Interval-valued Fuzzy Normal Subgroups

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Abstract

We study some properties of interval-valued fuzzy normal subgroups of a group. In particular, we obtain two characterizations of interval-valued fuzzy normal subgroups. Moreover, we introduce the concept of an interval-valued fuzzy coset and obtain several results which are analogous of some basic theorems of group theory.

Key Words: interval-valued fuzzy normal subgroup, interval-valued fuzzy coset, interval-valued fuzzy quotient group.

1. Introduction and Preliminaries

In 1975, Zadeh[11] introduced the concept of interval-valued fuzzy sets as the generalization of fuzzy sets introduced by himself[10]. After that time, Biswas[1] applied the notion of interval-valued fuzzy set to group theory, and Samanta and Montal[9] to topology. Recently, Choi et al.[2] introduced the concept of interval-valued smooth topological spaces and studied some of its properties. Hur et al.[3] investigated interval-valued fuzzy relations, Kang and Hur[6] applied the concept of interval-valued fuzzy sets to algebra. In particular, Kang[7] studied interval-valued fuzzy subgroups preserved by homomorphisms. In this paper, we investigate some properties of interval-valued fuzzy normal subgroups of a group. In particular, we obtain two characterizations of interval-valued fuzzy normal subgroups. introduce the concept of interval-valued fuzzy subgroups. Moreover, we introduce the concept of an interval-valued fuzzy coset and obtain several results which are analogous of some basic theorems of group theory.

Now, we will list some concepts and results related to interval-valued fuzzy set theory and needed in next sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and upper end points respectively. Especially, we denoted $0 = [0, 0], 1 = [1, 1], \alpha = [a, b]$ for every $\alpha \in (0, 1)$.

We also note that

\begin{enumerate}
  \item $(\forall M, N \in D(I)) (M = N \iff M^L = N^L, M^U = N^U)$,
  \item $(\forall M, N \in D(I)) (M \leq N \iff M^L \leq N^L, M^U \leq N^U)$.
\end{enumerate}

For every $M \in D(I)$, the complement of $M$, denoted by $M^c$, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$ (See [9]).

Definition 1.1 [9, 11]. A mapping $A : X \to D(I)$ is called an interval-valued fuzzy set (in short, IVS) in $X$ and is denoted by $A = [A^L, A^U]$. Thus $A(x) = [A^L(x), A^U(x)]$, where $A^L(x)$[resp. $A^U(x)$] is called the lower[resp. upper] end point of $x$ to $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy set $A$ in $X$ defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $[a, b]$ and if $a = b$, then the IVS $[a, b]$ is denoted by simply $\tilde{a}$. In particular, $0$ and $1$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in $X$, respectively.

We will denote the set of all IVSs in $X$ as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 1.2 [9]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

\begin{enumerate}
  \item $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.
  \item $A = B$ iff $A \subset B$ and $B \subset A$.
  \item $A^C = [1 - A^U, 1 - A^L]$.
  \item $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.
  \item $\bigvee_{\alpha \in \Gamma} A_\alpha = \left[ \bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha \right]$.
  \item $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.
  \item $\bigwedge_{\alpha \in \Gamma} A_\alpha = \left[ \bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha \right]$.
\end{enumerate}

Result 1. A [9, Theorem 1]. Let $A, B, C \in D(I)^X$ and let
An interval-valued fuzzy set
Definition 1.3 [6]. An interval-valued fuzzy set
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\text{Definition 1.3 [6]. An interval-valued fuzzy set}
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\begin{align*}
\text{Definition 1.3 [6]. An interval-valued fuzzy set} & \\
\end{align*}
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\[
\begin{align*}
\text{Result 1.C [6, Proposition 4.7].} & \\
\end{align*}
\]

\[
\begin{align*}
\text{Let } A & \text{ be an IVG of a group } G, \text{ then for each } [\lambda, \mu] \in D(I) \text{ with } A(e) \geq [t, s], \text{ i.e., } A^L(e) \geq t \text{ and } A^U(e) \geq s, \text{ the level subset } A^{[\lambda, \mu]} \text{ is a subgroup of } G. \text{ If Im } A = \{[t_0, s_0], [t_1, s_1], \ldots, [t_n, s_n]\}, \text{ the family of level subgroups } A^{[\lambda_i, \mu_i]} : 0 \leq i \leq n \text{ constitutes the complete list of level subgroups of } A. \text{ If the image set of the IVG } A \text{ of a finite group } G \text{ consists of } \{[t_0, s_0], [t_1, s_1], \ldots, [t_n, s_n]\}, \text{ where } t_0 > t_1 > \cdots > t_n \text{ and } s_0 > s_1 > \cdots > s_n, \text{ then, by Results 1.D and 1.E, the level subgroups of } A \text{ form a chain:}
\]

\[
A^{[t_0, s_0]} \subset A^{[t_1, s_1]} \subset \cdots \subset A^{[t_n, s_n]} = G,
\]

where } A(e) = [t_0, s_0].

\[
\text{Notation. } N < G \text{ denotes that } N \text{ is a normal subgroup of } G.
\]

\[
\begin{align*}
\text{2. Interval-valued fuzzy normal subgroups} & \text{ and interval-valued fuzzy cosets} \\
\end{align*}
\]

\[
\text{Lemma 2.1. If } A \text{ is an IVGP of a finite group } G, \text{ then } A \text{ is an IVG of } G.
\]

\[
\text{Proof. Let } x \in G. \text{ Since } G \text{ is finite, } x \text{ has finite order, say } n. \text{ Then } x^n = e, \text{ where } e \text{ is the identity of } G. \text{ Thus }
\]

\[
x^{-1} = x^{n-1}. \text{ Since } A \text{ is an IVGP of } G,
\]

\[
A^L(x^{-1}) = A^L(x^{n-1}) = A^L(x^{n-2}) \geq A^L(x)
\]

and

\[
A^U(x^{-1}) = A^U(x^{n-1}) = A^U(x^{n-2}) \geq A^U(x).
\]

Hence } A \text{ is an IVG of } G.

\]

\[
\text{Lemma 2.2. Let } A \text{ be an IVG of a group } G \text{ and let } x \in G.
\]

\[
\text{Then } A(xy) = A(y), \text{ for each } y \in G \text{ if and only if }
\]

\[
A(x) = A(e).
\]

\[
\text{Proof. (⇒): Suppose } A(xy) = A(y), \text{ for each } y \in G. \text{ Then clearly }
\]

\[
A(x) = A(e).
\]

\[
\text{⇐: Suppose } A(x) = A(e). \text{ Then, by Result 1.B(b), } A^L(y) \leq A^L(x) \text{ and } A^U(y) \leq A^U(x) \text{ for each } y \in G.
\]

\[
\text{Since } A \text{ is an IVG of } G, \text{ then } A^L(xy) \geq A^L(x) \wedge A^L(y) \text{ and } A^U(xy) \geq A^U(x) \wedge A^U(y). \text{ Thus } A^L(xy) \geq A^L(y)
\]

and

\[
A^U(xy) \geq A^U(y) \text{ for each } y \in G.
\]

\[
\text{On the other hand, by the hypothesis and Result 1.B(b), }
\]

\[
A^L(y) = A^L(x^{-1}y) \geq A^L(x) \wedge A^L(xy) \text{ and } A^U(y) = A^U(xy^{-1}) \geq A^U(x) \wedge A^U(xy).
\]

\[
\text{Since } A(x) \geq A^L(y) \text{ for each } y \in G, A^L(xy) \wedge A^L(xy) = A^L(xy) \text{ and } A^U(x) \wedge A^U(xy) = A^U(xy). \text{ So }
\]

\[
A^L(y) \geq A^L(xy) \text{ and } A^U(y) \geq A^U(xy) \text{ for each } y \in G. \text{ Hence } A(xy) = A(y) \text{ for each } y \in G.
\]

\]

\[
\text{Remark 2.3. It is easy to see that if } A(x) = A(e), \text{ then }
\]

\[
A(xy) = A(xy) \text{ for each } y \in G.
\]
Definition 2.4. Let $A$ be an IVS of a group $G$ and let $x \in G$. We define two mappings $Ax, xA : G \rightarrow D(I)$ as follows, respectively: For each $g \in G$, $Ax(g) = A(gx^{-1})$ and $xA(g) = A(g^{-1}x)$. Then $Ax$ [resp. $xA$] is called the interval-valued fuzzy right [resp. left] coset of $A$ determined by $x$ and $A$.

Remark 2.5. Definition 2.4 extends in a natural way to usual definition of a "coset" of a group. This is seen as follows: Let $H$ be a subgroup of a group $G$ and let $A = [\chi_H, \chi_H]$, where $\chi_H$ is the characteristic function of $H$. Let $x, y \in G$. Then $Ax = [\chi_H, \chi_H]$. Suppose $g \in H$. Then
\[
Ax(gx) = [\chi_H(g), \chi_H(gx)] \\
= [\chi_H(gxx^{-1}), \chi_H(gxx^{-1})] \\
= [\chi_H(g), \chi_H(g)] \\
= [1, 1].
\]
Suppose $g \notin H$. Then
\[
Ax(gx) = [\chi_H(g), \chi_H(gx)] \\
= [\chi_H(gxx^{-1}), \chi_H(gxx^{-1})] \\
= [\chi_H(g), \chi_H(g)] \\
= [0, 0].
\]
So $Ax = [\chi_H, \chi_H]$. The following is the immediate result of Definition 2.4.

Proposition 2.6. Let $A$ be an IVG of a group $G$. Then
(a) $(xy)A = x(yA)$.
(b) $A(xy) = (Ax)y$.
(c) $xA = A$ if $A(x) = [1, 1]$.

We know that any two left [resp. right] cosets of a subgroup $H$ of a group $G$ are equal or disjoint. However this fact is not valid in the interval-valued fuzzy case as shown in the following example.

Example 2.7. Let $G = \{e, a, b, c, d\}$ be the Klein’s four group and let $A$ be the IVG of $G$ defined by:
$A(a) = [1, 1], A(b) = [t_1, t_1], A(c) = A(d) = [t_2, t_2]$, where $0 < t_2 \leq t_1 < 1$. Then $ba \neq cb$.

Definition 2.8 [6]. Let $A$ be an IVG(G). Then $A$ is called an interval-valued fuzzy normal subgroup (in short, IVNG) of $G$ if $A(xy) = A(yx)$, for any $x, y \in G$.

We will denote the set of all IVNGs of a group $G$ as $\text{IVNG}(G)$. The following is the immediate result of Definitions 2.4 and 2.8.

Theorem 2.9. Let $A$ be an IVG of a group $G$. Then the followings are equivalent:
(a) $A^*(xyx^{-1}) \geq A^*(y)$ and $A^*(xyx^{-1}) \geq A^*(y)$, for any $x, y \in G$.
(b) $A(xy) = A(y)$ for any $x, y \in G$.
(c) $A \in \text{IVNG}(G)$.
(d) $xA = Ax, \forall x \in G$.
(e) $xA^{-1} = A, \forall x \in G$.

Remark 2.10. Let $G$ be a group.
(a) If $A$ is a fuzzy normal subgroup of $G$, then $[A, A] \in \text{IVNG}(G)$.
(b) If $A = [A^L, A^U] \in \text{IVNG}(G)$, then $A^L$ and $A^U$ are fuzzy normal subgroups of $G$.

Let $G$ be a group and $a, b \in G$. We say that $a$ is conjugate to $b$ if there exists $x \in G$ such that $b = x^{-1}ax$. It is well-known that conjugacy is an equivalence relation on $G$. The equivalence classes in $G$ under the relation of conjugacy are called conjugate classes [4].

Theorem 2.11. Let $A$ be an IVG of a group $G$. Then $A \in \text{IVNG}(G)$ if and only if $A$ is constant on the conjugate classes of $G$.

Proof. ($\Rightarrow$) : Suppose $A \in \text{IVNG}(G)$ and let $x, y \in G$. Then $A(y^{-1}xy) = A(xy^{-1}) = A(x)$. Hence $A$ is constant on the conjugate classes.

($\Leftarrow$) : Suppose the necessary condition holds and let $x, y \in G$. Then $A(xy) = A(xyx^{-1}) = A(xyx^{-1}) = A(yx^{-1}) = A(yx)$. Hence $A \in \text{IVNG}(G)$.

Let $G$ be a group and $x, y \in G$. Then the element $x^{-1}y^{-1}xy$ is usually denoted by $[x, y]$ and called the commutator of $x$ and $y$. It is clear that if $x$ and $y$ commute with each other, then clearly $[x, y] = e$. Let $H$ and $K$ be two subgroups of a group $G$. Then the subgroup $[H, K]$ is defined as the subgroup generated by the elements $\{[x, y] : x \in H, y \in K\}$. It is well-known that $N \triangleleft G$ if and only if $[N, G] \leq N$.

The following is the generalization of the above result using interval-valued fuzzy sets.

Theorem 2.12. Let $A$ be an IVG of a group $G$. Then $A \in \text{IVNG}(G)$ if and only if $A^L([x, y]) \geq A^L(x)$ and $A^U([x, y]) \geq A^U(x)$, for any $x, y \in G$.

Proof. ($\Rightarrow$) : Suppose $A \in \text{IVNG}(G)$ and let $x, y \in G$. 207
By the similar arguments, we have that $A(x, z) \in \text{IVNG}(G)$. This completes the proof.

\[ A^L([x, y]) = A^L(x^{-1}y^{-1}xy) = A^L(y^{-1}x y x^{-1}) \quad \text{(By the hypothesis)} \]
\[ \geq A^L(y^{-1}x y) \land A^L(x^{-1}) \quad \text{(Since } A \in \text{IVNG}(G)) \]
\[ = A^L(x) \land A^L(x) \quad \text{(By Theorem 2.9 and Result 1.B(a))} \]
\[ = A^L(x). \]

By the similar arguments, we have that $A^U([x, y]) \geq A^U(x)$. Hence the necessary conditions hold.

(\(\Leftarrow\)): Suppose the necessary conditions hold and let $x, z \in G$. Then
\[ A^L(x^{-1}z x) = A^L(z^{-1}x^{-1}z x) \]
\[ \geq A^L(z) \land A^L(z x) \land A^L(x^{-1}) \quad \text{(Since } A \in \text{IVNG}(G)) \]
\[ = A^L(z) \land A^L(z) \quad \text{(By the hypothesis)} \]
\[ = A^L(z). \]

By the similar arguments, we have that $A^U(x^{-1}z x) \geq A^U(z)$. On the other hand,
\[ A^L(z) = A^L(x x^{-1}z x x^{-1}) \]
\[ \geq A^L(x) \land A^L(x^{-1}z x) \land A^L(x^{-1}) \quad \text{(Since } A \in \text{IVNG}(G)) \]
\[ = A^L(x) \land A^L(x^{-1}z x). \quad \text{(By Result 1.B(a))} \]

By the similar arguments, we have that $A^U(z) \geq A^U(x) \land A^U(x^{-1}z x)$.

Case(i): Suppose $A^L(x) \land A^L(x^{-1}z x) = A^L(x)$ and $A^U(x) \land A^U(x x^{-1}z x) = A^U(x)$. Then $A^L(z) \geq A^L(x)$ and $A^U(z) \geq A^U(x)$ for any $x, z \in G$. Thus $A$ is a constant mapping. So $A(xy) = A(yx)$ for any $x, z \in G$, i.e., $A \in \text{IVNG}(G)$.

Case(ii): Suppose $A^L(x) \land A^L(x^{-1}z x) = A^L(x^{-1}z x)$ and $A^U(x) \land A^U(x^{-1}z x) = A^U(x^{-1}z x)$. Then $A^L(z) \geq A^L(x^{-1}z x)$ and $A^U(z) \geq A^U(x^{-1}z x)$ for any $x, z \in G$, i.e., $A(x^{-1}z x) = A(z)$ for any $x, z \in G$. So $A$ is constant on the conjugate classes. By Theorem 2.11, $A \in \text{IVNG}(G)$. Hence, in either cases, $A \in \text{IVNG}(G)$.

This completes the proof. \(\Box\)

Proposition 2.13. Let $A$ be an IVNG of a group $G$ and let $[\lambda, \mu] \in D(I)$ such that $\lambda \leq A^e(e), \mu \leq A^U(e)$, where $e$ denotes the identity of $G$. Then $A^{[\lambda, \mu]} \triangleleft G$.

\[ A^{[\lambda, \mu]} \triangleleft G. \]

Let $A$ be an IVNG of a finite group $G$ with $\text{Im} A = \{[t_0, s_0], [t_1, s_1], \ldots, [t_r, s_r]\}$, where $t_0 > t_1 > \cdots > t_r$, and $s_0 > s_1 > \cdots > s_r$. Then it follows from Theorem 2.7 that the level subgroups of $A$ form a chain of normal subgroups:
\[ A^{[t_0, s_0]} \subset A^{[t_1, s_1]} \subset \cdots \subset A^{[t_r, s_r]} = G. \quad (2.1) \]

The following is the immediate result of Proposition 2.13.

Corollary 2.13 [6, Proposition 5.4]. Let $A$ be an IVNG of a group $G$ with identity $e$. Then $G_A \triangleleft G$, where $G_A = \{x \in G : A(x) = A(e)\}$.

The following is the converse of Proposition 2.13.

Proposition 2.14. If $A$ is an IVNG of a finite group $G$ such that all the level subgroups of $A$ are normal in $G$, then $A \in \text{IVNG}(G)$.

Proof. Let $\text{Im} A = \{[t_0, s_0], [t_1, s_1], \ldots, [t_r, s_r]\}$, where $t_0 > t_1 > \cdots > t_r$, and $s_0 > s_1 > \cdots > s_r$. Then the family $\{A^{[i, s_i]} : 0 \leq i \leq r\}$ is the complete set of level subgroups of $G$. By the hypothesis, $A^{[i, s_i]} \triangleleft G$ for each $0 \leq i \leq r$. From the definition of the level subgroup, it is clear that $A^{[i, s_i]} \setminus A^{[i-1, s_{i-1}]} = \{x \in G : A(x) = [t_i, s_i]\}$. Since a normal subgroup of a group is a complete union of conjugate classes, it follows that in the given chain (2.1) of normal subgroups, each $A^{[i, s_i]} \setminus A^{[i-1, s_{i-1}]}$ is a union of some conjugate classes. Since $A$ is constant on the level subgroups $A^{[i, s_i]} \setminus A^{[i-1, s_{i-1}]}$, it follows that $A$ must be constant on each conjugate class of $G$. Hence, by Theorem 2.11, $A \in \text{IVNG}(G)$. \(\Box\)

Example 2.15. Let $G$ be the group of all symmetries of a square. Then $G$ is a group of order 8 generated by a rotation through $\pi/2$ and a reflection along a diagonal of the square. Let us denote the elements of $G$ by $\{1, 2, 3, 4, 5, 6, 7, 8\}$, where 1 is the identity, 2 is rotation through $\pi/2$ and 5 is a reflection along a diagonal: the multiplication table of $G$ is as shown in Table 1.

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Table 1.
We can easily see that the conjugate classes of $G$ are \{1, 3, 5, 7, 6, 8\}, \{2, 4\}.

Let $H = \{1, 3\}$ and let $K = \{1, 2, 3, 4\}$. Then clearly, $H \triangleleft G$ and $K \triangleleft G$ (in fact, $H$ is the center of $G$). Thus we have a chain of normal subgroups given by

$$\{1\} \subset H \subset K \subset G. \quad (2.2)$$

Now we will construct an IVG of $G$ whose level subgroups are precisely the members of the chain (2.2). Let $[t_0, s_0] \in D(I), 0 \leq i \leq 3$ such that $t_0 > t_1 > t_2 > t_3$ and $s_0 > s_1 > s_2 > s_3$. Define a mapping $A : G \rightarrow D(I)$ as follows:

$A(1) = [t_0, s_0], A(H \setminus \{1\}) = [t_1, s_1], A(K \setminus H) = [t_2, s_2], A(G \setminus K) = [t_3, s_3]$.

From the definition of $A$, it is clear that $A(x) = A(x^{-1})$ for each $x \in G$. Also, we can easily check that for any $x, y, G$,

$A^L(xy) \geq A^L(x) \land A^L(y)$ and $A^U(xy) \geq A^U(x) \land A^U(y)$.

Furthermore, it is clear that $A$ is constant on the conjugate classes. Hence, by Theorem 2.11, $A \in IVNG(G)$. \hfill \Box

The following can be easily proved and the proof is omitted.

**Lemma 2.16.** Let $A$ be an IVG of a group and let $x \in G$. Then $A(x) = [\lambda, \mu]$ if and only if $x \in A^{[\lambda, \mu]}$ and $x \notin A^{[t,s]}$ for each $[t, s] \in D(I)$ such that $t > \lambda$ and $s > \mu$.

It is well-known that if $N$ is a normal subgroup of a group $G$, then $xy \in N$ if and only if $yx \in N$ for any $x, y \in G$.

The following result is the generalization of Proposition 2.14.

**Proposition 2.17.** Let $A$ be an IVG of a group $G$. If $A^{[\lambda, \mu]}, [\lambda, \mu] \in \text{Im} \, A$, is a normal subgroup of $G$, then $A \in IVNG(G)$.

**Proof.** For any $x, y \in G$, let $A(x, y) = [\lambda, \mu]$ and let $A(xy) = [t, s]$ be such that $t > \lambda$ and $s > \mu$. Then, by Lemma 2.16, $xy \in A^{[\lambda, \mu]}$ and $x \notin A^{[t,s]}$. Thus $yx \in A^{[\lambda, \mu]}$ and $y \notin A^{[t,s]}$. So $A(yx) = [\lambda, \mu]$, i.e., $A(xy) = A(yx)$. Hence $A \in IVNG(G)$. \hfill \Box

**3. Homomorphisms**

**Definition 3.1** [9]. Let $f : X \rightarrow Y$ be a mapping, let $A = [A^L, A^U] \in D(I)^X$ and let $B = [B^L, B^U] \in D(I)^Y$. Then

(a) the image of $A$ under $f$, denoted by $f(A)$, is an IVS in $Y$ defined as follows: For each $y \in Y$,

$$f(A^L(y)) = \begin{cases} \bigvee_{y \in f(x)} A^L(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$f(A^U(y)) = \begin{cases} \bigwedge_{y \in f(x)} A^U(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

(b) the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is an IVS in $X$ defined as follows: For each $y \in Y$,

$$f^{-1}(B^L(y)) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B^U(y)) = (B^U \circ f)(x) = B^U(f(x)).$$

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

**Result 3.A** [9, Theorem 2]. Let $f : X \rightarrow Y$ be a mapping and $g : Y \rightarrow Z$ be a mapping. Then

(a) $f^{-1}(B^c) = f^{-1}(B^c), \forall B \in D(I)^Y$.

(b) $[f(A)]^c \subset f(A^c), \forall A \in D(I)^Y$.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.

(e) $f(f^{-1}(B)) \subset B, \forall B \in D(I)^Y$.

(f) $A \subset f(f^{-1}(A)), \forall A \in D(I)^Y$.

(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)), \forall C \in D(I)^Z$.

(h) $f^{-1}(\bigcup_{\alpha \in \Gamma} B_{\alpha}) = \bigcup_{\alpha \in \Gamma} f^{-1}B_{\alpha}$, where $\{B_{\alpha}\}_{\alpha \in \Gamma} \subset D(I)^Y$.

(i) $f^{-1}(\bigcap_{\alpha \in \Gamma} B_{\alpha}) = \bigcap_{\alpha \in \Gamma} f^{-1}B_{\alpha}$, where $\{B_{\alpha}\}_{\alpha \in \Gamma} \subset D(I)^Y$.

**Proposition 3.2.** Let $f : X \rightarrow Y$ be a groupoid homomorphism. If $A \in IVGP(X)$, then $f(A) \in IVGP(Y)$.

**Proof.** For each $y \in Y$, let $X_y = f^{-1}(y)$. Since $f$ is a homomorphism, it is clear that $X_yX_{y'} \subset X_{yy'}$ for any $y, y' \in Y$. \hfill (*)

Let $y, y' \in Y$.

Case (i): Suppose $yy' \notin f(A)$. Then clearly $f(A)(yy') = [0, 0]$. Since $yy' \notin f(X), X_{yy'} = \emptyset$. By (*)& $X_y = \emptyset$ or $X_{y'} = \emptyset$. Thus $f(A)(y) = [0, 0]$ or $f(A)(y') = [0, 0]$. So

$$f(A)(yy') = [0, 0]$$

and

$$f(A)(y) \land f(A)(y') = f(A)(y') \land f(A)(y') = [f(A)^L(y'), f(A)^U(y')] = [0, 0].$$

$$f(A)^L(y) \land f(A)^U(y') = [f(A)^L(y), f(A)^U(y')] = [0, 0].$$
Case (ii): Suppose $yy' \in f(X)$. Then $X_{yy'} \not= \emptyset$. If $X_y = \emptyset$ and $X_{y'} = \emptyset$, then $f(A)(y) = [0, 0]$ and $f(A)(y') = [0, 0]$. Thus

$$f(A)^L(yy') \geq f(A)^L(y) \land f(A)^L(y')$$

and

$$f(A)^U(yy') \geq f(A)^U(y) \land f(A)^U(y').$$

If $X_y \not= \emptyset$ or $X_{y'} \not= \emptyset$, then, by (*),

$$f(A)^L(yy') = \bigvee_{x \in X_{yy'}} A^L(z) \geq \bigvee_{x \in X_y, x' \in X_{y'}} A^L(x')$$

$$\geq \bigvee_{x \in X_y, x' \in X_{y'}} (A^L(x) \land A^L(x'))$$

(Since $A \in \text{IVGP}(X)$)

$$= \bigvee_{x \in X_y, x' \in X_{y'}} A^L(x') = f(A)^L(y') \land f(A)^L(y').$$

By the similar arguments, we have that

$$f(A)^U(yy') \geq f(A)^U(y) \land f(A)^U(y').$$

Consequently,

$$f(A)^U(yy') \geq f(A)^U(y) \land f(A)^L(y')$$

and

$$f(A)^U(yy') \geq f(A)^U(y') \land f(A)^U(y').$$

Hence $f(A) \in \text{IVGP}(Y)$. \hfill \Box

**Definition 3.3 [1, 6].** Let $A$ be an IVS in a groupoid $G$. Then $A$ is said to have the sup-property if for any $T \in P(G)$, there exists a $t_0 \in T$ such that $A(t_0) = \bigcup_{t \in T} A(t)$, i.e., $A^L(t_0) = \bigvee_{t \in T} A^L(t)$ and $A^U(t_0) = \bigvee_{t \in T} A^U(t)$, where $P(G)$ denotes the power set of $G$.

**Result 3.B [6, Proposition 4.11].** Let $f : G \rightarrow G'$ be a group homomorphism, let $A \in \text{IVG}(G)$ and let $B \in \text{IVG}(G')$. Then the followings hold:

(a) If $A$ has the sup property, then $f(A) \in \text{IVG}(G')$.

(b) $f^{-1}(B) \in \text{IVG}(G)$.

**Proposition 3.4.** Let $f : X \rightarrow Y$ be a group[resp. ring, algebra and field] homomorphism. If $A \in \text{IVG}(X)[resp.\ IVR(X), \text{IVA}(X) and \text{IVF}(X)]$, then $f(A) \in \text{IVG}(Y)[resp. \ IVR(Y), \text{IVA}(Y) and \text{IVF}(Y)]$, where $\text{IVG}(X)[resp. \ IVR(X), \text{IVA}(X) and \text{IVF}(X)]$ denotes the set of all interval-valued fuzzy subgroups[resp. subrings, subalgebras and subfields] of a group[resp. ring, algebra and field] $X$.

**Proof.** Suppose $f : X \rightarrow Y$ is a group homomorphism and let $A \in \text{IVG}(X)$. Then, we need only to show that $f(A)^L(y^{-1}) \geq f(A)^L(y)$ and $f(A)^U(y^{-1}) \geq f(A)^U(y)$ for each $y \in Y$. Let $y \in Y$.

Case (i): Suppose $y^{-1} \not\in f(X)$. Then $y \not\in f(X)$. Thus $f(A)(y^{-1}) = [0, 0] = f(A)(y)$.

Case (ii): Suppose $y^{-1} \in f(X)$. Then $y \in f(X)$. Thus

$$f(A)^L(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} A^L(t) \geq A^L(y) = f(A)^L(y)$$

and

$$f(A)^U(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} A^U(t) \geq A^U(y) = f(A)^U(y).$$

Hence $f(A) \in \text{IVG}(Y)$. The proofs of the rest are omitted. This completes the proof. \hfill \Box

**Another Proof :** Let $[\lambda, \mu] \in \text{Im} f(A)$. Then there exists a $y \in Y$ such that

$$f(A)(y) = \bigvee_{x \in f^{-1}(y)} A^L(x), \bigvee_{x \in f^{-1}(y)} A^U(x) = [\lambda, \mu].$$

Since $A \in \text{IVG}(X)$, by Result 1.B(b), $\lambda \leq A^L(e)$ and $\mu \leq A^U(e)$.

Case (i): Suppose $[\lambda, \mu] = [0, 0]$. Then clearly $(f(A)[\lambda, \mu]) = Y$. So, by Result 1.D, $f(A) \in \text{IVG}(Y)$.

Case (ii): Suppose $\lambda > 0$. Then $z \in (f(A)[\lambda, \mu]) \Leftrightarrow f(A)^L(z) \geq \lambda$ and $f(A)^U(z) \geq \mu \Leftrightarrow \bigvee_{x \in f^{-1}(z)} A^L(x) \geq \lambda$ and $\bigvee_{x \in f^{-1}(z)} A^U(x) \geq \mu \Leftrightarrow$ there exists an $x \in X$ such that $f(x) = z$, $A^L(x) \geq \lambda$ and $A^U(x) \geq \mu \Leftrightarrow z \in (f(A)[\lambda, \mu])$.

Thus $(f(A)[\lambda, \mu]) = f(A[\lambda, \mu])$. Since $f$ is a homomorphism and $A[\lambda, \mu]$ is a subgroup of $X$, $f(A[\lambda, \mu])$ is a subgroup of $Y$. So, by Result 1.D, $f(A) \in \text{IVG}(Y)$. Hence, in all, $f(A) \in \text{IVG}(X)$. \hfill \Box

**Remark 3.5.** In Result 3.B, A has the sup property but in Proposition 3.4, there is no restriction on $A$.

**Proposition 3.6.** Let $f : G \rightarrow G'$ be a group homomorphism, let $A \in \text{IVNG}(G)$ and let $B \in \text{IVNG}(G')$. Then the followings hold:

(a) If $f$ is surjective, then $f(A) \in \text{IVNG}(G')$.

(b) $f^{-1}(B) \in \text{IVNG}(G)$.

**Proof.** (a) By Proposition 3.4, $f(A) \in \text{IVG}(G')$. Let $[\lambda, \mu] \in \text{Im} f(A)$. From the process of the another proof of Proposition 3.4, it is clear that $\lambda \leq A^L(e)$, $\mu \leq A^U(e)$ and $(f(A)[\lambda, \mu]) = f(A[\lambda, \mu])$. Since $A \in \text{IVNG}(G)$, by Proposition 2.13, $A[\lambda, \mu] \subset G$. Since $f$ is an epimorphism, $(f(A)[\lambda, \mu]) = f(A[\lambda, \mu]) \subset G'$. Hence, by Proposition 2.17, $f(A) \in \text{IVG}(G')$. 

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Thus $A$ is constant on the conjugate classes of $G$. So, by Theorem 2.11, $A \in \text{IVNG}(G)$.

Now let $g \in N$. Then $A(g) = \tilde{B}(N_g) = \tilde{B}(N) = A(e)$. Thus $g \in G_A$. So $N \subseteq G_A$. Let $x \in G_A$. Then $A(x) = A(e)$. Thus $\tilde{B}(N) = \tilde{B}(N)$. So $x \in N$, i.e., $G_A \subset N$. Hence $N = G_A$. Furthermore, $\tilde{A} = \tilde{B}$. This completes the proof.

\hfill $\Box$

4. Interval-valued fuzzy Lagrange’s Theorem

Let $A$ be an IVS in a group $G$ and for each $x \in G$, $xf : G \rightarrow G$ [resp. $fx : G \rightarrow G$] be a mapping defined as follows, respectively: For each $g \in G$,

$x(f(g)) = xg$ [resp. $f(x)(g) = gx$].

**Proposition 4.1.** Let $A$ be an IVG of a group $G$. Then $xf(A) = xA$ [resp. $f_x(A) = Ax$] for each $x \in G$.

**Proof.** Let $g \in G$. Then

$$f_x(A)^L(g) = \bigvee_{g' \in f_x^{-1}(g)} A^L(g')$$

$$= \bigvee_{g' \neq g} A^L(g') = A^L(gx^{-1})$$

and

$$f_x(A)^U(g) = \bigvee_{g' \in f_x^{-1}(g)} A^U(g')$$

$$= \bigvee_{g' \neq g} A^U(g') = A^U(gx^{-1})$$.

Hence, $f_x(A) = Ax$. Similarly, we can see that $xf(A) = xA$.

\hfill $\Box$

**Theorem 4.2.** Let $A$ be an IVG of a group $G$ and let $g_1, g_2 \in G$. Then $g_1A = g_2A$ [resp. $Ag_1 = Ag_2$] if and only if $A(g_1^{-1}g_2) = A(g_2^{-1}g_1) = A(e)$ [resp. $A(g_1g_2^{-1}) = A(g_2g_1^{-1}) = A(e)$].

**Proof.** ($\Rightarrow$): Suppose $g_1A = g_2A$. Then $g_1A(g_1) = g_2A(g_2)$ and $g_1A(g_1^{-1}g_2) = g_2A(g_2^{-1}g_1)$. Hence $A(g_1g_2^{-1}) = A(g_2g_1^{-1}) = A(e)$.

($\Leftarrow$): Suppose $A(g_1^{-1}g_2) = A(g_2^{-1}g_1) = A(e)$. Let $x \in G$. Then $g_1A(x) = A(g_1^{-1}x) = A(g_1^{-1}g_2g_2^{-1}x)$. Since $A$ is a IVS in $G$,

$$A^L(g_1^{-1}x) = A^L(g_1^{-1}xg_2g_2^{-1}x)$$

$$= A^L(g_1^{-1}g_2) \land A^L(g_2^{-1}x)$$

$$= A^L(g_2^{-1}x).$$

(By Result 1.B(b))
By the similar arguments, we have that $A^U(g_1^{-1}x) \geq A^U(g_2^{-1}x)$. Thus $g_2A \subset g_1A$. Similarly, we have that $g_1A \subset g_2A$. Hence $g_1A = g_2A$. This completes the proof.

**Proposition 4.3.** Let $A$ be an IVNG of a group $G$. If $Ag_1 = Ag_2$ for any $g_1, g_2 \in G$, then $A(g_1) = A(g_2)$.

**Proof.** Suppose $Ag_1 = Ag_2$ for any $g_1, g_2 \in G$. Then $Ag_1(g_2) = Ag_2(g_2)$. Thus $A(g_2g_1^{-1}) = A(e)$. Hence, by Result 1.C, $A(g_1) = A(g_2)$. □

**Proposition 4.4.** Let $A$ be an IVNG of a group $G$. If $A^{[\lambda, \mu]}(x) = A^{[\lambda, \mu]}y$ for any $x, y \in G \setminus A^{[\lambda, \mu]}$ and each $[\lambda, \mu] \in D(I)$, then $A(x) = A(y)$.

**Proof.** Suppose $A^{[\lambda, \mu]}(x) = A^{[\lambda, \mu]}y$ for any $x, y \in G \setminus A^{[\lambda, \mu]}$ and each $[\lambda, \mu] \in D(I)$. Then $yx^{-1} \in A^{[\lambda, \mu]}$. Thus $A^U(yx^{-1}) \geq \lambda$ and $A^U(yx^{-1}) \geq \mu$. Since $x \in G \setminus A^{[\lambda, \mu]}$, $A^L(x) \geq \lambda$ and $A^U(x) \leq \mu$. On the other hand,

$$A^L(y) = A^L(yx^{-1}x) \geq A^L(yx^{-1}) \geq A^L(x)$$

and

$$A^U(y) = A^U(yx^{-1}x) \geq A^U(yx^{-1}) \geq A^U(x).$$

Thus $A^L(y) \geq A^L(x)$ and $A^U(y) \geq A^U(x)$. By the similar arguments, we have that $A^L(y) \leq A^L(x)$ and $A^U(y) \leq A^U(x)$. Hence $A(x) = A(y)$. □

**Proposition 4.5.** Let $A$ be an IVNG of a group $G$ and let $x \in G$. Then $Ax(xg) = Ax(gx) = A(g)$ for each $g \in G$.

**Proof.** Let $g \in G$. Then

$$Ax(xg) = [A^L(xg), A^U(xg)]$$

$$= [A^L(xgx^{-1}x), A^U(xgx^{-1}x)]$$

$$= [A^L(xgx^{-1}x), A^U(xgx^{-1}x)]$$

(By the definition of $Ax$)

$$= [A^L(xgx^{-1}), A^U(xgx^{-1})]$$

$$= [A^L(g), A^U(g)]$$

(By Theorem 2.11)

$$= A(g).$$

Similarly, we have that $Ax(gx) = A(g)$. This completes the proof.

**Remark 4.6.** Proposition 4.5 is analogous to the result in group theory that if $N \triangleleft G$, then $N：<x \in G$. For IVNG, we have the analogous result:

**Proposition 4.7.** Let $A$ be an IVNG of a group $G$ and let $H/A$ be the set of all the interval-valued fuzzy cosets of $A$. We define an operation $*$ on $G/A$ as follows: For any $x, y \in G$, $Ax * Ay = Axy$. Then $(G/A, *)$ is a group. In this case, $A$ is called the interval-valued fuzzy quotient group induced by $A$.

**Proof.** Let $x, y, z, x_0, y_0 \in G$ such that $Ax = Ax_0$ and $Ay = Ay_0$, and let $g \in G$. Then $Ax_0y(g) = A(gy^{-1}x^{-1})$ and $Ax_0y_0(g) = A(gy_0^{-1}x_0^{-1})$. On the other hand,

$$A^L(gy^{-1}x^{-1}) = A^L(gy_0^{-1}y_0^{-1}x^{-1})$$

$$A^L(gy_0^{-1}x_0^{-1}x_0) = A^L(gy_0^{-1}x^{-1}) \\ A^L(gy_0^{-1}x^{-1}) \geq A^L(gy_0^{-1}x_0^{-1}) \wedge A^L(gy_0^{-1}x^{-1}).$$

(See $A \in IVG(G))$ (4.1)

By the similar arguments, we have that

$$A^U(gy^{-1}x^{-1}) \geq A^U(gy_0^{-1}x_0^{-1}) \wedge A^U(gy_0^{-1}x_0^{-1}).$$

(4.2)

Since $Ax = Ax_0$ and $Ay = Ay_0$, $A(gx^{-1}) = A(gx_0^{-1})$ and $A(gy^{-1}) = A(gy_0^{-1})$. In particular,

$$A(x_0y_0^{-1}x_0^{-1}) = A(x_0y_0^{-1}x_0^{-1})$$

$$A(Ay_0^{-1}) (Since A \in IVNG(G))$$

$$= A(e).$$

So $[A^L(gy_0^{-1}x^{-1}), A^U(gy_0^{-1}x_0^{-1})] = [A^L(e), A^U(e)]$. By Result 1.B(b), $A^L(e) \geq A^L(gy_0^{-1}x_0^{-1})$ and $A^U(e) \geq A^U(gy_0^{-1}x_0^{-1})$. Thus, by (4.1) and (4.2),

$$A^L(gy^{-1}x^{-1}) \geq A^L(gy_0^{-1}x_0^{-1})$$

and

$$A^U(gy_0^{-1}x_0^{-1}) \geq A^L(gy_0^{-1}x_0^{-1}).$$

(See $A \in IVNG(G)$)

$$A(gy_0^{-1}x_0^{-1}) = A(gy_0^{-1}x_0^{-1}).$$

(i) * is associative.

(ii) $Ax^{-1}$ is the inverse of $Ax$ for each $x \in G$.

(iii) $Ae = A$ is the identity of $G/A$. Therefore $(G/A, *)$ is a group. This completes the proof. □

**Proposition 4.8.** Let $A$ be an IVNG of a group $G$. We define a mapping $\bar{A} : G/A \rightarrow D(I)$ as follows: For each $x \in G$, $\bar{A}(Ax) = Ax$. Then $\bar{A}$ is an IVG of $G/A$. In this case, $\bar{A}$ is called the interval-valued fuzzy subquotient group determined by $A$.

**Proof.** From the definition of $\bar{A}$, it is clear that $\bar{A} \in$...
Let \( x, y \in G \). Then
\[
D(I)^G_A := \{ (Ax) \mid x \in G \}.
\]
By the similar arguments, we have that \( B^U(xy) \geq B^U(x) \wedge B^U(y) \). Since \( A^* \in \text{IVG}(G/A) \),
\[
A^*(Ax) = A^*(Ay).
\]
We define a mapping \( \overline{H} \). Hence
\[
\overline{H} : G/A \rightarrow \overline{G}/A,
\]
Let \( \pi \in \text{ker}(\overline{H}) \). We define a mapping \( \overline{K} \).
Thus
\[
\overline{K} : A \rightarrow \overline{A}
\]
and
\[
\overline{A}(x) = \overline{A}(y).
\]
Hence \( A \in \text{IVG}(G/A) \).

**Proposition 4.9.** Let \( A \) be an IVNG of a group \( G \). We define a mapping \( \pi : G \rightarrow G/A \) as follows: For each \( x \in G \), \( \pi(x) = Ax \). Then \( \pi \) is a homomorphism with \( \text{ker}(\pi) = G_A \). In this case, \( \pi \) is called the natural homomorphism.

**Proof.** Let \( x, y \in G \). Then \( \pi(xy) = Axy = Ax \ast Ay = \pi(x) \ast \pi(y) \). So \( \pi \) is a homomorphism. Furthermore,
\[
\text{ker}(\pi) = \{ x \in G : \pi(x) = Ae \}
\]
\[
= \{ x \in G : A(x) = Ae \}
\]
\[
= \{ x \in G : Ax(x) = Ae(x) \}
\]
\[
= \{ x \in G : A(e) = A(e) \}
\]
\[
= G_A.
\]
This completes the proof.

Now we obtain for interval-valued fuzzy subgroups an analogous result of the “Fundamental Theorem of Homomorphism of Groups”.

**Proposition 4.10.** Let \( A \in \text{IVNG}(G) \). Then each interval-valued fuzzy(normal) subgroup of \( G/A \) corresponds in a natural way to an interval-valued fuzzy(normal) subgroup of \( G \).

**Proof.** Let \( A^* \) be an interval-valued fuzzy subgroup of \( G/A \). Define a mapping \( B : G \rightarrow D(I) \) as follows: For each \( x \in G \), \( B(x) = A^*(Ax) \). By the definition of \( B \), it is clear that \( B \in D(I)^G \). Let \( x, y \in G \). Then
\[
B^U(xy) = A^U(Axy)
\]
\[
= A^*(Ax) \ast Ay
\]
\[
\geq A^L(Ax) \wedge A^L(Ay) \quad \text{(Since } A^* \in \text{IVG}(G/A) \text{)}
\]
\[
= B^L(x) \wedge B^L(y).
\]
Since \( k \) divides the order of \( G \), \( |G/A| \) also divides the order of \( G \). This completes the proof.

References


