Abstract—In this paper, we investigate the correlation properties of the squared envelope of a class of autocorrelation-ergodic (AE) sum-of-cisoids (SOC) simulation models for mobile Rayleigh fading channels. Novel closed-form expressions are presented for both the ensemble and the time-averaged autocorrelation functions (ACFs) of the simulation model's squared envelope. Basing on those expressions, we show that under certain conditions, the squared envelope of the SOC model is itself an AE random process. In addition, we evaluate the performance of three fundamental methods for the computation of the model's parameters—namely the generalized method of equal areas (GMEA), the $L_p$-norm method (LPNM), and the Riemann sum method (RSM)—regarding their accuracy for emulating the squared envelope ACF of the channel. The obtained results can be used as a basis to design efficient simulators for the performance analysis of mobile communication systems sensitive to the correlation properties of the channel's squared envelope.

Keywords—Channel simulators, ergodic processes, mobile communications, Rayleigh fading channels, squared envelope, sum-of-cisoids, sum-of-sinusoids.

I. INTRODUCTION

Simulation models basing on a finite sum of complex sinusoids (cisoids) have widely been in use in the literature as a basis for the design of efficient simulators for mobile radio channels, e.g., see [1]–[5]. Applications of sum-of-cisoids (SOC) models span from the simulation of narrowband single-input single-output (SISO) channels [1], [2], to the development of narrowband [3] and wideband [4], [5] multiple-input multiple-output (MIMO) fading channel simulators. SOC models are able to produce complex-valued waveforms with inphase and quadrature (IQ) components having specified autocorrelation and cross-correlation properties [6, Ch. 3]. Owing to this characteristic, the SOC models can be applied to the simulation of fading channels characterized not only by symmetrical Doppler power spectral densities (DPSDs), but also by asymmetrical DPSDs. This is a noteworthy feature, as it has been observed from measured data that the DPSD of real-world channels is in general asymmetrical [7]–[9].

Based on the random or deterministic nature of the cisoids' parameters—gains, Doppler frequencies, and phases, we can identify eight basic classes of SOC simulation models [10]: Seven classes of stochastic SOC models and one class of deterministic SOC models, with the latter class being a superset of the other seven. Among the seven classes of stochastic SOC models, only the class comprising cisoids with constant gains, constant Doppler frequencies, and random phases enables the design of autocorrelation-ergodic (AE) channel simulators [6, Ch. 3]. The autocorrelation-ergodicity property of such a class of SOC models is highly convenient, as it allows to efficiently approximate the channel’s autocorrelation function (ACF) over a given range of interest without the need of averaging across multiple simulation runs.

Some important statistical properties of AE SOC simulators for Rayleigh fading channels have been studied in [11] and [12]. The autocorrelation and spectral characteristics of the underlying SOC model, as well as the probability density functions (PDFs) of its envelope and phase, are analyzed in [11]. The level-crossing rate (LCR) and the average duration of fades (ADF) of the model’s envelope are investigated in [12]. Despite its relevance, the correlation properties of the squared envelope of AE SOC models have not been studied so far. Closing this gap is necessary not only for a better characterization of the simulator, but also to carry out a reliable software-assisted performance evaluation of systems sensitive to the squared envelope ACF of the channel, such as the systems described in [13]–[16]. It is the aim of this paper to shed some light on the correlation properties of the squared envelope of AE SOC simulation models for mobile Rayleigh fading channels.

The outline of the paper is as follows. In Section II, we review some relevant statistical properties of a mobile Rayleigh fading channel model that we will adopt as a reference model throughout the paper. In Section III, we analyze the correlation characteristics of the squared envelope of an AE SOC model that is well suited for the simulation of the reference model. Novel closed-form expressions are presented in that section for the ensemble and the time-averaged ACFs of the simulation model’s squared envelope. In Section IV, we evaluate the performance of three fundamental methods for the computation of the SOCs’ parameters—namely the generalized method of equal areas (GMEA) [1], the $L_p$-norm method (LPNM) [4], and the Riemann sum method (RSM) [2]—with respect to their accuracy for emulating the squared envelope ACF of the reference model. Finally, we summarize our conclusions in Section V.
II. THE REFERENCE MODEL

The complex envelope of our reference narrowband mobile Rayleigh fading channel model is mathematically represented in the equivalent baseband by a complex random process

$$\mu(t) = \mu(t) + j\mu_Q(t), \quad j = \sqrt{-1}$$

(1)

where $\mu(t)$ and $\mu_Q(t)$ are stationary zero-mean Gaussian processes with variance $\sigma_{\mu}^2/2$. In line with the two-dimensional scattering propagation model proposed by Clarke [17], we define the random processes in (1) such that their ACFs and cross-correlation functions (CCFs) satisfy the equations:

$$r_{\mu_1\mu_1}(\tau) = r_{\mu_2\mu_2}(\tau)$$

$$= \sigma_{\mu}^2 \int_{0}^{\pi} g_\alpha(\alpha) \cos(2\pi f_{\max} \cos(\alpha) \tau) d\alpha$$

(2)

$$r_{\mu_1\mu_Q}(\tau) = r_{\mu_2\mu_Q}(\tau)$$

$$= \sigma_{\mu}^2 \int_{0}^{\pi} g_\alpha(\alpha) \sin(2\pi f_{\max} \cos(\alpha) \tau) d\alpha$$

(3)

$$r_{\mu\mu}(\tau) = 2 \{ r_{\mu_1\mu_1}(\tau) + j r_{\mu_1\mu_Q}(\tau) \}$$

$$= 2 \sigma_{\mu}^2 \int_{0}^{\pi} g_\alpha(\alpha) \exp(j2\pi f_{\max} \cos(\alpha) \tau) d\alpha$$

(4)

where $r_{xy} = E\{x(t)y(t+\tau)\}$, with $x(t)$ and $y(t)$ denoting two arbitrary random processes. The operators $E\{\cdot\}$ and $\cdot^*$ indicate statistical expectation and complex conjugate, respectively. In (2)–(4), $f_{\max}$ designates the maximum Doppler shift experienced by the channel’s multipath components, and $g_\alpha(\alpha) \triangleq \left[ p_\alpha(\alpha) + p_\alpha(-\alpha) \right]/2$ is the even part of the probability density function (PDF) $p_\alpha(\alpha)$ characterizing the angle-of-arrival (AOA) statistics of the channel.

We can observe from (2)–(4) that if the IQ components of $\mu(t)$ are uncorrelated, meaning that $r_{\mu_1\mu_Q}(\tau) = r_{\mu_2\mu_Q}(\tau) = 0$, then the reference model’s ACF $r_{\mu\mu}(\tau)$ is a real-valued even function. On the other hand, if $\mu_1(t)$ and $\mu_Q(t)$ are mutually correlated, then $r_{\mu\mu}(\tau)$ is a complex-valued hermitian symmetric function. Based on the properties of the Fourier transform [18, Sec. 3.6], we can further observe that if $\mu_1(t)$ and $\mu_Q(t)$ are uncorrelated, then the channel’s DPSD $S_{\mu\mu}(f) \triangleq \int_{-\infty}^{\infty} r_{\mu\mu}(\tau) \exp(-j2\pi f \tau) d\tau$ is symmetrical with respect to the origin. Otherwise, if the IQ components of $\mu(t)$ are cross-correlated, then the DPSD $S_{\mu\mu}(f)$ is asymmetrical.

With respect to the squared envelope $\zeta^2(t) \triangleq |\mu(t)|^2$ of the reference model, one can easily show that $\zeta^2(t)$ is a wide-sense stationary (WSS) process with mean value equal to $E\{\zeta^2(t)\} = \sigma_{\zeta}^2$ and ACF given as [6, Appx. B]

$$r_{\zeta^2\zeta^2}(\tau) = \sigma_{\zeta}^4 + 4 \{ r_{\mu_1\mu_1}(\tau) + r_{\mu_1\mu_Q}(\tau) \}$$

$$= \sigma_{\mu}^4 + 4 r_{\mu\mu}(\tau)^2.$$  

(5)

The operator $|\cdot|$ denotes the complex absolute value.

III. THE SOC SIMULATION MODEL

A. THE STOCHASTIC SOC SIMULATION MODEL

The AE SOC simulation model under analysis is characterized by a random process of the form

$$\tilde{\mu}(t) = \sum_{n=1}^{N} \tilde{c}_n \exp \{ j(2\pi f_n t + \tilde{\theta}_n) \}.$$  

(6)

The random phases $\tilde{\theta}_n$, introduced above are assumed to be mutually independent and uniformly distributed over $[-\pi, \pi]$, the gains $\tilde{c}_n$ satisfy $\sum_{n=1}^{N} \tilde{c}_n^2 = \sigma_{\mu}^2$, and the Doppler frequencies $f_n$ are defined as $f_n \triangleq f_{\max} \cos(\alpha_n)$, where $\alpha_n \in [-\pi, \pi]$, $n = 1, 2, \ldots, N$. The numerical results presented in [1], [2], and [11] indicate that the correlation properties, the spectral characteristics, and the PDFs of the envelope and phase of the reference model can efficiently be approximated by means of the SOC model defined in (6).

Some important first-order and second-order statistics of $\tilde{\mu}(t)$ are analyzed in [11] and [12]. For the purposes of this paper, it is only relevant to know that if the Doppler frequencies $f_n$ satisfy the inequalities

$$f_n \neq 0, \quad \forall n$$

$$f_n \neq f_m, \quad n \neq m$$

then $\tilde{\mu}(t)$ is a zero-mean WSS process with variance $\sigma_{\tilde{\mu}}^2$ for which the correlation properties can be summarized as [11]:

$$r_{\tilde{\mu}_1\tilde{\mu}_1}(\tau) = r_{\tilde{\mu}_2\tilde{\mu}_2}(\tau)$$

$$= \sum_{n=1}^{N} \tilde{c}_n^2 \cos(2\pi f_n \tau)$$

(7a)

$$r_{\tilde{\mu}_1\tilde{\mu}_Q}(\tau) = -r_{\tilde{\mu}_Q\tilde{\mu}_1}(\tau)$$

$$= \sum_{n=1}^{N} \tilde{c}_n^2 \sin(2\pi f_n \tau)$$

(7b)

$$r_{\tilde{\mu}\tilde{\mu}}(\tau) = 2 \{ r_{\tilde{\mu}_1\tilde{\mu}_1}(\tau) + j r_{\tilde{\mu}_1\tilde{\mu}_Q}(\tau) \}$$

$$= \sum_{n=1}^{N} \tilde{c}_n^2 \exp \{ j2\pi f_n \tau \}.$$  

(10)

In the previous equations, $\tilde{\mu}_1(t) = \Re\{\tilde{\mu}(t)\}$ and $\tilde{\mu}_Q(t) = \Im\{\tilde{\mu}(t)\}$ are the IQ components of $\tilde{\mu}(t)$.

For a proper simulation of fading channels characterized by symmetrical DPSDs, the SOC model in (6) has to be parameterized in such a way that its IQ components are mutually uncorrelated, implying that $r_{\tilde{\mu}_1\tilde{\mu}_Q}(\tau) = r_{\tilde{\mu}_Q\tilde{\mu}_1}(\tau) = 0$. Assuming the fulfillment of the inequalities in (7), we can deduce from (9) that the random processes $\tilde{\mu}_1(t)$ and $\tilde{\mu}_Q(t)$ are uncorrelated if and only if (iff):

**Condition 1:** The number of cosines $N$ is even, i.e., $N = 2M$, where $M$ is a positive integer, and for each pair of parameters $(\tilde{c}_n, f_n)$, there exists one and only one pair $(\tilde{c}_m, f_m)$, such that $\tilde{c}_n = \tilde{c}_m$ and $f_n = -f_m$ hold for $n \neq m$ and $n, m = 1, 2, \ldots, N$.

The previous condition has a strong influence on the time-averaged ACF of the simulation model’s squared envelope.
\(\hat{\xi}^2(t) \triangleq |\hat{\mu}(t)|^2\), as will be shown in Subsection III-D. Before we proceed to study the time-averaged characteristics of \(\hat{\xi}^2(t)\), let us analyze first the correlation properties of the ensemble.

**B. Ensemble ACF of the SOC Model’s Squared Envelope**

From the definition of the ACF \(r_{\hat{\xi}\hat{\xi}}(\tau)\) of the simulation model’s squared envelope \(\hat{\xi}^2(t)\), we have

\[
r_{\hat{\xi}\hat{\xi}}(\tau) = E\{(|\hat{\mu}(t)|^2|\hat{\mu}(t+\tau)|^2)\}
= \sum_{l=1}^{N} \sum_{m=1}^{N} \sum_{n=1}^{N} \hat{c}_l \hat{c}_m \hat{c}_n \hat{c}_p
\times \exp \{j2\pi(\hat{f}_l - \hat{f}_p)\tau\}
\times \exp \{j2\pi(\hat{f}_n - \hat{f}_p)(t + \tau)\}
\times E\{j(\hat{\theta}_l - \hat{\theta}_m + \hat{\theta}_n - \hat{\theta}_p)\}. \tag{11}\]

Since the random phases \(\hat{\theta}_n\) are mutually independent and uniform over \([-\pi, \pi)\), the expectation in (11) is different from zero only when: \(l = m = n = p, l = m, n = p, l \neq n;\) and \(l = p, m = n, l \neq m\). Bearing this in mind, we obtain

\[
r_{\hat{\xi}\hat{\xi}}(\tau) = \sum_{l=1}^{N} c_l^4 + \sum_{m=1}^{N} \sum_{n=1}^{N} c_m^2 c_n^2
\times \exp \{j2\pi \hat{f}_p \tau\} \exp \{j2\pi \hat{f}_n \tau\}. \tag{12}\]

The previous result can be rearranged as follows

\[
r_{\hat{\xi}\hat{\xi}}(\tau) = \sum_{l=1}^{N} c_l^4 + \left[ \sum_{m=1}^{N} c_m^2 \right]^2 - \sum_{n=1}^{N} c_n^4
\times \exp \{j2\pi \hat{f}_p \tau\} \exp \{j2\pi \hat{f}_n \tau\} - \sum_{p=1}^{N} c_p^4. \tag{13}\]

Taking into account that \(\sum_{p=1}^{N} c_p^2 = \sigma_{\mu}^2\), and given that \(r_{\hat{\mu}\hat{\mu}}(\tau) = \sum_{n=1}^{N} c_n^2 \exp \{j2\pi \hat{f}_n \tau\}\), we can finally write

\[
r_{\hat{\xi}\hat{\xi}}(\tau) = \sigma_{\mu}^4 + |r_{\hat{\mu}\hat{\mu}}(\tau)|^2 - \sum_{n=1}^{N} c_n^4. \tag{14}\]

One can easily verify that the mean value of \(\hat{\xi}^2(t)\) is equal to \(E\{\hat{\xi}^2(t)\} = \sigma_{\mu}^2\). Thus, we can conclude that the value of \(\hat{\xi}^2(t)\) is a WSS process, since its mean value is constant over time and its ACF \(r_{\hat{\xi}\hat{\xi}}(\tau)\) is time-shift insensitive [cf. (14)]. In addition, by comparing (14) with (5), we can observe that in order to accurately emulate the squared envelope ACF \(r_{\hat{\xi}\hat{\xi}}(\tau)\) of the reference model, the SOC model in (6) has to be parameterized in such a way that \(r_{\hat{\mu}\hat{\mu}}(\tau) \approx r_{\hat{\mu}\hat{\mu}}(\tau)\) and \(\sum_{n=1}^{N} c_n^4 \approx 0\). Several different parameter computation methods that render a good approximation to \(r_{\hat{\mu}\hat{\mu}}(\tau)\) have been proposed in the literature, such as those described in [1] and [2]. However, none of the existing methods is designed to minimize the factor \(\sum_{n=1}^{N} c_n^4\). For this reason, a large number of cisoids, say \(N \geq 50\), has to be considered to ensure that the value of such a factor is negligible.

**C. The Deterministic SOC Simulation Model**

In practice, the simulation of \(\mu(t)\) is performed by generating sample functions of \(\hat{\mu}(t)\). The output of the simulator can therefore be represented by a deterministic process

\[
\hat{\mu}^{(k)}(t) = \sum_{n=1}^{N} \hat{c}_n \exp \{j(2\pi \hat{f}_n t + \hat{\theta}_n^{(k)})\} \tag{15}\]

where \(k\) is a positive integer and \(\hat{\theta}_n^{(k)}\) is the outcome of \(\hat{\theta}_n\), associated to the \(k\)th sample function of \(\hat{\mu}(t)\). The time-averaged characteristics2 of \(\hat{\mu}^{(k)}(t)\) are investigated in [1], [6]. Here, it is only important to know that if the inequalities in (7) hold, then

\[
r_{\hat{\mu}\hat{\mu}}^{(k)}(\tau) = \sum_{n=1}^{N} \hat{c}_n \exp \{j2\pi \hat{f}_n \tau\}, \forall k \tag{16}\]

where \(r_{\hat{g}\hat{g}}(\tau) \triangleq \langle x^*(t)g(t+\tau)\rangle\), with \(x(t)\) and \(g(t)\) denoting two arbitrary functions of time.

**D. Time-Averaged ACF of the SOC Model’s Squared Envelope**

With respect to the time-averaged ACF \(r_{\hat{\xi}\hat{\xi}}^{(k)}(\tau)\) of the sample functions \(\hat{\xi}^2(k)(t)\) of \(\hat{\xi}^2(t)\), we have

\[
r_{\hat{\xi}\hat{\xi}}^{(k)}(\tau) = \langle |\hat{\mu}^{(k)}(t)|^2|\hat{\mu}^{(k)}(t+\tau)|^2\rangle
= \sum_{l=1}^{N} \sum_{m=1}^{N} \sum_{n=1}^{N} \hat{c}_l \hat{c}_m \hat{c}_n \hat{c}_p
\times \exp \{j2\pi (\hat{f}_l - \hat{f}_p)\tau\} \exp \{j(\hat{\theta}_l - \hat{\theta}_m + \hat{\theta}_n - \hat{\theta}_p)\}
\times \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \exp \{j2\pi (\hat{f}_l - \hat{f}_p) t\} dt. \tag{17}\]

In order to find a compact expression for \(r_{\hat{\xi}\hat{\xi}}^{(k)}(\tau)\), we will assume that in addition to the fulfillment of the inequalities in (7), the following condition is met:

**Condition 2:** If \(N \geq 4\), then:

\[
\hat{f}_l + \hat{f}_n = \hat{f}_m + \hat{f}_p, \quad \text{iff} \quad \left\{ \begin{array}{l}
l = m = n = p; \\
or l = m, n = p, l \neq n; \\
or l = p, m = n, l \neq m.
\end{array} \right. \tag{18}\]

Under such considerations, the integral in (17) proves to be different from zero only when \(l = m = n = p, l = m, n = p, l \neq n;\) and \(l = p, m = n, l \neq m\). By solving (17) for such

2The time average of a function \(x(t)\) is denoted by \(\langle x(t) \rangle\) and defined as \(\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt.\)
three cases, we find that [cf. (12)]:

\[
\begin{align*}
\hat{r}_{\xi_{\mu}^{(k)}\xi_{\mu}^{(k)}}(\tau) &= \sum_{l=1}^{N} \hat{c}_l^2 + \sum_{m=1}^{N} \sum_{n=1}^{N} \hat{c}_m^2 \hat{c}_n^2 \\
&+ \sum_{p=1}^{N} \sum_{q=1}^{N} \hat{c}_p^2 \hat{c}_q^2 \exp \{ -j2\pi f_p \tau \} \exp \{ j2\pi f_q \tau \} \\
&= \sigma^4_{\mu} + |\hat{r}_{\hat{\mu}(k)\hat{\mu}(k)}(\tau)|^2 - \sum_{n=1}^{N} \hat{c}_n^4. 
\end{align*}
\]

Notice that the results presented in (19) and (14) are equivalent to each other, since \( r_{\hat{\mu}(k)\hat{\mu}(k)}(\tau) = r_{\hat{\mu}\hat{\mu}}(\tau) \) [cf. (10) and (16)].

We pointed out in Subsection III-A that the IQ components of \( \hat{\mu}(t) \) are mutually uncorrelated iff the Doppler frequencies \( f_n \) and the gains \( \hat{c}_n \) satisfy the Condition 1. In this particular case, the solution given in (19) for \( r_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(\tau) \) is not valid, since the Condition 1 is not compatible with the Condition 2. This is clear, since the equation \( \hat{f}_\ell + \hat{f}_m = \hat{f}_\ell + \hat{f}_m \) has more solutions than the ones specified by the Condition 2 when the Condition 1 is fulfilled.

Since the simulation of fading channels having uncorrelated IQ components is relevant for many practical purposes, e.g., for analyzing the system performance under isotropic scattering conditions, we present (without proof) a solution for \( r_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(\tau) \) by assuming the fulfillment of the Condition 1. Therefore, we suppose without loss of generality that the Doppler frequencies \( \hat{f}_n \) are indexed in such a way that \( \hat{f}_n < \hat{f}_m \) \( \forall n < m \). The solution for this particular case is the following

\[
\begin{align*}
\hat{r}_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(\tau) &= \sigma^4_{\mu} + |\hat{r}_{\hat{\mu}(k)\hat{\mu}(k)}(\tau)|^2 - \sum_{n=1}^{N} \hat{c}_n^4 \\
&+ 4 \left\{ \sum_{m=1}^{M} \hat{c}_m^2 \cos(2\pi \hat{f}_m \tau) \exp \{ j(\hat{\theta}_m^{(k)} + \hat{\theta}_m^{(k)} + \hat{\theta}_m^{(k)} + \hat{\theta}_m^{(k)}) \} \right\}^2 \\
&- \sum_{k=1}^{M} \hat{c}_k^2 \cos^2(2\pi \hat{f}_k \tau) 
\end{align*}
\]

where \( M = N/2 \) (\( N \) is even). It should be noticed that the expression presented above depends on the cisoids phases \( \hat{\theta}_n^{(k)} \). This is in contrast to the solution given in (19), which is not influenced by \( \hat{\theta}_n^{(k)} \).

E. The Ergodicity of the SOC Model’s Squared Envelope

On the basis of the results presented so far, we can analyze the autocorrelation-ergodicity property of the simulation model’s squared envelope \( \hat{\xi}^2(t) \). To start with, we recall that a WSS random process is said to be AE if the time-averaged ACFs of its sample functions are equal to the ACF of the ensemble [19, Sec. 6.6]. Clearly, the simulation model described by \( \hat{\mu}(t) \) is AE, since \( \hat{r}_{\hat{\mu}(k)\hat{\mu}(k)}(\tau) = r_{\hat{\mu}\hat{\mu}}(\tau) \) \( \forall k \).

From the results presented in Subsections III-B and III-D, we can conclude that if the Condition 2 is fulfilled, then \( \hat{\xi}^2(t) \) is an AE process, as

\[
r_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(\tau) = r_{\hat{\xi}\hat{\xi}}(\tau) \forall k \text{ if such a condition is met. However, if the IQ components of the simulation model are uncorrelated, implying that the Condition 1 is satisfied, then the time-averaged ACF } r_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(t) \text{ of the } k \text{th sample function of } \hat{\xi}^2(t) \text{ proves to be a function that depends on the phases } \hat{\theta}_n^{(k)}. \text{ In such a case, } \hat{\xi}^2(t) \text{ is not an AE process, since the ACF of the ensemble does not depend on the cisoids' phases [see (14) and (10)].}
\]

IV. PERFORMANCE EVALUATION OF THE EXISTING PARAMETER COMPUTATION METHODS

The accuracy of the SOC simulation model to emulate the statistical properties of the reference model is ultimately determined by the method employed to compute the cisoids’ gains \( \hat{c}_n \) and Doppler frequencies \( \hat{f}_n \). The GMEA [1], the LPNM [4], and the RSM [2] have been proposed in the literature as suitable parameter computation methods for the design of SOC simulators for mobile fading channels characterized by any given (symmetrical or asymmetrical) DPSSs. In this section, we evaluate the performance of these three methods in terms of the emulation of the squared envelope ACF of the reference model. In addition, we present some simulation results that demonstrate the correctness of the expressions derived in the previous section for \( r_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(\tau) \) and \( r_{\hat{\xi}_{\mu}^{(k)}\hat{\xi}_{\mu}^{(k)}}(\tau) \).

A. Description of the Parameter Computation Methods

1) The GMEA: To allow for a proper emulation of the channel’s envelope distribution, the GMEA defines the gains \( \hat{c}_n \) as follows [1], [6]

\[
\hat{c}_n = \frac{\sigma_{\mu}}{\sqrt{N}}, \quad n = 1, 2, \ldots, N. \tag{21}
\]

The Doppler frequencies \( \hat{f}_n \) are to be computed in such a way that the underlying AOAs \( \hat{\alpha}_n \) satisfy the equation

\[
\int_{0}^{2\pi} g_\alpha (\alpha) d\alpha = \frac{1}{2N} \left( n - \frac{1}{2} \right), \quad n = 1, 2, \ldots, N. \tag{22}
\]

We recall that \( \hat{f}_n = f_{\text{max}} \cos(\hat{\alpha}_n) \) [Sec. III-A].

2) The LPNM: For this method, the gains are given as in (21). However, to maximize the quality of the approximation \( \hat{r}_{\hat{\mu}\hat{\mu}}(\tau) \approx r_{\hat{\mu}\hat{\mu}}(\tau) \) over a given interval of interest centered at the origin, say \( \tau \in [\tau_{\text{min}}, \tau_{\text{max}}] \), the Doppler frequencies \( \hat{f}_n \) are computed as minimizers of the \( L_p \)-norm [3], [4]

\[
\epsilon_{\hat{r}_{\mu\mu}}^{(p)} \triangleq \left\{ \frac{1}{\tau_{\text{max}}} \int_{0}^{\tau_{\text{max}}} [r_{\hat{\mu}\hat{\mu}}(\tau) - r_{\hat{\mu}\hat{\mu}}(\tau)]^p d\tau \right\}^{1/p}
\]

where \( p \) is a positive integer. The minimization of \( \epsilon_{\hat{r}_{\mu\mu}}^{(p)} \) has to be done by applying a numerical optimization algorithm, e.g., [20].
3) The RSM: For the RSM, it is assumed that the PDF \( p_\alpha(\alpha) \) of the AOA is defined in such a way that its even part \( g_\alpha(\alpha) \) has not more than one maximum in \([0, \pi]\). Under this assumption, the parameters \( \hat{c}_n \) and \( \hat{\alpha}_n \) are defined as [2]:

\[
\hat{c}_n = \sigma_\mu \sqrt{\frac{g_\alpha(\hat{\alpha}_n)}{\sum_{m=1}^{N} g_\alpha(\hat{\alpha}_m)}}
\]

\[
\hat{\alpha}_n = \alpha_\ell + \frac{\alpha_u - \alpha_\ell}{N} \left( n - \frac{1}{2} \right), \quad \alpha_u > \alpha_\ell
\]

for \( n = 1, 2, \ldots, N \). In (25), \( \alpha_\ell \) and \( \alpha_u \) designate the lower and the upper boundaries of the subinterval \( \mathcal{I}_U \) of \([0, \pi]\) inside of which the function \( g_\alpha(\alpha) \) is above a given threshold \( \gamma \in (0, \sup \{g_\alpha(\alpha)\}_{\alpha \in [0, \pi])} \), where \( \sup \{ \} \) denotes the supremum.

It is worth mentioning that the Doppler frequencies \( f_\alpha \) obtained by applying any of these three methods satisfy in general the inequalities in (7) [6, Ch. 4]. The methods are therefore well suited for the design of AE SOC channel simulators. Moreover, if the DPSD \( S_{\mu\mu}(f) \) of the reference model is asymmetrical, then the application of such methods results in the majority of cases in a set of Doppler frequencies that meet the Condition 2. On the other hand, for the GMEA and the RSM, it is shown in [6, Appx. E] that the Condition 1 is always fulfilled if the channel’s DPSD \( S_{\mu\mu}(f) \) is symmetrical. Thus, the GMEA and the RSM guarantee the uncorrelatedness of the IQ components of \( \hat{\mu}(t) \) if \( S_{\mu\mu}(f) = S_{\mu\mu}(-f) \).

B. Simulation Set-Up

We evaluate the methods’ performance by assuming that the AOA statistics of the channel follow the von Mises distribution [21]. The von Mises PDF is given as \( p_\alpha(\alpha) = \exp \{ \kappa \cos(\alpha - m_\alpha) \}/(2\pi I_0(\kappa)) \), \( \alpha \in [-\pi, \pi] \), where \( m_\alpha \in [-\pi, \pi] \) designates the mean AOA, \( \kappa \geq 0 \) determines the channel’s angular spread, and \( I_0(\cdot) \) is the zeroth-order modified Bessel function of the first kind. For the von Mises distribution, the ACF \( r_{\mu\mu}(\tau) \) of the reference model can be written in closed form as \( r_{\mu\mu}(\tau) = \sigma_\mu^2 I_0(\kappa^2/2) + \kappa^2 I_0(\kappa) \) [21]. We carry out our investigations by considering the following values for the parameters \( m_\alpha \) and \( \kappa \): \( (m_\alpha = 0^\circ, \kappa = 10) \); \( (m_\alpha = 30^\circ, \kappa = 10) \); and \( (m_\alpha = 90^\circ, \kappa = 10) \). The first two pairs of parameters are representative of fading channels having cross-correlated IQ components, whereas the last pair corresponds to a channel with uncorrelated IQ components [6, Sec. 2.5.1].

For the simulations, we consider \( N = 20 \), \( f_{\max} = 91 \) Hz, and \( \sigma_\mu^2 = 1 \). With respect to the parameter computation methods, we choose \( \gamma = 1 \times 10^{-3} \) for the RSM. For the LPNM, we set \( p = 2 \) and \( \tau_{\max} = N/(4f_{\max}) \). The Doppler frequencies \( \tilde{f}_n \) obtained by applying the GMEA are taken as initial values for the minimization of the \( L_2 \)-norm \( \epsilon_{\mu\mu}^{(2)} \).

C. Results and Analysis

In Fig. 1, we present a comparison between the squared envelope ACF of the reference model, \( r_{\hat{\mu}\hat{\mu}}(\tau) \), and the squared envelope ACF of the simulation model, \( r_{\mu\mu}(\tau) \). The figure shows both theoretical and empirical curves of the ACF of \( \hat{\mu}(t) \). The theoretical graphs were produced by evaluating (14), whereas the empirical curves were generated by averaging over the measured ACFs of 60 samples functions of \( \hat{\mu}(t) \). Such an averaging was necessary, since the process \( \hat{\mu}(t) \) is not always AE\(^3\). With the purpose of demonstrating the correctness of the solutions presented in Section III-D for the time-averaged ACF \( r_{\xi\xi}(\tau) \hat{\xi}(\tau) \) of the sample functions of \( \hat{\mu}(t) \), we present in Fig. 2 analytical and empirical graphs of \( r_{\hat{\theta}\hat{\theta}}(\tau) \hat{\theta}(\tau) \) generated by considering the RSM. One can observe from Figs. 1 and 2 that the results obtained in practice are in excellent agreement with the ones predicted by the theory. It can also be noticed from the graphs depicted in Fig. 2 that the empirical ACF of the sample functions of \( \hat{\theta}(t) \) changes from one realization to another when \( S_{\mu\mu}(f) \) is symmetrical. In contrast, the ACF of \( \hat{\theta}(t) \) is the same for all sample functions when \( S_{\mu\mu}(f) \) is asymmetrical.

With respect to the methods’ performance, we can see in Fig. 1 that the three methods provide a reasonably good approximation to \( r_{\xi\xi}(\tau) \hat{\xi}(\tau) \). However, the best fitting to the ACF of \( \hat{\mu}(t) \) is produced by the GMEA and the LPNM. A quick inspection of the curves drawn in Fig. 1 reveals that regardless of the parameter computation method, there exists an offset between the ACFs of \( \hat{\mu}(t) \) and \( \hat{\theta}(t) \). Such an offset, which can clearly be distinguished at \( \tau = 0 \), is caused by the negative term \( -\sum_{n=1}^{N} \hat{c}_n^2 \) affecting the ACF \( r_{\hat{\xi}\hat{\xi}}(\tau) \) of \( \hat{\mu}(t) \) [see (14)]. It is evident from Fig. 1 that the smallest offset is produced by the GMEA and the LPNM, and the largest by the RSM.

\(^3\)We pointed out in Section III-E that \( \hat{\xi}(t) \) is non-AE when the IQ components of \( \mu(t) \) are uncorrelated. Such a situation ensues in the case of the GMEA and the RSM when \( m_\alpha = 90^\circ \) and \( \kappa = 10 \).
the RSM and the von Mises PDF of the AOA. envelope ACFs of the simulation model's sample functions by considering Fig. 2. Comparison between the theoretical and the empirical squared process. We also evaluated the performance of the GMEA, Condition 2 is met, then
\[ \hat{\mu}^2(t) \] under non-isotropic scattering conditions." in Proc. 50th IEEE Global Communications Conference (Globecom 2007), Washington, DC, Nov. 2007, pp. 3842–3846.

REFERENCES

V. CONCLUSIONS
In this paper, we analyzed the correlation properties of the squared envelope of an AE SOC simulation model for Rayleigh fading channels. We showed that if the IQ components of the SOC model are uncorrelated, then the squared envelope \( \hat{\zeta}^2(t) \) of \( \hat{\mu}(t) \) is a non-AE process. However, if the Condition 2 is met, then \( \hat{\zeta}^2(t) \) proves to be an AE ergodic process. We also evaluated the performance of the GMEA, the LPNM, and the RSM with respect to their accuracy for emulating the squared envelope ACF of the reference model. The obtained results show that the three methods provide a reasonably good approximation to the channel's squared envelope ACF, although the GMEA and the LPNM perform better than the RSM.

ACKNOWLEDGMENTS
This work was financed in part by the Mexican Council for Science and Technology (CONACYT). The author is grateful to Prof. Matthias Pätzold at the Department of Information and Communication Technology, University of Agder, Grimstad, Norway, for useful discussions on the topics analyzed in this paper and critical reading of the manuscript.