Generalized Unequal Length Lapped Orthogonal Transform for Subband Image Coding

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***Abstract*—In this paper, generalized linear phase lapped orthogonal transforms with unequal length basis functions (GULLOTs) are considered. The length of each basis of the proposed GULLOT can be different from each other, whereas all the bases of the conventional GenLOT are of equal length. In general, for image coding application, the long basis for a low-frequency band and the short basis for a high-frequency one are desirable to reduce the blocking and the ringing artifact simultaneously. Therefore, the GULLOT is suitable especially for a subband image coding. In order to apply the GULLOT to a subband image coding, we also investigate the size-limited structure to process the finite length signal, which is important in practice. Finally, some design and image coding examples are shown to confirm the validity of the proposed GULLOT.**

***Index Terms*—GenLOT, subband image coding, symmetric extension method, unequal length basis functions.**

I. Introduction

LAPPED orthogonal transforms (LOTs) play an important role in transform coding and have been extensively studied [1]–[10]. In [5]–[7], Malvar and Staelin have formalized its design, implementation, and size-limited structure. It is well known that the LOT is a particular class of linear-phase paraunitary filterbanks, where the length of each filter L is restricted to twice the number of channels M $(L = 2M)$. In [8], the LOT has been generalized as the generalized lapped orthogonal transforms (GenLOTs: $L = NM$, N is a positive integer). The GenLOT can considerably reduce the blocking effect, which is a shortcoming of the DCT; however, it tends to spread the quantization error out to neighboring blocks. This causes more severe ringing artifacts around stronger edges of an image because of the long basis. One of the solutions to this problem is to use the GenLOT with unequal length basis functions, which are defined as the generalized linear phase unequal length lapped orthogonal transforms (GULLOTs). The GULLOT should have longer filters in the lower frequency bands to reduce the blocking, whereas the lengths of higher frequency bands must be shorter to suppress the ringing. This is similar to the discrete wavelet transform that has unequal length basis functions. The widths of frequency bands are unequal as well, whereas the GULLOT divides the signal uniformly in the frequency domain. Such GULLOT's have been reported by Tran and Nguyen [9], who showed that the same lattice structure as the GenLOT can be used and derived some conditions for the first and the last lattice blocks. Such a structure produces both longer and shorter filters; however, the fast algorithm is not involved. This is because the first block cannot be the DCT. Therefore, the computational complexity is the same as the equal length case since the GULLOT has the identical structure with that of the equal length filterbank. This means that this structure is not computationally efficient. Moreover, the unequal length property may be lost, i.e., outside region of the shorter filter's support is not absolutely zero, when the parameters (plane rotation angles) are quantized. This happens because all coefficients located outside of the shorter filter's support are forced to be zero by restricting the parameters.

In this paper, we propose a GULLOT that has an efficient fast algorithm. Unlike the GULLOT in [9], our proposed GULLOT produces both longer and shorter filters structurally. Therefore, the unequal length property is completely guaranteed in spite of the parameter's quantization. Since no restriction is required for the parameters, the first lattice block can be DCT. This means the fast algorithm for the DCT is applicable. The proposed GULLOT can be considered as a generalization of the GenLOT, including unequal length filters.

We also investigate the implementation over finite-length signals, namely, the symmetric extension method for the GULLOT. For fast implementation, the size-limited structure for the GULLOT is derived based on the symmetric extension method. Some design examples and the results of image coding application show that our proposed GULLOT is valid.

This paper is organized as follows. The GenLOT and the LOT with unequal length basis functions (ULLOT) is briefly reviewed in the next section. Section III discusses the GULLOT. In Section IV, the symmetric extension method and the size-limited structure for the GULLOT are investigated. Some design and coding examples are shown in Section V. Finally, conclusions are presented in Section VI.

Notations: The bold-faced capital and small letters indicate matrices and vectors, respectively. $\mathbf{A}^T$ denotes the transpose of matrix $\mathbf{A}$. Matrices $\mathbf{I}_a$, $\mathbf{J}_a$ and $\mathbf{0}_{a,b}$ represent identity matrix of size $a \times a$, counter identity matrix of size $a \times a$, and null matrix

Manuscript received October 29, 1999; revised August 3, 2000. This work was supported in part by the Ministry of Education, Science, Sports, and Culture, Government of Japan, Encouragement of Young Scientists (A), 11750313, 1999. The associate editor coordinating the review of this paper and approving it for publication was Dr. Paulo J. S. G. Ferreira.

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Publisher Item Identifier S 1053-587X(00)10149-7.

of size $a \times b$, respectively. The $a \times a$ permutation matrix $\mathbf{R}_a$ is defined as

$$\mathbf{R}_a = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \vdots & & \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & \vdots & & \end{bmatrix}.$$

We often abbreviate the subscript when the size is obvious. diag[$\mathbf{x}$] represents the diagonal matrix whose diagonal entry is vector $\mathbf{x}$. $\lfloor x \rfloor$ and $\lceil x \rceil$ round the positive real number x to the nearest integer toward minus infinity and infinity, respectively.

II. Preliminaries

A. GenLOT

In this subsection, the GenLOT is briefly reviewed. See [5]–[8] and [10] for more details.

Let M and $L = NM$ be the block size (the number of channels) and the length of each basis (filter), respectively. The GenLOTs transform an infinite input signal $\mathbf{x}$ into an infinite transform coefficient $\mathbf{y}$ as

$$\mathbf{y} = \begin{bmatrix} \ddots & & & & \mathbf{0} \\ & \mathbf{P} & & & \\ & \mathbf{0}_M & \mathbf{P} & & \\ & & \mathbf{0}_M & \mathbf{P} & \\ \mathbf{0} & & & & \ddots \end{bmatrix} \mathbf{x} \tag{1}$$

where $\mathbf{P}$ is an $M \times NM$ matrix that contains the basis functions. When $N = 2$, the transform is called the LOT. The GenLOTs extend the overlapping factor N to $N > 2$. The orthogonality of the basis functions is achieved if and only if [5]

$$\mathbf{P}\mathbf{S}^m\mathbf{P}^T = \delta(m)\mathbf{I}, \quad m = 0, 1, \ldots, N-1 \tag{2}$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{(N-1)M,M} & \mathbf{I}_{(N-1)M} \\ \mathbf{0}_{M,M} & \mathbf{0}_{M,(N-1)M} \end{bmatrix}.$$

Since the above time-domain conditions are difficult to handle, it is better to represent GenLOTs by the filterbank's notation. It is well known that the GenLOT is a particular class of linear-phase paraunitary filterbank (LPPUFB) that has been factorized in [4]. The GenLOTs can be represented as the polyphase transfer matrix (PTM)

$$\mathbf{E}(z) = \mathbf{R}_M^T \mathbf{K}_{N-1}(z)\mathbf{K}_{N-2}(z)\ldots\mathbf{K}_1(z)\mathbf{R}_M\mathbf{C}_M^{\mathrm{II}}\mathbf{J}_M \tag{3}$$

where

$$\mathbf{K}_i(z) = \frac{1}{2}\mathbf{\Phi}_i\mathbf{W}\mathbf{\Lambda}(z)\mathbf{W}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix}$$

$$\mathbf{\Lambda}(z) = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{M/2} \end{bmatrix}$$

$$\mathbf{\Phi}_i = \begin{bmatrix} \mathbf{U}_{M/2,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{M/2,i} \end{bmatrix}.$$

$\mathbf{C}_M^{\mathrm{II}}$ denotes the M-point DCT-II matrix, and the matrices $\mathbf{U}_{M/2,i}$ and $\mathbf{V}_{M/2,i}$ are $M/2 \times M/2$ real orthogonal matrices. The cascade structure of GenLOTs is illustrated in Fig. 1. For fast implementation, $\mathbf{U}_{M/2,i}$ and $\mathbf{V}_{M/2,i}$ are chosen as

$$\mathbf{U}_{M/2,i} = \mathbf{I}_{M/2}, \quad \mathbf{V}_{M/2,i} = \mathbf{\Theta}_{M/2-1,i}\ldots\mathbf{\Theta}_{2,i}\mathbf{\Theta}_{1,i} \tag{4}$$

where

$$[\mathbf{\Theta}_{j,i}]_{k,l} = \begin{cases} 1, & \text{for } k = l\ (\neq j \text{ and } \neq j+1) \\ \cos\theta_{j,i}, & \text{for } k = l\ (=j \text{ or } =j+1) \\ \sin\theta_{j,i}, & \text{for } k = j \text{ and } l = j+1 \\ -\sin\theta_{j,i}, & \text{for } k = j+1 \text{ and } l = j \\ 0, & \text{otherwise.} \end{cases}$$

The parameters $\theta_{j,i}$s are determined to maximize (or minimize) a particular cost function using a nonlinear optimization technique. The maximization of the stopband attenuation of the filters is one of the criteria for designing a GenLOT. Another possible cost function to be maximized is the following coding gain, which reflects coding efficiency of the transform over PCM

$$G_{TC} = \frac{\sum_{k=0}^{M-1}\sigma_{y_k}^2}{M\left(\prod_{k=0}^{M-1}\sigma_{y_k}^2\right)^{\frac{1}{M}}} \tag{5}$$

where $\sigma_{y_k}^2$ denotes a variance of the kth subband channel [1].

B. ULLOT

In [15], we developed the ULLOT. The PTM of the Fast ULLOT can be written as

$$\mathbf{E}(z) = \hat{\mathbf{\Phi}}_\alpha\mathbf{A}_\alpha\hat{\mathbf{W}}_\alpha\mathbf{\Lambda}(z)\hat{\mathbf{W}}_\alpha\mathbf{R}_M\mathbf{C}_M^{\mathrm{II}}\mathbf{J}_M \tag{6}$$

where

$$\hat{\mathbf{\Phi}}_\alpha = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{M/2-\alpha,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_\alpha \end{bmatrix}$$

$$\hat{\mathbf{W}}_\alpha = \begin{bmatrix} \mathbf{I}_{M/2-\alpha} & \mathbf{0} & \mathbf{I}_{M/2-\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_\alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{M/2-\alpha} & \mathbf{0} & -\mathbf{I}_{M/2-\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_\alpha \end{bmatrix}$$

$$\mathbf{A}_\alpha = \mathrm{diag}[\underbrace{1/2 \cdots 1/2}_{M/2-\alpha}\ \underbrace{1 \cdots 1}_{\alpha}\ \underbrace{1/2 \ldots 1/2}_{M/2-\alpha}\ \underbrace{1 \ldots 1}_{\alpha}].$$

The system generated by (6) has 2α shorter bases $(L_i = M)$ and $M - 2\alpha$ longer ones $(L_i = 2M)$. In the next section, the above ULLOT is generalized as the GULLOT.

III. Formulation of the GULLOT

We will derive the structure of GULLOTs in this section. We assume $M = 2^n$ since the fast algorithm for the DCT is expected. The length of ith basis is denoted as $L_i = N_iM$. In this case, the necessary conditions for the existence of LPPUFBs are as follows [2], [10]:

a) There are $M/2$ symmetric and $M/2$ anti-symmetric filters: *Symmetry Polarity Condition*

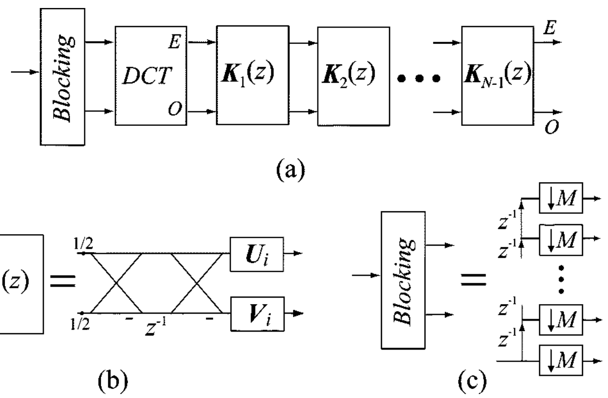

Fig. 1. Cascade structure of GenLOTs. Each line carries $M/2$ samples. E and O represent even and odd coefficients, respectively. (a) Whole structure of analysis GenLOT. (b) Details of the ith block. (c) Details of the blocking.

b) $\sum_{i=0}^{M-1} N_i$ must be even: *Length Condition*
In this paper, we impose the following restriction on GULLOTs.

c) There exists a pair of symmetric and anti-symmetric filters with same N_i.

It can be easily seen that requirement c) contains both conditions a) and b) since $\sum_{i=0}^{M-1} N_i = 2\sum_{i=0}^{M/2-1} N_i$. Moreover, c) is more restrictive than a) and b). For example, although the overlapping factor $[N_0\ N_1\ N_2\ N_3] = [3\ 2\ 2\ 1]$ is permissible under conditions a) and b), conditon c) does not allow such a choice. As we will see later, the structure of our proposed GULLOT is based on the butterfly shown in Fig. 3(b). It reminds us that there exists at least one pair of symmetric and anti-symmetric filters with same N_i. Even if the LPPUFB that does not satisfy requirment c) exists, some restrictions on the parameters $\theta_{j,i}$ must be required to implement it. We ignore such a class of LPPUFBs since our goal is to impose the unequal length property on GULLOTs structurally.

We assume that $N_0 \geq N_1 \geq \cdots \geq N_{M-1}$ and $\max(N_i) = N_0 = N$. The basic idea for constructing the GULLOT is to create some longer filters by adding some extra lattice blocks to the LOT/ULLOT. Here, we classify the GULLOTs into four types as follows.

1) Type A: All N_is are odd numbers.
2) Type B: All N_is are even numbers.
3) Type C: N_is consist of both odd and even numbers and N is odd.
4) Type D: N_is consist of both odd and even numbers, and N is even.

These four types of GULLOTs are illustrated in Fig. 2. It should be mentioned that types C and D have two different centers of symmetry, that is, one is for odd length filters, and the other one is for even length filters. The distances between them are always $M/2$.

A. Types A and B

Our proposed GULLOT transforms an input signal in the same way as (1). The finite length version of the transform will be disucussed later. Our generalization allows the basis functions of the GULLOT to have variable lengths of support. There-

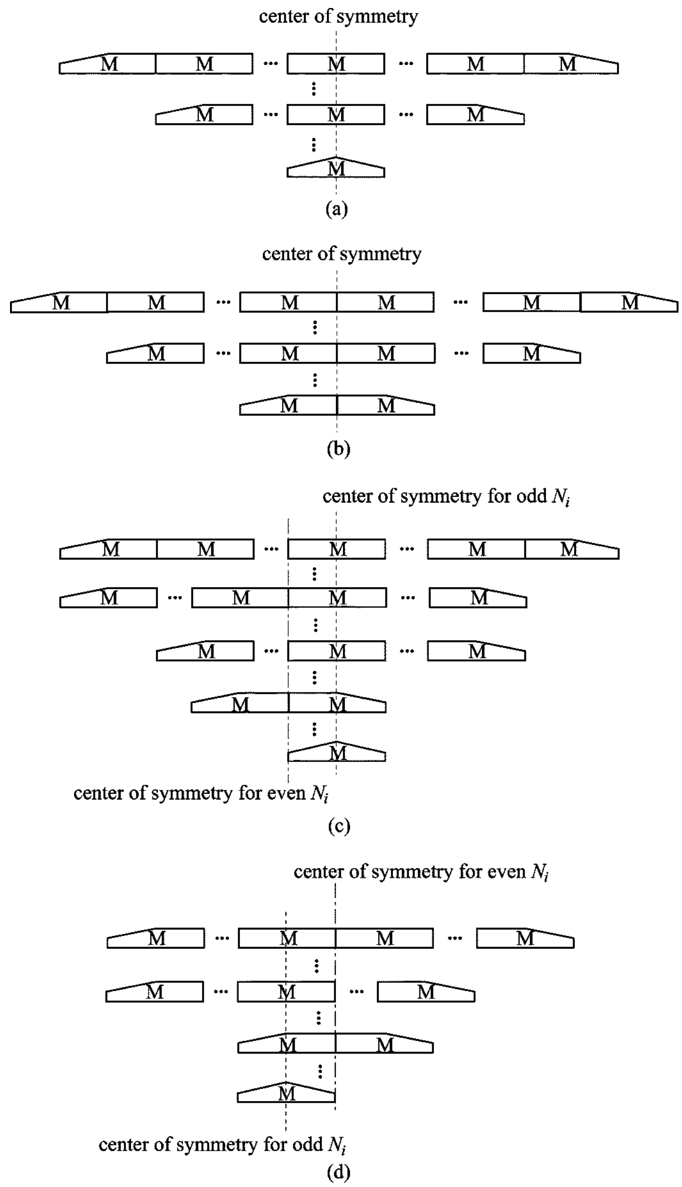

Fig. 2. Four types of GULLOT's. (a) Type A. (b) Type B. (c) Type C. (d) Type D.

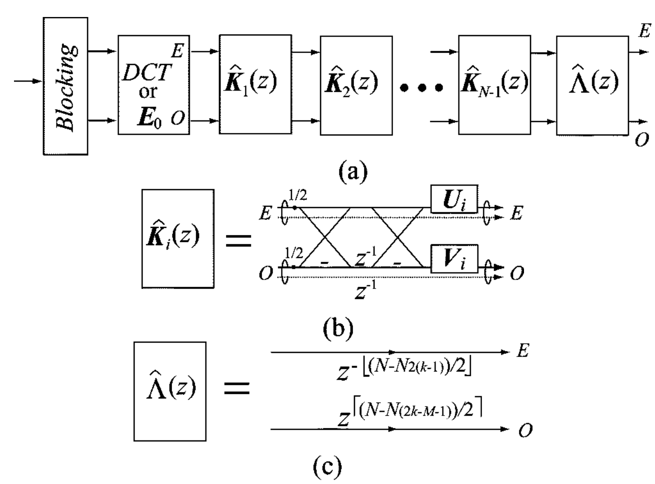

Fig. 3. Cascade structure of the GULLOT's. (a) Whole structure of the analysis GULLOT. Each line carries $M/2$ samples. (b) Details of the ith block. (c) Details of $\hat{\Lambda}(z)$.

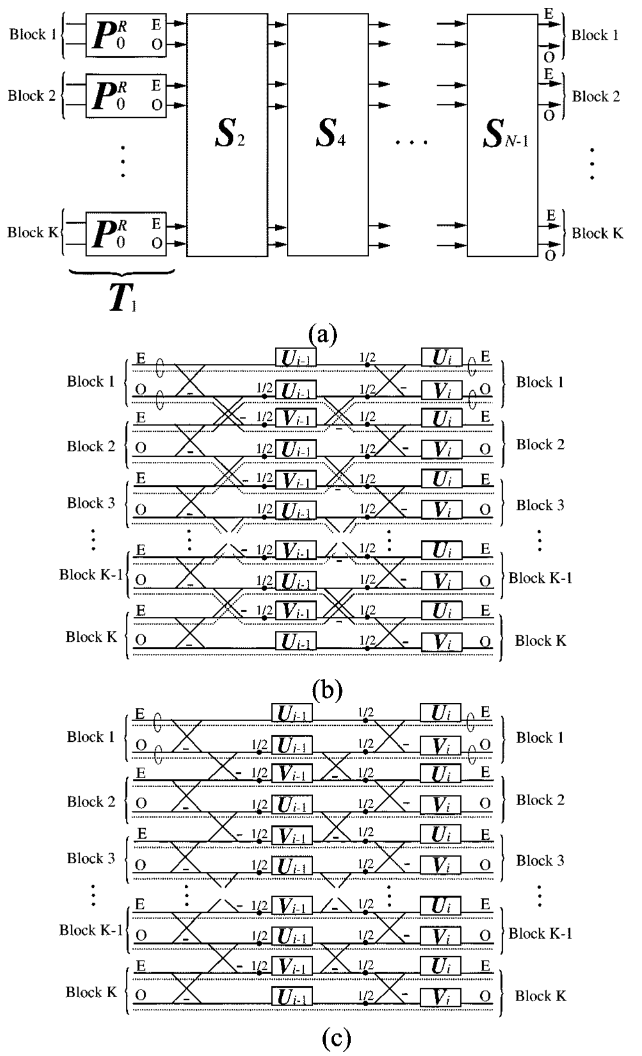

Fig. 4. Flowgraph for implementation of Type A GULLOTs. (a) Whole structure of Type A GULLOTs. Each line carries $M/2$ samples. (b) Flowgraph for implementation of each block $\mathbf{S}_i$. Each solid line carries $M/2-\alpha_i$ samples, and each broken line is a set of α_i samples. (c) Simplified implementation of each block $\mathbf{S}_i$.

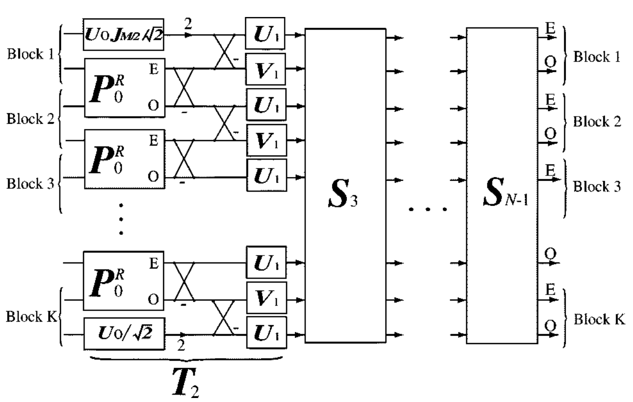

Fig. 5. Flowgraph for implementation of Type B GULLOT's. Each line carries $M/2$ samples.

fore, the matrix $\mathbf{P}$ in (1) looks like Fig. 2. Of course, the outside regions of the shorter basis's support are padded by zeros to keep the transform matrix rectangle. Here, we also use the filterbank's representation, where the orthogonality simply means a cascade of orthogonal matrices.

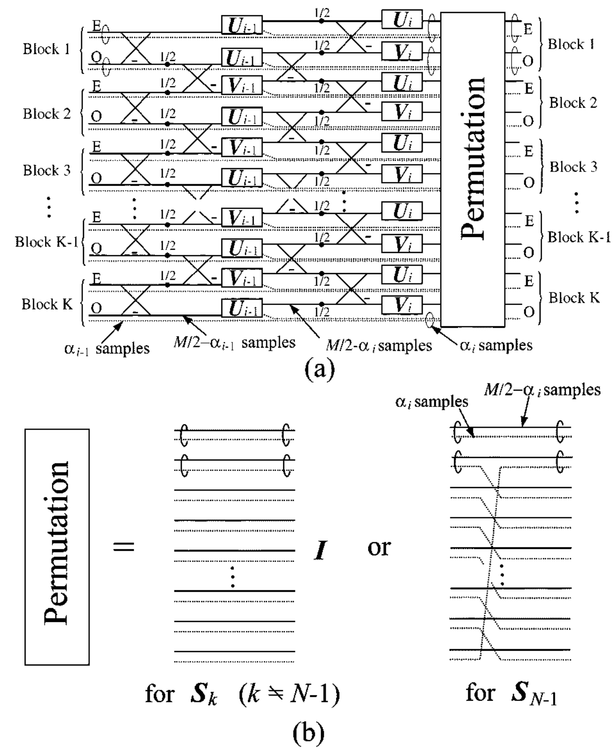

Fig. 6. Flowgraph for implementation of each block $\mathbf{S}_i$ for the Type C GULLOT. (a) Structure of $\mathbf{S}_i$. (b) Details of the permutation.

We first define the PTM of the GULLOT for all types as

$$\mathbf{E}(z) = \mathbf{R}_M^T \hat{\mathbf{\Lambda}}(z) \hat{\mathbf{K}}_{N-1}(z) \hat{\mathbf{K}}_{N-2}(z) \ldots \hat{\mathbf{K}}_1(z) \mathbf{R}_M \mathbf{C}_M^{\mathrm{II}} \mathbf{J}_M \tag{7}$$

where

$$\hat{\mathbf{K}}_i(z) = \hat{\mathbf{\Phi}}_i \mathbf{A}_{\alpha_i} \hat{\mathbf{W}}_{\alpha_i} \mathbf{\Lambda}(z) \hat{\mathbf{W}}_{\alpha_i}$$

$$\hat{\mathbf{\Phi}}_i = \begin{bmatrix} \mathbf{U}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\alpha_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\alpha_i} \end{bmatrix} \equiv \begin{bmatrix} \bar{\mathbf{U}}_i & \mathbf{0}_{\frac{M}{2},\frac{M}{2}} \\ \mathbf{0}_{\frac{M}{2},\frac{M}{2}} & \bar{\mathbf{V}}_i \end{bmatrix}.$$

The matrices $\mathbf{U}_i$ and $\mathbf{V}_i$ can be any real orthogonal matrices of size $(M/2-\alpha_i) \times (M/2-\alpha_i)$. However, the choice of (4) is suitable for fast implementation. In the above equation, α_i represents

$$\alpha_i = \frac{M - \mathcal{N}(\mathbf{m}_i)}{2}, \quad 1 \le i \le N-1 \tag{8}$$

where $\mathcal{N}(\mathbf{a})$ denotes the number of non-negative (zero or positive) elements of $\mathbf{a}$, and

$$\mathbf{m}_i = [N_0 - i - 1 \; N_1 - i - 1 \; \ldots \; N_{M-1} - i - 1].$$

It seems that the number α_i may come out fractional; however, this is not true. From requirement c), it turns out that $\mathcal{N}(\mathbf{m}_i)$ is

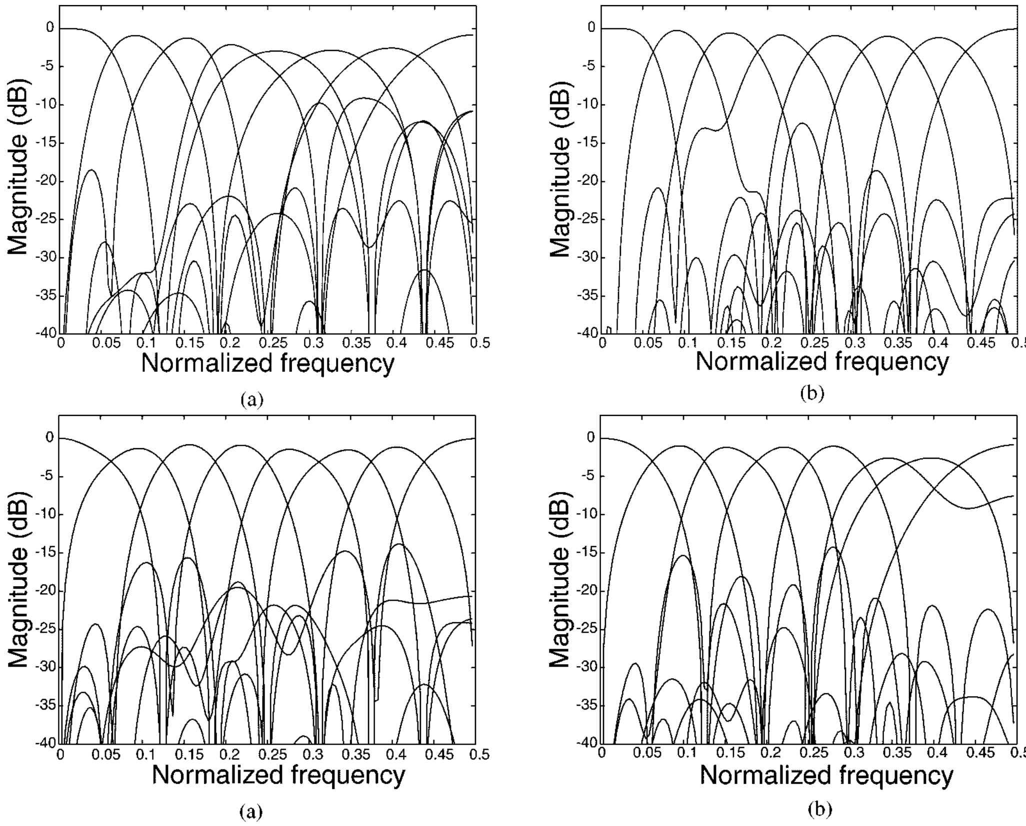

Fig. 7. Frequency responses of eight-channel GULLOTs. (a) GULA. (b) GULB. (c) GULC. (d) GULD.

always even. This fact shows that α_i is always an integer. The matrix $\hat{\mathbf{\Lambda}}(z)$ is a diagonal matrix whose entries are

$$[\hat{\mathbf{\Lambda}}(z)]_{k,k} = \begin{cases} z^{-\lfloor (N-N_{2(k-1)})/2 \rfloor}, & \text{for } k \le \frac{M}{2} \\ z^{\lceil (N-N_{(2k-M-1)})/2 \rceil}, & \text{for } k > \frac{M}{2}. \end{cases} \tag{9}$$

For Types A and B, the above matrix $\hat{\mathbf{\Lambda}}(z)$ aligns the center of all filters. Therefore, the PTM (7) must satisfy the following relationship for the linear phase (LP) property [10]:

$$\mathbf{E}_R(z) = z^{-(N-1)}\mathbf{D}_M\mathbf{E}_R(z^{-1})\mathbf{J}_M \tag{10}$$

where

$$\mathbf{E}_R(z) = \mathbf{R}_M\mathbf{E}(z), \quad \mathbf{D}_M = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{M/2} \end{bmatrix}.$$

The first block $\mathbf{R}_M\mathbf{C}_M^{\text{II}}\mathbf{J}_M$ can be replaced by the orthogonal matrix $\mathbf{E}_0$ such that

$$\mathbf{E}_0 = \mathbf{\Phi}_0\mathbf{W}\mathbf{Q} \tag{11}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & \mathbf{J}_{M/2} \end{bmatrix}.$$

In this case, the transform is called the LPPUFB with unequal length filters. Although the coding gain of the LPPUFB is slightly higher than that of the GenLOT, we prefer to use the DCT matrix for the first block since the fast algorithm is available.

To show that the PTM realized by (7) is the valid GULLOT, the orthogonality, the unequal length property, and the LP property of (7) must be proven. Since the orthogonality and the unequal length property of the PTM are obvious, we would like to prove the following theorem.

Theorem 1: The GULLOT generated by (7) has the LP property.

Proof: Let $\mathbf{E}(z)$ be the GULLOT (Type A or Type B) or the GenLOT that satisfies (10). The overlapping factor of $\mathbf{E}(z)$ is assumed to be $\mathbf{n} = [N_0\ N_1\ \ldots\ N_{M-1}]$. By adding two more blocks, we expect to obtain the new GULLOT whose overlapping factor $\bar{\mathbf{n}}$ would be

$$\begin{aligned} \bar{\mathbf{n}} &= [\bar{N}_0\ \bar{N}_1\ \ldots\ \bar{N}_{M-1}] \\ &= \left[\underbrace{N_0+2\ \ldots\ N_{\beta-1}+2}_{\beta}\ \underbrace{N_\beta\ \ldots\ N_{M-1}}_{M-\beta}\right] \end{aligned} \tag{12}$$

where β is a positive even integer. At this time, the PTM of the new GULLOT can be written as follows:

$$\begin{aligned} \bar{\mathbf{E}}_R(z) &= \bar{\mathbf{\Lambda}}(z)\hat{\mathbf{K}}_{N+1}(z)\hat{\mathbf{K}}_N(z)\hat{\mathbf{\Lambda}}(z^{-1})\mathbf{E}_R(z) \\ &= \mathbf{Z}(z)\hat{\mathbf{K}}_{N+1}(z)\hat{\mathbf{K}}_N(z)\mathbf{E}_R(z) \end{aligned} \tag{13}$$

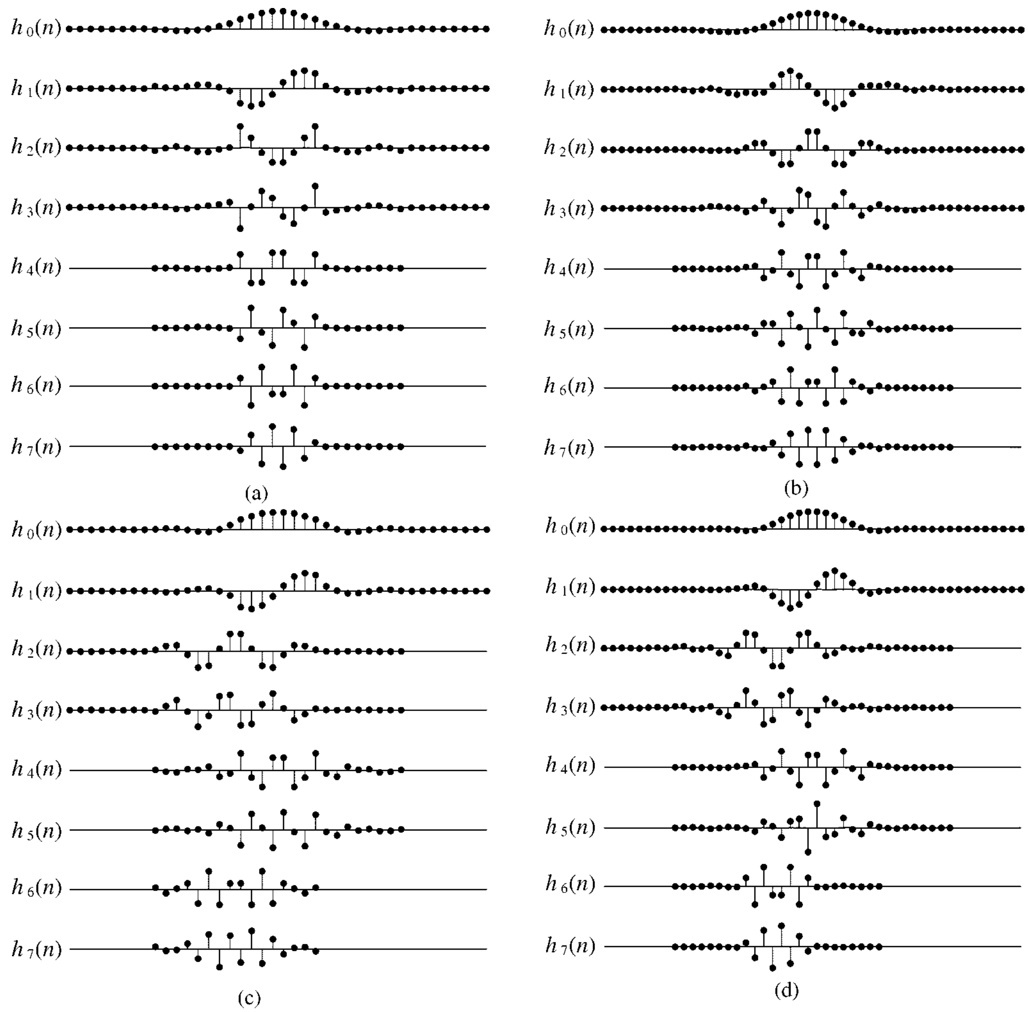

Fig. 8. Impulse responses of eight-channel GULLOTs. $h_0(n)$ corresponds to the basis for the lowest frequency band, and $h_7(n)$ denotes the basis for the highest frequency one. (a) GULA. (b) GULB. (c) GULC. (d) GULD.

TABLE I
COEFFICIENTS OF GULA

h0(n)	h1(n)	h2(n)	h3(n)	h4(n)	h5(n)	h6(n)	h7(n)
-0.000422	-0.000472	0.001017	-0.000995	0	0	0	0
0.001007	-0.001916	0.001672	-0.000374	0	0	0	0
0.002660	-0.001672	-0.000150	0.002075	0	0	0	0
0.000140	0.001474	-0.002113	0.001521	0	0	0	0
-0.003509	0.004987	-0.003550	-0.000223	0	0	0	0
-0.000646	0.005085	-0.006270	0.003725	0	0	0	0
0.004964	0.000541	-0.005215	0.007180	0	0	0	0
0.006613	-0.004634	0.000269	0.004726	0	0	0	0
-0.014196	0.053366	-0.067653	0.037837	0.001853	-0.001822	-0.000194	0.000186
-0.022922	0.020552	-0.010200	0.002045	0.005809	-0.005751	-0.000317	0.000290
-0.039688	0.014082	0.034714	-0.042061	0.005565	-0.005551	0.001334	-0.001361
-0.043088	0.045382	-0.009162	-0.035223	-0.004068	0.004000	0.000125	-0.000106
-0.025132	0.084470	-0.093058	0.011961	-0.018332	0.018189	-0.002365	0.002452
0.008004	0.093337	-0.101030	0.041263	-0.014413	0.014209	0.000545	-0.000477
0.066900	0.030453	-0.039814	0.072882	-0.004112	0.003977	0.002345	-0.002326
0.128639	-0.059680	0.004790	0.121768	0.015939	-0.015908	-0.001605	0.001529
0.232919	-0.359907	0.523516	-0.516451	0.335762	-0.265918	0.193141	-0.100230
0.303605	-0.428942	0.244899	0.012615	-0.355250	0.478238	-0.463967	0.282739
0.383222	-0.373432	-0.118605	0.394947	-0.344706	-0.114913	0.460060	-0.417078
0.425142	-0.145186	-0.354056	0.224967	0.375953	-0.431841	-0.189101	0.485842

where

$$[\bar{\mathbf{\Lambda}}(z)]_{k,k} = \begin{cases} z^{-(\bar{N}_0 - \bar{N}_{2(k-1)})/2} & \text{for } k \le \frac{M}{2} \\ z^{(\bar{N}_0 - \bar{N}_{(2k-M-1)})/2} & \text{for } k > \frac{M}{2} \end{cases}$$

$$\hat{\mathbf{K}}_{N+1}(z) = \hat{\mathbf{\Phi}}_{N+1} \hat{\mathbf{W}}_{\frac{M-\beta}{2}} \mathbf{\Lambda}(z) \hat{\mathbf{W}}_{\frac{M-\beta}{2}}$$

$$\hat{\mathbf{K}}_{N}(z) = \hat{\mathbf{\Phi}}_{N} \hat{\mathbf{W}}_{\frac{M-\beta}{2}} \mathbf{\Lambda}(z) \hat{\mathbf{W}}_{\frac{M-\beta}{2}}$$

$$\mathbf{Z}(z) = \text{diag}\left[\underbrace{1\ldots1}_{\beta/2}\underbrace{z^{-1}\ldots z^{-1}}_{(M-\beta)/2}\underbrace{1\ldots1}_{\beta/2}\ \underbrace{z\ldots z}_{(M-\beta)/2}\right].$$

If the new GULLOT in (13) satisfies (10), namely

$$\bar{\mathbf{E}}_R(z) = z^{-(\bar{N}_0 - 1)} \mathbf{D}_M \bar{\mathbf{E}}_R(z^{-1}) \mathbf{J}_M \tag{14}$$

it can be said that the PTM of (7) has LP property. Substituting (10) and (13) into (14), we can obtain

$$\bar{\mathbf{E}}_R(z) = \mathbf{Z}(z)\hat{\mathbf{Z}}(z)\mathbf{D}_M \hat{\mathbf{K}}_{N+1}(z^{-1})\hat{\mathbf{K}}_N(z^{-1})\mathbf{D}_M \mathbf{E}_R(z) \tag{15}$$

where

$$\hat{\mathbf{Z}}(z) = \text{diag}\left[\underbrace{z^{-2}\ldots z^{-2}}_{\beta/2}\ \underbrace{1\ldots1}_{(M-\beta)/2}\ \underbrace{z^{-2}\ldots z^{-2}}_{\beta/2}\underbrace{z^{-4}\ldots z^{-4}}_{(M-\beta)/2}\right].$$

Using the relationship (see the Appendix)

$$\hat{\mathbf{Z}}(z)\mathbf{D}_M \hat{\mathbf{K}}_{N+1}(z^{-1})\hat{\mathbf{K}}_N(z^{-1})\mathbf{D}_M = \hat{\mathbf{K}}_{N+1}(z)\hat{\mathbf{K}}_N(z) \tag{16}$$

TABLE II
COEFFICIENTS OF GULB

h0(n)	h1(n)	h2(n)	h3(n)	h4(n)	h5(n)	h6(n)	h7(n)
-0.000358	-0.000165	-0.000144	0.000349	0	0	0	0
0.001200	0.000493	0.000573	-0.001235	0	0	0	0
0.002054	0.000853	0.000966	-0.002103	0	0	0	0
0.000115	0.000080	0.000006	-0.000083	0	0	0	0
-0.002284	-0.000872	-0.001187	0.002422	0	0	0	0
-0.000866	-0.000281	-0.000523	0.000972	0	0	0	0
0.004030	0.001757	0.001772	-0.004037	0	0	0	0
0.007812	0.003366	0.003493	-0.007867	0	0	0	0
0.010331	0.010402	0.003385	-0.007076	0.000435	0.000385	0.000208	0.000012
0.007241	-0.012903	0.000152	-0.012252	0.001637	0.007503	0.000179	-0.004264
0.000931	-0.032904	0.004052	-0.016113	0.001988	0.011185	0.000232	-0.006396
-0.012577	-0.007972	-0.003652	0.010041	-0.001443	-0.003916	-0.000341	0.001937
-0.031098	0.025213	-0.018473	0.047455	-0.005719	-0.023212	-0.001443	0.012155
-0.044018	-0.009467	-0.012681	0.043315	-0.004768	-0.015812	-0.000923	0.008350
-0.049276	-0.087696	-0.011327	0.015994	-0.000041	0.010097	-0.000888	-0.007052
-0.042831	-0.125426	-0.031527	0.002858	0.002975	0.026483	-0.000573	-0.016452
-0.024009	-0.077239	0.048287	-0.089253	0.044419	0.017250	0.043435	0.035588
0.018397	-0.094516	0.159875	0.006243	0.064255	-0.124733	-0.083648	-0.030074
0.076774	-0.072033	0.158247	0.176727	-0.220076	0.124127	0.022435	-0.030053
0.155824	0.101533	-0.080088	-0.053516	-0.036304	0.111861	0.144300	0.106835
0.243573	0.355105	-0.358545	-0.386775	0.397020	-0.367326	-0.333857	-0.192837
0.318680	0.450972	-0.341402	-0.053997	-0.130697	0.349390	0.440196	0.322667
0.371962	0.324272	0.040297	0.438432	-0.419403	0.024866	-0.377583	-0.412258
0.402610	0.096905	0.438445	0.328819	0.305725	-0.443638	0.148272	0.416719

(15) comes to the definition (13). Hence, we can conclude that (14) holds. Taking into account the LP property of the first block $\mathbf{R}_M \mathbf{C}_M^{\rm II} \mathbf{J}_M$, the proof is completed. ■

It should be noted that when $N_i = N$, i.e., $\mathbf{n} = [N \dots N]$, α_i and $\hat{\mathbf{\Lambda}}(z)$ becomes $\alpha_i = 0$ and $\hat{\mathbf{\Lambda}}(z) = \mathbf{I}_M$, respectively. In this case, (7) is equivalent to (3). The structure of the proposed GULLOT is shown in Fig. 3.

B. Types C and D

As mentioned in the foregoing subsection, Type C and D GULLOTs can be factorized as (7). The proof of LP property is omitted since it is possible to prove in a similar way as above.

C. Transform Matrix

It is useful to consider the transform matrix $\mathbf{P}_{N-1}$ of the GULLOT $\mathbf{E}(z)$. This transform matrix $\mathbf{P}_{N-1}$ has size $M \times NM$, and the transformed coefficients $\mathbf{y}_i$ can be computed from the blocked input $\mathbf{x}_i$ as

$$\mathbf{y}_i = \mathbf{P}_{N-1}\mathbf{x}_i. \tag{17}$$

The transform matrix $\mathbf{P}_{N-1}$ can be obtained by the following recursion ($1 \leq i \leq N-1$):

$$\mathbf{P}_0 = \mathbf{C}_M^{\rm II}\mathbf{J}_M, \quad \text{or} \quad \mathbf{P}_0 = \mathbf{E}_0 \tag{18}$$

$$\mathbf{P}_i = \mathbf{R}_M^T \mathbf{P}_i^R \tag{19}$$

$$\mathbf{P}_i^R = \hat{\mathbf{\Phi}}_i \bar{\mathbf{Q}}_i \begin{bmatrix} \mathbf{P}_{i-1}^R & \mathbf{0}_{M,M} \\ \mathbf{0}_{M,M} & \mathbf{P}_{i-1}^R \end{bmatrix}. \tag{20}$$

The matrix $\bar{\mathbf{Q}}_i$ is

$$\bar{\mathbf{Q}}_i = \mathbf{D}_i \hat{\mathbf{W}}_{\alpha_i} \mathbf{A}_{\alpha_i} \bar{\mathbf{W}}_{\alpha_i} \tag{21}$$

where we have (22), shown at the bottom of the page. The matrices $\mathbf{W}_{\alpha_i}^{++}$, $\mathbf{W}_{\alpha_i}^{+-}$, $\mathbf{W}_{\alpha_i}^{-+}$, and $\mathbf{W}_{\alpha_i}^{--}$ are parts of the $M \times M$ matrix $\hat{\mathbf{W}}_{\alpha_i}$ such that

$$\begin{aligned}
&\mathbf{W}_{\alpha_i}^{++} \\
&\quad = [\,\mathbf{I}_{M/2-\alpha_i} \quad \mathbf{0}_{M/2-\alpha_i,\alpha_i} \quad \mathbf{I}_{M/2-\alpha_i} \quad \mathbf{0}_{M/2-\alpha_i,\alpha_i}\,] \\
&\mathbf{W}_{\alpha_i}^{+-} \\
&\quad = [\,\mathbf{0}_{\alpha_i,M/2-\alpha_i} \quad \mathbf{I}_{\alpha_i} \quad \mathbf{0}_{\alpha_i,M/2}\,], \\
&\mathbf{W}_{\alpha_i}^{-+} \\
&\quad = [\,\mathbf{I}_{M/2-\alpha_i} \quad \mathbf{0}_{M/2-\alpha_i,\alpha_i} \quad -\mathbf{I}_{M/2-\alpha_i} \quad \mathbf{0}_{M/2-\alpha_i,\alpha_i}\,] \\
&\mathbf{W}_{\alpha_i}^{--} = [\,\mathbf{0}_{\alpha_i,M-\alpha_i} \quad \mathbf{I}_{\alpha_i}\,].
\end{aligned}$$

The matrix $\mathbf{D}_i$ is the following $M \times M$ diagonal matrix:

$$\mathbf{D}_i = \text{diag}\left[\underbrace{1\dots1}_{M/2}\underbrace{-1\dots-1}_{M/2-\alpha_i}\underbrace{1\dots1}_{\alpha_i}\right]. \tag{23}$$

When $i = N-1$ (the last block), the matrix $\bar{\mathbf{W}}_{\alpha_{N-1}}$ in (22) must be slightly modified. Assuming that N is odd and N_{2k-1}, $(1 < k \leq M/2)$ is even, or vice versa. Then, the $(k+M/2)$th row of $\bar{\mathbf{W}}_{\alpha_{N-1}}$ is replaced with the following vector:

$$\mathcal{R}\left(\bar{\mathbf{W}}_{\alpha_{N-1}}, k+M/2\right) = [\mathbf{0}_{1,M/2+k-1}\; 1\; \mathbf{0}_{1,3M/2-k}]$$

where $\mathcal{R}(\mathbf{A}, i)$ denotes the ith row of the matrix $\mathbf{A}$.

IV. SIZE-LIMITED STRUCTURES FOR THE GULLOTS

To process finite-length signals, special care is required for the signal's boundary to avoid the border distortion. The symmetric extension method, which reflects the signal at its boundaries, is often used to overcome this problem

$$\bar{\mathbf{W}}_{\alpha_i} = \begin{cases} \begin{bmatrix} \mathbf{W}_{\alpha_i}^{-+} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\alpha_i}^{+-} \\ \mathbf{0} & \mathbf{W}_{\alpha_i}^{++} \\ \mathbf{W}_{\alpha_i}^{--} & \mathbf{0} \end{bmatrix} & \text{for } \begin{cases} N \text{ is odd} & \text{and} \quad i \text{ is odd} \\ N \text{ is even} & \text{and} \quad i \text{ is even} \end{cases} \\ \begin{bmatrix} \mathbf{W}_{\alpha_i}^{-+} & \mathbf{0} \\ \mathbf{W}_{\alpha_i}^{+-} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\alpha_i}^{++} \\ \mathbf{0} & \mathbf{W}_{\alpha_i}^{--} \end{bmatrix} & \text{for } \begin{cases} N \text{ is odd} & \text{and} \quad i \text{ is even} \\ N \text{ is even} & \text{and} \quad i \text{ is odd} \end{cases} \end{cases} \tag{22}$$

[11]–[14]. However, most of the symmetric extension methods presented in the literature use filterbanks with equal length. Thus, we show the nonexpansive symmetric extension method for the GULLOTs. Moreover, the symmetric extension methods have a disadvantage that the computational complexity for the transform and inverse transform increases because of the extended samples. In particular, for image processing, these computational overheads are not negligible. A reasonable solution would be to have the size-limited structure as mentioned in [14]. In the following section, the size-limited structures for the GULLOT will be discussed.

Let the finite signal be $\mathbf{x} = [x(0)\ x(1)\ \ldots\ x(L_x-1)]^T$ $(L_x = KM)$. The symmetric extension method expands an input signal at each boundary as

$$\Bigg[\underbrace{x\left(\frac{M(N-1)}{2}-1\right)\ldots x(0)}_{M(N-1)/2}\underbrace{x(0)\ldots x(L_x-1)}_{L_x}\underbrace{x(L_x-1)\ldots x\left(\frac{L_x-M(N-1)}{2}\right)}_{M(N-1)/2}\Bigg]^T = \mathbf{\Lambda}^J_{\frac{M(N-1)}{2}}\mathbf{x} \qquad (24)$$

where

$$\mathbf{\Lambda}^J_n = \begin{bmatrix} \mathbf{J}_n & & \mathbf{0} \\ & \mathbf{I}_{L_x} & \\ \mathbf{0} & & \mathbf{J}_n \end{bmatrix}. \qquad (25)$$

Then, the expanded signal is transformed by the $L_x \times (L_x + (N-1)M)$ matrix $\bar{\mathbf{T}}_N$, and transformed coefficients $\mathbf{y}$ can be obtained as

$$\begin{aligned} \mathbf{y} &= \bar{\mathbf{T}}_N \mathbf{\Lambda}^J_{\frac{M(N-1)}{2}}\mathbf{x} \\ &= \begin{bmatrix} \mathbf{P}^R_{N-1} & \mathbf{0}_M & & \\ \mathbf{0}_M & \mathbf{P}^R_{N-1} & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{P}^R_{N-1} \end{bmatrix} \mathbf{\Lambda}^J_{\frac{M(N-1)}{2}}\mathbf{x}. \end{aligned} \qquad (26)$$

Since the permutation matrix $\mathbf{R}^T_M$ can be applied after the transformation, $\mathbf{P}^R_i$ is considered instead of $\mathbf{P}_i$ for the sake of convenience. From (26), one can see that the matrix $\mathbf{T}_N$ has size $L_x \times L_x$. Thus, the input signal does not have to be extended for the transform. The purpose of this section is to find the structure of the matrix $\mathbf{T}_N$ for the GULLOT based on the implementation described in the foregoing section (see Fig. 3).

A. Symmetric Extension Method for GULLOTs

Before developing the size-limited structure, we show the nonexpansive symmetric extension method for GULLOTs.

TABLE III
COEFFICIENTS OF GULC

h0(n)	h1(n)	h2(n)	h3(n)	h4(n)	h5(n)	h6(n)	h7(n)
0.000014	-0.000014	-0.000212	0.000214	0	0	0	0
-0.001309	0.001309	-0.001874	0.001676	0	0	0	0
-0.002239	0.002239	-0.001539	0.001200	0	0	0	0
0.000714	-0.000714	0.002546	-0.002438	0	0	0	0
0.004324	-0.004324	0.003433	-0.002779	0	0	0	0
0.002843	-0.002843	0.007588	-0.007158	0	0	0	0
-0.002553	0.002553	-0.001660	0.001274	0	0	0	0
-0.004422	0.004422	0.000833	-0.001502	0	0	0	0
0.004234	-0.004202	0.039308	-0.034709	0.014352	-0.014154	0.046898	0.053377
0.029852	-0.029583	0.131262	0.096907	-0.033794	0.025314	-0.092077	-0.047958
0.022910	-0.022703	0.148025	0.243367	-0.046716	0.031443	0.023102	-0.024127
-0.012858	0.012481	-0.078553	-0.041512	0.026632	-0.022702	0.149525	0.128253
-0.048982	0.048512	-0.389366	-0.400377	0.017569	0.011714	-0.340867	-0.234363
-0.057751	0.056635	-0.345416	-0.139106	0.094735	-0.077918	0.438837	0.350611
0.008007	-0.007786	0.063614	0.335967	-0.149391	0.131931	-0.369862	-0.439699
0.109441	-0.109617	0.422011	0.368584	-0.076007	0.044069	0.144443	0.326543
0.244286	-0.389646	--	--	0.415209	-0.366969	--	--
0.319556	-0.434021	--	--	-0.170368	0.390650	--	--
0.387790	-0.351255	--	--	-0.401572	0.051147	--	--
0.410356	-0.130997	--	--	0.309353	-0.426860	--	--

TABLE IV
COEFFICIENTS OF GULD

h0(n)	h1(n)	h2(n)	h3(n)	h4(n)	h5(n)	h6(n)	h7(n)
-0.000079	0.000079	-0.001298	-0.001282	0	0	0	0
-0.000126	0.000126	-0.002057	-0.002056	0	0	0	0
0.000369	-0.000369	0.006080	0.005965	0	0	0	0
0.000070	-0.000070	0.001143	0.001141	0	0	0	0
-0.000555	0.000555	-0.009152	-0.008963	0	0	0	0
0.000264	-0.000264	0.004310	0.004306	0	0	0	0
0.000786	-0.000786	0.012930	0.012726	0	0	0	0
-0.000512	0.000512	-0.008363	-0.008346	0	0	0	0
0.000519	-0.000254	0.030920	0.031370	0.001357	-0.001194	0.001049	0.000041
-0.004771	0.004795	0.042959	0.039196	0.006298	0.002678	-0.003397	-0.002762
-0.003659	0.001788	-0.027809	-0.038413	0.000251	0.012872	-0.011000	-0.002816
0.002309	-0.002341	-0.025579	-0.023630	-0.003302	-0.001270	0.001180	0.000918
0.007258	-0.004172	0.011532	0.030795	-0.003213	-0.022502	0.018387	0.004615
0.008205	-0.008260	-0.118754	-0.110508	-0.013146	-0.005557	0.005969	0.004628
-0.007158	0.003826	-0.189718	-0.203396	-0.006246	0.019902	-0.017248	-0.003372
-0.023083	0.023373	0.071279	0.060376	0.023599	0.008670	-0.013947	-0.012510
-0.040793	0.045678	0.369403	0.411366	0.030643	-0.018636	0.204240	0.101626
-0.027311	0.093474	0.327228	0.119465	0.071384	-0.087455	-0.441295	-0.260503
0.026611	0.021207	-0.055168	-0.404316	-0.244272	0.145370	0.466971	0.402929
0.121261	-0.154048	-0.439884	-0.299885	-0.032048	0.038148	-0.210909	-0.509148
0.223211	-0.331139	--	--	0.392116	-0.235107	--	--
0.321763	-0.452826	--	--	-0.096387	0.154460	--	--
0.392132	-0.360441	--	--	-0.424989	0.217976	--	--
0.417502	-0.140336	--	--	0.297954	-0.584453	--	--

1) Types A and B: As illustrated in Fig. 2(a) and (b), the symmetry center of all bases are located at the same point in these cases. This fact shows that the same symmetric extension method used in the filterbanks with equal length is applicable to the GULLOTs of types A and B.

2) Types C and D: The main difficulty in developing the symmetric extension method for these classes comes from the fact that the GULLOTs of Types C and D have two different centers of symmetry. For example, let us consider the case where $M = 4$ and $\mathbf{n} = [3\ 3\ 2\ 2]$. In this case, each subband sequence looks like the first expression at the bottom of the next page, where HS, HA, WS, and WA denote half sample symmetry, half sample anti-symmetry, whole sample symmetry, and whole sample anti-symmetry, respectively. All samples enclosed by $\square$ must be transmitted to reconstruct the original sequence. Since the number of the samples that must be sent is greater than that of the input signal, it seems that the nonexpansive extension is impossible. Fortunately, from the property of an antisymmetric filter, the samples $y_3(0)$ and $y_3(K)$ are always zero. Therefore, $y_2(K)$ can be sent instead of sending $y_3(0)$, and $y_3(K)$ can be discarded without loss of information. This means that the nonexpansive symmetric extension is possible in this case. To complete the first block, the sample $y_2(K)$ is moved to the first block. Hence, the first block becomes $[y_0(0)\ \ y_1(0)\ \ y_2(0)\ \ y_2(K)]$. This shift is a permutation matrix $\mathbf{B}$s task.

This symmetric extension method is applicable to every GULLOT defined in the foregoing section. This is because the structure we have derived guarantees an existence of a set of symmetric and antisymmetric filters with shorter length. To formalize the above symmetric extension method, (26) must be modified. Let the overlapping factor $\mathbf{n}$ be as shown in the second expression at the bottom of the page, where N_i^e and N_i^o denote even positive and odd positive integers, respectively. We only consider the above $\mathbf{n}$ for simplicity from now on. When N_i^e and N_i^o are in reciprocal order, the appropriate permutation matrix makes the above assumption valid. The symmetric extension method for this class can be written as

$$\mathbf{y} = \mathbf{B}\begin{bmatrix} \hat{\mathbf{P}}_{N-1}^R & & & \mathbf{0} \\ \mathbf{0}_M & \mathbf{P}_{N-1}^R & & \\ & & \ddots & \\ & & & \mathbf{P}_{N-1}^R \\ \mathbf{0} & & & \bar{\mathbf{P}}_{N-1}^R \end{bmatrix} \mathbf{\Lambda}^J_{\frac{M(N-1)}{2}}\mathbf{x} = \mathbf{B}\bar{\mathbf{T}}_N^{*}\mathbf{\Lambda}^J_{\frac{M(N-1)}{2}}\mathbf{x} = \mathbf{T}_N^{*}\mathbf{x} \tag{27}$$

where $\hat{\mathbf{P}}_{N-1}^R$ is a $(M-\alpha)\times NM$ matrix whose rows consist of all symmetric bases and $(M/2-\alpha)$ longer antisymmetric ones. The matrix $\bar{\mathbf{P}}_{N-1}^R$ is an $\alpha \times (N-1)M$ matrix that contains α shorter symmetric bases. The matrix $\mathbf{B}$ is a permutation matrix that shifts the last α samples to the first block. To reconstruct the input signal, transformed coefficients $\mathbf{y}$ are permuted by $\mathbf{B}^T$, extended, inverse transformed, and then truncated. This inverse transform can be written as follows:

$$\mathbf{x} = \mathbf{\Xi}\begin{bmatrix} \mathbf{P}_{N-1}^{R^T} & \mathbf{0}_M & & \\ \mathbf{0}_M & \mathbf{P}_{N-1}^{R^T} & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{P}_{N-1}^{R^T} \end{bmatrix}\mathbf{\Gamma}\mathbf{B}^T\mathbf{y} \tag{28}$$

where matrices $\mathbf{\Xi}$ and $\mathbf{\Gamma}$ represent windowing operation and extension of the transformed coefficients, respectively. The above equation can be rewritten using the forward transform $\mathbf{T}_N^*$ as

$$\mathbf{x} = \mathbf{T}_N^{*^T}\mathbf{\Psi}\mathbf{y} \tag{29}$$

where $\mathbf{\Psi}$ is a following diagonal matrix

$$\mathbf{\Psi} = \mathrm{diag}\left[\underbrace{1\ldots1}_{M/2-\alpha}\underbrace{1/2\ldots1/2}_{\alpha}\underbrace{1\ldots1}_{M/2-\alpha}\underbrace{1/2\ldots1/2}_{\alpha}\underbrace{1\ldots1}_{L_x-M}\right]. \tag{30}$$

In the following subsections, the size-limited structures for GULLOTs are shown, based on the above symmetric extension methods.

B. Size-Limited Structure for Type A GULLOT

Now, we define the size $(L_x+(N-i-1)M)\times(L_x+(N-i+1)M)$ matrix $\bar{\mathbf{S}}_i$ and the square matrix $\bar{\mathbf{T}}_1^N$ of size $(L_x+(N-1)M)$

$$\bar{\mathbf{S}}_i = \begin{bmatrix} \mathbf{G}_i & \mathbf{0}_M & & \mathbf{0} \\ \mathbf{0}_M & \mathbf{G}_i & & \\ & & \ddots & \\ \mathbf{0} & & \mathbf{0}_M & \mathbf{G}_i \end{bmatrix},$$

$$\begin{array}{lcc|cccc|cc|cl} y_0: & \cdots\ y_0(1) & y_0(0) & y_0(0) & y_0(1) & \cdots & y_0(K-1) & y_0(K-1) & & y_0(K-2)\ \cdots & \text{(HS)} \\ y_1: & \cdots -y_1(1) & -y_1(0) & y_1(0) & y_1(1) & \cdots & y_1(K-1) & -y_1(K-1) & & -y_1(K-2)\ \cdots & \text{(HA)} \\ y_2: & \cdots\ y_2(2) & y_2(1) & y_2(0) & y_2(1) & \cdots & y_2(K-1) & y_2(K) & & y_2(K-1)\ \cdots & \text{(WS)} \\ y_3: & \cdots -y_3(2) & -y_3(1) & y_3(0) & y_3(1) & \cdots & y_3(K-1) & y_3(K) & & -y_3(K-1)\ \cdots & \text{(WA)} \end{array}$$

$$\begin{cases} \mathbf{n} = \left[\underbrace{N_0^o\ N_1^o\ \ldots\ N_{M-2\alpha-1}^o}_{M-2\alpha}\underbrace{N_{M-2\alpha}^e\ \ldots\ N_{M-2}^e\ N_{M-1}^e}_{2\alpha}\right] & \text{for Type C} \\ \mathbf{n} = \left[\underbrace{N_0^e\ N_1^e\ \ldots\ N_{M-2\alpha-1}^e}_{M-2\alpha}\underbrace{N_{M-2\alpha}^o\ \ldots\ N_{M-2}^o\ N_{M-1}^o}_{2\alpha}\right] & \text{for Type D} \end{cases}$$

TABLE V
COMPARISON OF CODING GAIN (dB) AND FLOPS

	DCT8	GULA		GULB		GULC	
angle set		full	reduced	full	reduced	full	reduced
coding gain (dB)	8.826	9.372	9.287	9.474	9.471	9.243	9.157
flops per block	42	174	138	226	172	136	112

GULD		GLOT48		GLOT40		GLOT32		GLOT24	
full	reduced	full	reduced	full	reduced	full	reduced	full	reduced
9.415	9.326	9.506	9.496	9.404	9.338	9.351	9.340	9.178	9.119
188	170	302	212	250	178	198	144	146	110

$$\bar{\mathbf{T}}_1^N = \begin{bmatrix} \mathbf{P}_0^R & \mathbf{0}_M & & \\ \mathbf{0}_M & \mathbf{P}_0^R & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{P}_0^R \end{bmatrix}.$$

The matrix $\mathbf{G}_i$ relates $\mathbf{P}_i^R$ to $\mathbf{P}_{i-2}^R$ as

$$\mathbf{P}_i^R = \mathbf{G}_i \begin{bmatrix} \mathbf{P}_{i-2}^R & \mathbf{0}_M & \mathbf{0}_M \\ \mathbf{0}_M & \mathbf{P}_{i-2}^R & \mathbf{0}_M \\ \mathbf{0}_M & \mathbf{0}_M & \mathbf{P}_{i-2}^R \end{bmatrix}$$

$$\mathbf{G}_i = \hat{\mathbf{\Phi}}_i \bar{\mathbf{Q}}_i \begin{bmatrix} \hat{\mathbf{\Phi}}_{i-1}\bar{\mathbf{Q}}_{i-1} & \mathbf{0}_M \\ \mathbf{0}_M & \hat{\mathbf{\Phi}}_{i-1}\bar{\mathbf{Q}}_{i-1} \end{bmatrix}.$$

By using the above matrices, one can verify that the matrix $\mathbf{T}_N$ in (26) can be expressed as

$$\mathbf{T}_N = \bar{\mathbf{S}}_{N-1}\bar{\mathbf{S}}_{N-3}\ldots\bar{\mathbf{S}}_2\bar{\mathbf{T}}_1^N \mathbf{\Lambda}^J_{\frac{M(N-1)}{2}}. \tag{31}$$

The above structure, however, is not a size-limited one because the expanded samples must be processed. Hence, the problem here is to find the way to express the above matrix $\mathbf{T}_N$ using a cascade of $L_x \times L_x$ matrices $\mathbf{S}_i$s such that

$$\mathbf{T}_N = \mathbf{S}_{N-1}\mathbf{S}_{N-3}\ldots\mathbf{S}_2\mathbf{T}_1, \tag{32}$$

$$\mathbf{S}_i = \begin{bmatrix} \mathbf{G}_i^u\mathbf{0}_M & & & \mathbf{0} \\ & \mathbf{G}_i & \mathbf{0}_M & \\ & & \ddots & \\ & & \mathbf{0}_M & \mathbf{G}_i \\ \mathbf{0} & & & \mathbf{0}_M\ \mathbf{G}_i^d \end{bmatrix} \tag{33}$$

where $\mathbf{G}_i^u$ and $\mathbf{G}_i^d$ are $M \times 2M$ matrices. In other words, an appropriate choice of $\mathbf{G}_i^u$ and $\mathbf{G}_i^d$ guarantees the equivalence between (31) and (32). Let us define the matrix $\mathbf{\Lambda}_n^D$

$$\mathbf{\Lambda}_n^D = \begin{bmatrix} \bar{\mathbf{D}}_n & & \mathbf{0} \\ & \mathbf{I}_{L_x} & \\ \mathbf{0} & & \bar{\mathbf{D}}_n \end{bmatrix}, \quad \bar{\mathbf{D}}_n = \begin{bmatrix} \mathbf{0} & & \mathbf{D}_M \\ & \iddots & \\ \mathbf{D}_M & & \mathbf{0} \end{bmatrix}.$$

We found that if the matrices $\mathbf{G}_i^u$ and $\mathbf{G}_i^d$ are chosen as

$$\mathbf{G}_i^u = \mathbf{Z}_i \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\alpha_i}^{++} \\ \mathbf{0} & \mathbf{W}_{\alpha_i}^{+-} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}}_{i-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_{i-1} \end{bmatrix} \times \begin{bmatrix} \hat{\mathbf{W}}_{\alpha_{i-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\alpha_{i-1}}^{++} \\ \mathbf{0} & \mathbf{W}_{\alpha_{i-1}}^{+-} \end{bmatrix} \tag{34}$$

$$\mathbf{G}_i^d = \mathbf{Z}_i \begin{bmatrix} \mathbf{W}_{\alpha_i}^{-+} & \mathbf{0} \\ \mathbf{W}_{\alpha_i}^{--} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M/2} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{i-1} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{U}}_{i-1} \end{bmatrix} \times \begin{bmatrix} \mathbf{W}_{\alpha_{i-1}}^{-+} & \mathbf{0} \\ \mathbf{W}_{\alpha_{i-1}}^{--} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{W}}_{\alpha_{i-1}} \end{bmatrix} \tag{35}$$

where $\mathbf{Z}_i$ is

$$\mathbf{Z}_i = \hat{\mathbf{\Phi}}_i \mathbf{D}_i \hat{\mathbf{W}}_{\alpha_i} \mathbf{A}_{\alpha_i}.$$

Then, (31) can be rewritten as

$$\begin{aligned} \mathbf{T}_N &= \bar{\mathbf{S}}_{N-1}\bar{\mathbf{S}}_{N-3}\ldots\bar{\mathbf{S}}_2\mathbf{\Lambda}^D_{\frac{M(N-1)}{2}}\mathbf{T}_1 \\ &= \bar{\mathbf{S}}_{N-1}\bar{\mathbf{S}}_{N-3}\ldots\mathbf{\Lambda}^D_{\frac{M(N-3)}{2}}\mathbf{S}_2\mathbf{T}_1 \\ &= \bar{\mathbf{S}}_{N-1}\mathbf{\Lambda}^D_M\mathbf{S}_{N-3}\ldots\mathbf{S}_2\mathbf{T}_1 \\ &= \mathbf{S}_{N-1}\mathbf{S}_{N-3}\ldots\mathbf{S}_2\mathbf{T}_1. \end{aligned} \tag{36}$$

Consequently, the size-limited structure can be written as (32). The structure is illustrated in Fig. 4(a). Fig. 4(b) shows the structure of each block $\mathbf{S}_i$. From the figure, it can be seen that the $\mathbf{S}_i$ is further simplified as Fig. 4(c).

C. Size-Limited Structure for Type B GULLOT

The only difference between the size-limited structure for Type A and that for Type B is the first block $\mathbf{T}$. Now, let us define the $M \times (M + M/2)$ matrices $\mathbf{P}_1^u$, $\mathbf{P}_1^d$ and $L_x \times L_x$ matrix $\mathbf{T}_2$

$$\mathbf{P}_1^u = \mathbf{Z}_1 \begin{bmatrix} 2\mathbf{I}_{\frac{M}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_0^{++} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0\mathbf{J}_{\frac{M}{2}}/\sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^R \end{bmatrix} \tag{37}$$

$$\mathbf{P}_1^d = \mathbf{Z}_1 \begin{bmatrix} \mathbf{W}_0^{-+} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{I}_{\frac{M}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_0^R & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_0/\sqrt{2} \end{bmatrix} \tag{38}$$

$$\mathbf{T}_2 = \begin{bmatrix} \mathbf{P}_1^u & & \mathbf{0} \\ \mathbf{0}_{M,M/2}\mathbf{P}_1^R & & \\ & \ddots & \\ & & \mathbf{P}_1^R\mathbf{0}_{M,M/2} \\ \mathbf{0} & & \mathbf{P}_1^d \end{bmatrix}. \tag{39}$$

The transform matrix $\mathbf{T}_N$ can be written as

$$\mathbf{T}_N = \mathbf{S}_{N-1}\mathbf{S}_{N-3}\ldots\mathbf{S}_3\mathbf{T}_2. \tag{40}$$

The size-limited structure for Type B GULLOT is illustrated in Fig. 5. Each block $\mathbf{S}_i$ has the same structure as the one in Fig. 4(c).

D. Size-Limited Structure for Type C GULLOT

In this case, we must consider the transform matrix $\mathbf{T}_N^*$ in (27) instead of $\mathbf{T}_N$. Using similar principles, we can extend the size-limited structure for the Type A GULLOT in (36) to this class.

Although the whole structure of the Type C GULLOT is equal to that of Type A, the details of $\mathbf{S}_i$ are different from each other. This is because the situation $\alpha_{2k} < \alpha_{2k-1}$ occurs at some stage $\mathbf{S}_{2k}$ of the Type C GULLOT. Moreover, the last block $\mathbf{S}_{N-1}$ must contain the permutation matrix $\mathbf{B}$. Each block $\mathbf{S}_i$ for