y) = \frac{1}{3} G(v^1) + \frac{1}{4} G(v^2) + \frac{1}{6} G(v^3) + \frac{1}{4} G(v^4) \]
\[ = \frac{1}{3} (0) + \frac{1}{4} \left( \frac{1}{4} \right) + \frac{1}{6} \left( \frac{1}{2} \right) + \frac{1}{4} \left( \frac{3}{4} \right) \]
\[ = \frac{1}{3} \]

The proof of Theorem 1 suggests a simple way to compute the Gini index. That is, for an arbitrary income vector \( z \in \mathbb{R}_+^n \setminus \{0\} \), it is shown that the Gini index is obtained from
\[ G(z) = \sum_{i=1}^{n} \gamma_i y_i \]  
(4)
where \( \gamma_i = \frac{2i-n-1}{n} \) and \( y_i = \frac{z_i}{\sum_{j=1}^{n} z_j} \).

In order to compute the coefficients \( \{\gamma_i\}_{i=1}^{n} \), it is sufficient to calculate half of them, since the relation \( \gamma_i + \gamma_{n+1-i} = 0 \) holds for every \( i = 1,2,\ldots,n \). When \( n \) is an odd number, it turns out that the central coefficient \( \gamma_{\frac{n+1}{2}} \) vanishes.

**Example 3** Let us give the precise values of coefficients \( \gamma_i \) for the case \( n = 7 \):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \gamma_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( -\frac{6}{7} )</td>
</tr>
<tr>
<td>2</td>
<td>( -\frac{4}{7} )</td>
</tr>
<tr>
<td>3</td>
<td>( -\frac{2}{7} )</td>
</tr>
<tr>
<td>4</td>
<td>( 0 )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{2}{7} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{4}{7} )</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{6}{7} )</td>
</tr>
</tbody>
</table>

**Example 4** As a final example, we present the computation of the Gini index from expression (4) for the income distribution \( z = (75, 5, 10, 20, 6, 58, 6) \).

According to the previous discussion, \( y = \left( \frac{1}{36}, \frac{1}{36}, \frac{1}{36}, \frac{1}{18}, \frac{1}{5}, \frac{29}{90}, \frac{5}{12} \right) \) and we then get:
\[ G(z) = -\frac{6}{7} \left( \frac{1}{36} \right) - \frac{4}{7} \left( \frac{1}{30} \right) - \frac{2}{7} \left( \frac{1}{30} \right) + \frac{2}{7} \left( \frac{1}{9} \right) + \frac{4}{7} \left( \frac{29}{90} \right) + \frac{6}{7} \left( \frac{5}{12} \right) \]
\[ = \frac{164}{315} \approx 0.5206 \]

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\[ ^{11} \text{Silber [7] proposed an equivalent expression to compute the Gini coefficient:} \]
\[ G(z) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{z_i}{\sum_{j=1}^{n} z_j} \left( \frac{n-j}{n} \right) \left( \frac{j-1}{n} \right) \]

where \( z \) is arranged in a non-increasing order. Xu [9] noted that by simply reversing the order and making use of \( i = n-j+1 \), then the resulting expression is exactly the one given by (4).
References


