

KERR-SCHILD METRICS REVISITED II. THE COMPLETE VACUUM SOLUTION[†]

László Á. Gergely [‡] and Zoltán Perjés

Central Research Institute for Physics
H-1525 Budapest 114, P.O.Box 49, Hungary

ABSTRACT

The complete solution of Einstein's gravitational equations with a vacuum-vacuum Kerr-Schild pencil of metrics $g_{ab} + Vl_al_b$ is obtained. Our result generalizes the solution of the Kerr-Schild problem with a flat metric g_{ab} (represented by the Kerr theorem) to the case when g_{ab} is the metric of a curved space-time.

PACS classification: 04.20Jb

1. INTRODUCTION

The Kerr-Schild pencil of metrics in general relativity has the form¹

$$\tilde{g}_{ab} = g_{ab} + Vl_al_b \quad (1.1)$$

where the metrics g_{ab} and \tilde{g}_{ab} are both Lorentzian, the vector l is null with respect to both metrics, and V is a scalar function.

In part **I** of this series², we have established some generic properties of the Kerr-Schild pencil for which both the parent g_{ab} and \tilde{g}_{ab} are vacuum metrics. The vector l is then tangent to a null geodesic congruence.

In this paper we present the complete solution of the vacuum Kerr-Schild problem. The structure of our solution is as follows. A Kerr-Schild space-time is characterized by a real *deformation parameter* η . The deformation parameter vanishes for Kerr-Schild

[†] Research supported by OTKA fund no. 1826

[‡] Present address: *Research Group on Laser Physics of the Hungarian Academy of Sciences, H-6720 Szeged, Dóm tér 9, Hungary*

space-times the parent of which is a Minkowski space-time. By our theorem in **I**, the field quantities are severely restricted unless the $\sin\eta$ assumes either of the exceptional values $0, \pm 1, \pm\sqrt{2}/2$. Here, we obtain all the metrics with arbitrary values of the deformation parameter. These turn out to be Kóta-Perjés metrics. For the exceptional values of η , we establish that (a) the class with $\eta = 0$ is algebraically special, (b) the values $\sin\eta = \pm 1$ can occur only in automorphisms of Minkowski space-time, and (c) the class with $\sin\eta = \pm 1/\sqrt{2}$ contains the remaining Kóta-Perjés metrics. Our results, as an important implication, dash the hopes for a complex-analytic description of space-time within the framework of Kerr-Schild theory.

In **I**, the vacuum Kerr-Schild equations have been written down in a Newman-Penrose (NP) form³, choosing l a vector of the null tetrad. The geodesic condition reads:

$$\kappa = 0 . \tag{1.2}$$

Following **I**, we adopt the gauge with

$$\epsilon = 0 , \quad \pi = \alpha + \bar{\beta} . \tag{1.3}$$

We have been able to integrate a closed subset of the ‘radial’ field equations³ containing derivatives in the direction of the affine parameter r :

$$D = l^a \nabla_a = \partial / \partial r . \tag{1.4}$$

This has yielded the spin coefficient quantities

$$\rho = -\frac{1}{2r}(1 + \cos\eta C) , \quad \sigma = -\frac{\sin\eta}{2rC} \tag{1.5}$$

$$\Psi_0 = -\frac{\sin 2\eta}{4r^2} . \tag{1.6}$$

Here we introduce the complex phase factor

$$C = \frac{r^{\cos\eta} - iB}{r^{\cos\eta} + iB} \tag{1.7}$$

with the properties

$$\bar{C} = \frac{1}{C}, \quad DC = \frac{\cos\eta}{2r}(1 - C^2) . \tag{1.8}$$

The real integration functions B and η may depend on the coordinates (x^1, x^2, x^3) . A further integration function has been eliminated from ρ by the appropriate choice of the origin of the affine parameter. When $r \geq 0$, the real potential can be written in the form

$$V = V_0 \frac{r^{\cos\eta}}{r^{2\cos\eta} + B^2} . \quad (1.9)$$

The complex tetrad vector m has the r dependence

$$m = \frac{1}{2B} \left(1 - \frac{1}{C}\right) \left[iQ_1^j r^{\frac{\cos\eta - \sin\eta - 1}{2}} - Q_2^j r^{\frac{\cos\eta + \sin\eta - 1}{2}} \right] \frac{\partial}{\partial x^j} \quad j = 1, 2, 3 \quad (1.10)$$

where V_0, Q_1^j and Q_2^j are real integration functions. With this choice of the tetrad, the spacelike and null rotations have been completely used up (cf. **I**).

The real parameters η and B tune the amount of divergence and rotation of the null congruence with tangent l , respectively. B occurs in the phase factor C , giving rise to the imaginary part of the spin coefficient ρ . When $B = 0$, we have $C = 1$, and the congruence is curl-free. Similarly, for large values of the affine parameter r , C approaches the unit value, and the rotation dies out. The parameter η regulates the shear. For $\eta = 0$ or $\eta = 180^\circ$, the congruence is shear-free. When both $B = 0$ and $\eta = 0$, the rays are exactly spherical: $\rho = -1/r$. The rays become cylindrical, $\rho = -1/2r$, for $\eta = 90^\circ$. When $\eta = 180^\circ$, there is no expansion. We have shewn in **I** that the general shearing class does not contain the shear-free case as a smooth limit. In what follows, we shall consider the generic case.

The field equations in the NP form may be grouped in three sets. The first set of equations is a coupled system of linear homogeneous equations for the affine parameter dependence of the quantities $\pi, \tau, \alpha, \beta, \Psi_1$ and for their complex conjugates:

$$D\tau = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \Psi_1 \quad (1.11a)$$

$$\text{THE} \quad D\pi = 2\rho\pi + 2\bar{\sigma}\bar{\pi} + \bar{\Psi}_1 \quad (1.11b)$$

$$\Psi_1 \quad D\alpha = \rho(\pi + \alpha) + \bar{\sigma}(\bar{\pi} - \bar{\alpha}) \quad (1.11c)$$

$$\text{EQS.} \quad D\beta = \bar{\rho}\beta + \sigma(2\pi - \bar{\beta}) + \Psi_1 \quad (1.11d)$$

$$D\Psi_1 = 4\rho\Psi_1 + (\pi - 4\alpha)\Psi_0. \quad (1.11e)$$

This set, to be called the Ψ_1 equations, can be obtained by using (**I** 3.13) in (NP 4.2.c), (NP 4.2.d), (NP 4.2.e), and the first Bianchi equation (NP 4.5). It is complemented by

two algebraic equations, linear in the unknown functions. The latter conditions will be employed in Sec. 2 to get a quartet of coupled equations for the spin coefficients τ, π and their complex conjugates. The general solution of the Ψ_1 set is given in Sec. 2.

The second system of the field equations, for the affine parameter dependence of the fields λ, μ and Ψ_2 , is linear and inhomogeneous:

$$THE \quad D\lambda = \rho\lambda + \bar{\sigma}\mu + (\bar{\delta}\pi + 2\alpha\pi) \quad (1.12a)$$

$$\Psi_2 \quad D\mu = \bar{\rho}\mu + \sigma\lambda + \Psi_2 + (\delta\pi + 2\beta\pi) \quad (1.12b)$$

$$EQS. \quad D\Psi_2 = 3\rho\Psi_2 - \lambda\Psi_0 + (\bar{\delta}\Psi_1 + 2\bar{\beta}\Psi_1) . \quad (1.12c)$$

This set will be named the Ψ_2 equations. The complex conjugate fields $\bar{\lambda}, \bar{\mu}$ and $\bar{\Psi}_2$ are decoupled here, and the source terms in the brackets are provided nonlinearly by the solution of the Ψ_1 system. In Sec. 3 we find the general solution of the homogeneous part of the Ψ_2 system.

Thus the affine-parameter dependence of the field quantities is essentially governed by the Ψ_1 and the Ψ_2 equations. The third (‘nonradial’³) set of equations yields some constraints and also the integrals of the remaining field quantities γ, ν, Ψ_3 and Ψ_4 . In Section 4, we obtain the metrics with the trivial solution of the Ψ_1 system and with the general solution of the Ψ_2 system. We get a three-parameter pencil for which both the parent metric and the image of the Kerr-Schild map are Kóta-Perjés metrics.

In Sec. 5 we prove that there are no other metrics in the class with arbitrary values of η than those given in Section 4. The exceptional values of the parameter η are considered in Section 6. We find that all Kóta-Perjés metrics⁷ are explicit cases of Kerr-Schild space-times, either with a real deformation parameter or with $\sin \eta = 1/\sqrt{2}$. This paper is an expanded version of our preprint on the homogeneous integrals⁴.

2. THE Ψ_1 EQUATIONS

The Ψ_1 set of field equations [Eqs.(1.11)] is linear and homogeneous for the quantities $\tau, \pi, \alpha, \beta, \Psi_1$ and their complex conjugates. Two linear algebraic relations among these quantities follow from the adopted gauge, $\pi - \alpha - \bar{\beta} = 0$, and from the Kerr-Schild condition (I 4.1);

$$\Psi_1 = \rho\bar{\pi} + \sigma(\pi - 4\alpha) + (\rho - \bar{\rho})\tau \quad (2.1)$$

and Eq. (I 4.4) yields the second algebraic constraint:

$$4\sigma\alpha = \sigma(3\pi - \bar{\tau}) + \left(\rho - \frac{\Psi_0}{2\sigma}\right)\bar{\pi} + \left(\frac{\Psi_0}{2\sigma} - \bar{\rho}\right)\tau . \quad (2.2)$$

Using the Kerr-Schild constraint (2.1) for eliminating Ψ_1 in (1.11a) and (1.11b), we get the four coupled equations for π, τ ;

$$\begin{aligned} D\tau &= \left(2\rho - \frac{\Psi_0}{2\sigma}\right)\tau + \sigma(2\bar{\tau} - \pi) + \left(\rho + \frac{\Psi_0}{2\sigma}\right)\bar{\pi} \\ D\pi &= \left(2\rho + \frac{\Psi_0}{2\bar{\sigma}}\right)\pi + \bar{\sigma}\tau + \left(\bar{\rho} - \frac{\Psi_0}{2\bar{\sigma}}\right)\bar{\tau} \end{aligned} \quad (2.3)$$

with their complex conjugates. Eqs. (1.11c), (1.11d) and (1.11e) are a consequence of the algebraic relations and Eqs. (2.3). The solution of Eqs. (2.3) determines the solution of the complete system (1.11), (2.1) and (2.2).

We solve Eqs. (2.3) by separating an overall power of r from the unknown functions:

$$\pi = \frac{1}{r^{p/2}}\pi^o, \quad \tau = \frac{1}{r^{p/2}}\tau^o . \quad (2.4)$$

Here we consider π^o and τ^o to be functions of the complex phase factor C as the new independent variable. Thus the functions π^o and τ^o satisfy the equations

$$\begin{aligned} \cos\eta(C^2 - 1)\dot{\tau}^o &= (2 - p + 3\cos\eta C)\tau^o + 2\frac{\sin\eta}{C}\bar{\tau}^o - \frac{\sin\eta}{C}\pi^o + \bar{\pi}^o \\ \cos\eta(C^2 - 1)\dot{\bar{\pi}}^o &= (1 + 2\cos\eta C)\tau^o + \frac{\sin\eta}{C}\bar{\tau}^o + (2 - p + 2\frac{\cos\eta}{C} - \cos\eta C)\bar{\pi}^o \end{aligned} \quad (2.5)$$

and the equations for the complex conjugates. We denote the derivative with respect to C by a dot.

We seek the solution in the form of the finite series in C :

$$\pi^o = \sum_{-n}^n P_k C^k, \quad \tau^o = \sum_{-n}^n T_k C^k . \quad (2.6)$$

Substituting in (2.5) and collecting the like powers of C , we obtain a set of homogeneous algebraic equations for the coefficients P_k and T_k . Solutions exist when the determinant of the algebraic equations vanishes. For $n = 3$, the determinant is

$$\frac{\mathcal{D}}{9\cos^2\eta - \sin^2\eta(p-4)^2} \quad (2.7)$$

where \mathcal{D} , a 12^{th} order polynomial in p , can be factorized in the form

$$\begin{aligned}
\mathcal{D} = & (p^2 - 6\cos\eta p - 6p + 10\cos^2\eta + 18\cos\eta + 8) \times \\
& (p^2 + 6\cos\eta p - 6p + 10\cos^2\eta - 18\cos\eta + 8) \times \\
& (p^2 - 2\cos\eta p - 6p + 2\cos^2\eta + 6\cos\eta + 8) \times \\
& (p^2 + 2\cos\eta p - 6p + 2\cos^2\eta - 6\cos\eta + 8) \times \\
& (p^2 - 2\cos\eta p - 2p + 2\cos^2\eta + 2\cos\eta) \times \\
& (p^2 + 2\cos\eta p - 2p + 2\cos^2\eta - 2\cos\eta) .
\end{aligned} \tag{2.8}$$

Thus solutions exist in the twelve cases

$$\begin{aligned}
p &= \pm 3\cos\eta \pm \sin\eta + 3 \\
p &= \pm \cos\eta \pm \sin\eta + 3 \\
p &= \pm \cos\eta \pm \sin\eta + 1 .
\end{aligned} \tag{2.9}$$

In each case, the coefficient equations are satisfied by choosing the overall factor of the solution P_{-3} either real or pure imaginary. The correct choice of P_{-3} can be found in Table 1. The explicit forms of the solutions P_k and T_k are given in Tables 2 and 3. In each case the solution is a rational expression in $\cos\eta$ and $\sin\eta$ which, however, is elaborate for some values of k . The functions α and Ψ_1 may be given in a similar representation: $\alpha = \frac{1}{r^{p/2}} \sum_{-3}^3 A_k C^k$, $\Psi_1 = \frac{1}{r^{p/2+1}} \sum_{-2}^4 \psi_k C^k$. The coefficients A_k and ψ_k are determined by the linear algebraic equations (2.1) and (2.2).

Returning to the affine parameter r as the independent variable, the solutions for τ and π with $p = \pm 3\cos\eta \pm \sin\eta + 3$ and with $p = \pm \cos\eta \pm \sin\eta + 3$ respectively turn out to be pairwise equal up to a constant factor. Similarly, solutions with the opposite signs of $\cos\eta$ in p will pairwise coincide. Thus, in accordance with the general theory of linear differential equations⁶, we are left with the fundamental solution consisting of the four linearly independent cases

$$\begin{aligned}
p^{(1)} &= \cos\eta + \sin\eta + 1 \\
p^{(2)} &= \cos\eta - \sin\eta + 1 \\
p^{(3)} &= \cos\eta + \sin\eta + 3 \\
p^{(4)} &= \cos\eta - \sin\eta + 3 .
\end{aligned} \tag{2.10}$$

The four fundamental solutions for π and τ are:

$$\begin{aligned}
\pi^{(1)} &= \frac{C+1}{r^{\frac{\cos\eta+\sin\eta+1}{2}}} \left(C^{-3} - \frac{\cos\eta}{\sin\eta+1} C^{-2} - \frac{\sin\eta}{\sin\eta+1} C^{-1} \right. \\
&\quad \left. + \frac{5\sin^2\eta - 4\sin\eta + 3}{\cos\eta(\sin\eta - 3)} + \frac{3\sin\eta - 1}{\sin\eta+1} \frac{2\sin\eta + 3}{\sin\eta - 3} C + 2 \frac{\sin\eta}{\cos\eta} C^2 \right) \\
\pi^{(2)} &= i \frac{C+1}{r^{\frac{\cos\eta-\sin\eta+1}{2}}} \left(C^{-3} + \frac{\cos\eta}{\sin\eta-1} C^{-2} - \frac{\sin\eta}{\sin\eta-1} C^{-1} \right. \\
&\quad \left. + \frac{5\cos^2\eta - 4\sin\eta - 8}{\cos\eta(\sin\eta + 3)} + \frac{6\cos^2\eta + 7\sin\eta - 3}{\cos^2\eta - 2\sin\eta + 2} C - 2 \frac{\sin\eta}{\cos\eta} C^2 \right) \\
\pi^{(3)} &= i \frac{C-1}{r^{\frac{\cos\eta+\sin\eta+3}{2}}} (C+1)^2 \left(-C^{-3} - \frac{\sin\eta+1}{\cos\eta} C^{-2} + \frac{1}{\sin\eta-1} C^{-1} - 2 \frac{\sin\eta}{\cos\eta} \right) \\
\pi^{(4)} &= \frac{C-1}{r^{\frac{\cos\eta-\sin\eta+3}{2}}} (C+1)^2 \left(-C^{-3} - \frac{\cos\eta}{\sin\eta+1} C^{-2} - \frac{1}{\sin\eta+1} C^{-1} + 2 \frac{\sin\eta}{\cos\eta} \right) \\
&\hspace{20em} (2.11) \\
\tau^{(1)} &= \frac{C+1}{r^{\frac{\cos\eta+\sin\eta+1}{2}}} \left(\frac{\sin\eta}{\cos\eta} C^{-3} - \frac{\sin\eta}{\sin\eta+1} C^{-2} - \frac{\sin^2\eta + 8\sin\eta + 3}{\cos\eta(\sin\eta - 3)} \frac{\sin\eta - 1}{\sin\eta + 1} C^{-1} \right. \\
&\quad \left. - 3 \frac{\sin\eta}{\sin\eta+1} + \frac{2\sin\eta - 3}{\cos\eta} C + 2C^2 \right) \\
\tau^{(2)} &= i \frac{C+1}{r^{\frac{\cos\eta-\sin\eta+1}{2}}} \left(\frac{\sin\eta}{\cos\eta} C^{-3} + \frac{\sin\eta}{\sin\eta-1} C^{-2} - \frac{\sin\eta\cos^2\eta - 7\cos^2\eta + 4\sin\eta + 4}{\cos\eta(2\sin\eta - 2 - \cos^2\eta)} C^{-1} \right. \\
&\quad \left. + 3 \frac{\sin\eta}{\sin\eta-1} + \frac{2\sin\eta + 3}{\cos\eta} C - 2C^2 \right) \\
\tau^{(3)} &= i \frac{C-1}{r^{\frac{\cos\eta+\sin\eta+3}{2}}} (C+1)^2 \left(-\frac{\sin\eta}{\cos\eta} C^{-3} + \frac{\sin\eta}{\sin\eta-1} C^{-2} - \frac{2\sin\eta+1}{\cos\eta} C^{-1} - 2 \right) \\
\tau^{(4)} &= \frac{C-1}{r^{\frac{\cos\eta-\sin\eta+3}{2}}} (C+1)^2 \left(-\frac{\sin\eta}{\cos\eta} C^{-3} - \frac{\sin\eta}{\sin\eta+1} C^{-2} - \frac{2\sin\eta-1}{\cos\eta} C^{-1} + 2 \right). \\
&\hspace{20em} (2.12)
\end{aligned}$$

Hence we get the fundamental solutions for Ψ_1 :

$$\begin{aligned}\Psi_1^{(1)} &= \frac{C+1}{2r^{\frac{\cos\eta+\sin\eta+3}{2}}} \left(\sin\eta C^{-2} + (\sin\eta - 5) \frac{\sin\eta}{\cos\eta} C^{-1} + \frac{16\sin^2\eta - 15\sin\eta + 9}{\sin\eta - 3} \right. \\ &\quad \left. - \frac{2\sin^2\eta - 3\sin\eta - 3}{\cos\eta} C - 3(\sin\eta - 1)C^2 - 3\cos\eta C^3 \right) \\ \Psi_1^{(2)} &= i \frac{C+1}{2r^{\frac{\cos\eta-\sin\eta+3}{2}}} \left(\sin\eta C^{-2} - (\sin\eta + 5) \frac{\sin\eta}{\cos\eta} C^{-1} + \frac{16\sin^2\eta + 15\sin\eta + 9}{\sin\eta + 3} \right. \\ &\quad \left. + \frac{2\sin^2\eta + 3\sin\eta - 3}{\cos\eta} C - 3(\sin\eta + 1)C^2 + 3\cos\eta C^3 \right) \\ \Psi_1^{(3)} &= i \frac{(C-1)(C+1)^2}{2r^{\frac{\cos\eta+\sin\eta+5}{2}}} \left(3\cos\eta C + 3(\sin\eta + 1) - (\sin\eta - 3) \frac{\sin\eta}{\cos\eta} C^{-1} - \sin\eta C^{-2} \right) \\ \Psi_1^{(4)} &= \frac{(C-1)(C+1)^2}{2r^{\frac{\cos\eta-\sin\eta+5}{2}}} \left(-3\cos\eta C + 3(\sin\eta - 1) + (\sin\eta + 3) \frac{\sin\eta}{\cos\eta} C^{-1} - \sin\eta C^{-2} \right) .\end{aligned}\tag{2.13}$$

The cases labeled (3) and (4) are found to satisfy the relation $\tau = 2\beta$.

The general solution of the Ψ_1 system is given by constant linear combinations with *real* coefficients of the four fundamental solution vectors. This is not to say, though, that the general solution will satisfy the complete set of Einstein's vacuum equations. In fact, one can exclude some of the linear combinations, without solving the rest of the vacuum equations, merely by looking at the limit $\eta \rightarrow 0$. When $\eta = 0$, the parent metric is algebraically special. Certainly, the trivial solution of the Ψ_1 system is well-behaved in this limit. However, it is easy to see that none of the four fundamental solutions (2.13) for Ψ_1 vanishes in the limit considered. One can take linear combinations proportional to $\sin\eta$, but these solutions of the Ψ_1 system are trivial for $\eta = 0$. Nontrivial solutions can also be constructed. Remembering the relation (1.7) between the variables C and r , one finds that the four solution vectors pairwise coincide in the limit $\eta \rightarrow 0$. The solutions with $p = \cos\eta \pm \sin\eta + 1$ coincide, up to a constant multiplier, with the solutions with $p = 3\cos\eta \pm \sin\eta + 3$. Thus one can form the two linear combinations

$$\Psi_1^{(+)} = B\Psi_1^{(3)} - \Psi_1^{(1)} ,\tag{2.14}$$

$$\Psi_1^{(-)} = B\Psi_1^{(4)} - \Psi_1^{(2)}\tag{2.15}$$

with the correct limiting behavior.

3. THE Ψ_2 EQUATIONS

The Ψ_2 system (1.12) is a first-order linear and inhomogeneous set of equations for the field quantities λ, μ and Ψ_2 . The source terms are supplied by derivatives and quadratic algebraic expressions of the solutions of the Ψ_1 equations. Let us solve first the linear homogeneous system

$$D\lambda = \rho\lambda + \bar{\sigma}\mu \quad (3.1a)$$

$$D\mu = \bar{\rho}\mu + \sigma\lambda + \Psi_2 \quad (3.1b)$$

$$D\Psi_2 = 3\rho\Psi_2 - \lambda\Psi_0 \quad (3.1c)$$

which amounts to taking the trivial solution of the Ψ_1 Eqs.,

$$\tau = \pi = \Psi_1 = \alpha = \beta = 0. \quad (3.2)$$

We can decouple equation (3.1c) for Ψ_2 from the rest by considering first the homogeneous equation

$$D\Psi_2 = 3\rho\Psi_2. \quad (3.3)$$

The solution is

$$\Psi_2 = \Psi_2^o \left[\frac{r^{\frac{\cos\eta-1}{2}}}{r^{\cos\eta} + iB} \right]^3 \quad (3.4)$$

where Ψ_2^o is a constant of integration. We now turn to the method of variation of constants, and allow for the possibility that Ψ_2^o depends on r . Introducing the new functions λ^o and μ^o by putting

$$\lambda = \lambda^o \frac{r^{\frac{3\cos\eta+1}{2}}}{(r^{\cos\eta} + iB)^3}, \quad \mu = \mu^o \frac{r^{\frac{3\cos\eta+3}{2}}}{(r^{\cos\eta} + iB)^2 (r^{\cos\eta} - iB)}, \quad (3.5)$$

we can write the system (3.1) as

$$D\lambda^o = \frac{1}{r}(\cos\eta C - 1)\lambda^o - \frac{1}{2}\sin\eta \mu^o \quad (3.6.a)$$

$$D\mu^o = \frac{1}{r}C\cos\eta x - \frac{1}{2r^2}(4r\mu^o + \sin\eta \lambda^o) \quad (3.6.b)$$

$$D\Psi_2^o = \frac{1}{2}\sin\eta \cos\eta \lambda^o \quad (3.6.c)$$

where we denote

$$x = \mu^\circ + \frac{1}{\cos\eta} \frac{\Psi_2^\circ}{r^2}. \quad (3.7)$$

Comparing Eqs. (3.6.b) and (3.6.c), we find that the function x satisfies the uncoupled equation

$$Dx = -\frac{2}{r}x + \frac{\cos\eta}{r}Cx. \quad (3.8)$$

This has the solution

$$x = x^\circ \frac{(r^{\cos\eta} + iB)^2}{r^{\cos\eta+2}} \quad (3.9)$$

where x° is a function of integration. Thus Eqs. (3.6.a) and (3.6.b) become a pair of coupled inhomogeneous linear equations for λ° and μ° with the driving term $\frac{1}{r}C\cos\eta x$.

Let us introduce the functions L and N of the variable C by

$$\lambda^\circ = \frac{1}{1-C^2} \frac{L}{r}, \quad \mu^\circ = \frac{1}{1-C^2} \frac{N}{r^2}. \quad (3.10)$$

L and N satisfy the inhomogeneous equations

$$N = \frac{C^2 - 1}{\tan\eta} \dot{L} \quad (3.11)$$

$$\ddot{L} - \left(\frac{\tan\eta}{C^2 - 1}\right)^2 L = -A_0 \tan^3\eta \frac{C}{(C^2 - 1)^2} \quad (3.12)$$

where $A_0 = \frac{8iB}{\tan^2\eta} x^\circ$. The homogeneous part of Eq. (3.12) can be transformed to a Riccati equation by the substitution $L = L_o e^{\int y dC}$ where L_o is a constant. A particular solution for y is given by

$$y_1 = \frac{1}{2\cos\eta} \left(\frac{1 + \cos\eta}{1 + C} + \frac{1 - \cos\eta}{1 - C} \right). \quad (3.13)$$

Substituting next $y = y_1 + z$, the Riccati equation becomes a Bernoulli equation for z with the general solution

$$z = -\frac{2}{\cos\eta(1-C^2)(1+L_1r)}. \quad (3.14)$$

Here L_1 is a constant of integration. Thus the solution of the homogeneous part of Eq. (3.12) is

$$L_{hom} = L_o(C^2 - 1)^{\frac{\cos\eta-1}{2\cos\eta}} \left[(C-1)^{\frac{1}{\cos\eta}} + (iB)^{\frac{1}{\cos\eta}} L_1 (C+1)^{\frac{1}{\cos\eta}} \right] \quad (3.15)$$

and the solution of the inhomogeneous equation is⁶:

$$L = A_0 \tan \eta C + (1 - C^2)^{\frac{1}{2}} [A_1 r^{\frac{1}{2}} + A_2 r^{-\frac{1}{2}}] \quad (3.16)$$

where L_o , A_0 , A_1 and A_2 are real functions of integration.

Hence we get the solution of the homogeneous Ψ_2 system:

$$\lambda = -\sin \eta \frac{C-1}{8 \cos \eta B} \left\{ 2i \cos \eta (C+1) \left(A_1 + \frac{A_2}{r} \right) - A_0 r^{\frac{\cos \eta - 1}{2}} \frac{C}{B} \right\} \quad (3.17)$$

$$\mu = -i \frac{C-1}{8 B^2 C} \left\{ 2B(C+1) \left[\cos \eta C \left(A_1 + \frac{A_2}{r} \right) - \left(A_1 - \frac{A_2}{r} \right) \right] - A_0 \frac{(C-1)^2}{B} r^{\frac{3 \cos \eta - 1}{2}} \right\} \quad (3.18)$$

$$\begin{aligned} \Psi_2 = \frac{C-1}{4 \cos \eta B^2} \left\{ -\frac{1}{4} A_0 r^{\frac{\cos \eta - 3}{2}} (2 \cos^2 \eta C^2 - \cos^2 \eta - 1) \right. \\ \left. + i A_1 \frac{\cos^2 \eta}{r} B (\cos \eta C - 1)(C+1) \right. \\ \left. + i A_2 \frac{\cos^2 \eta}{r^2} B (\cos \eta C + 1)(C+1) \right\}. \end{aligned} \quad (3.19)$$

4. THE HOMOGENEOUS METRICS

In this section we carry through the solution procedure of the field equations for the general solution (3.17)-(3.19) of the homogeneous Ψ_2 system.

From the equation (NP 4.2.1)

$$\Psi_2 = \mu \rho - \lambda \sigma + \gamma (\rho - \bar{\rho}) \quad (4.1)$$

we obtain γ algebraically:

$$\begin{aligned} \gamma = \frac{1}{4B} \left\{ \frac{r^{\frac{\cos \eta - 1}{2}}}{2B \cos \eta} A_0 (1 - \cos \eta C)(1 - C) \right. \\ \left. - i A_1 \frac{1}{\cos \eta} [\sin^2 \eta + (1 - \cos \eta C)^2] - i A_2 \frac{\cos \eta}{r} (C^2 - 1) \right\}. \end{aligned} \quad (4.2)$$

We now find the r dependence of the tetrad vector n . This can be done by applying the commutator $[\Delta, \delta]$ to the each of coordinates r and x^j :

$$n = n^0 \partial / \partial r + n^j \partial / \partial x^j \quad (4.3)$$

where n^j are functions independent of r and

$$n^0 = 2Re \left\{ -A_0 \frac{i}{2B \cos \eta} \frac{r^{\frac{\cos \eta + 1}{2}}}{r^{\cos \eta} + iB} + A_1 \frac{ir}{2B \cos \eta} \left(1 - \cos \eta + \frac{2iB \cos \eta}{r^{\cos \eta} + iB} \right) - A_2 \frac{1}{r^{\cos \eta} + iB} \right\} - G. \quad (4.4)$$

The real integration function G is determined by the action of the commutator $[\bar{\delta}, \delta]$ on the coordinate r :

$$G = -\frac{\cos \eta + 1}{2B \cos \eta} Im A_2 \quad (4.5)$$

The integration functions are severely restricted by the NP equations involving the Δ derivative:

$$\begin{aligned} (NP4.5) \quad \Delta \Psi_0 &= (4\gamma - \mu) \Psi_0 + 3\sigma \Psi_2 \quad \longrightarrow A_0 = Re A_1 = Im A_2 = \Delta \cos \eta = 0 \\ (NP4.2.p) \quad -\Delta \sigma &= \mu \sigma + \bar{\lambda} \rho - (3\gamma - \bar{\gamma}) \sigma \quad \longrightarrow Im A_1 = \Delta B = 0. \end{aligned} \quad (4.6)$$

Hence B and η are constants. The commutator $[\delta, \Delta]$ applied to the coordinate r , together with (NP 4.2.o),

$$\delta \gamma = -\sigma \nu$$

yields

$$\delta A_2 = \nu = 0. \quad (4.7a)$$

Furthermore,

$$\begin{aligned} (NP4.2.i) \quad D\nu &= \Psi_3 \quad \longrightarrow \Psi_3 = 0 \\ (NP4.2.n) \quad \delta \nu - \Delta \mu &= \mu^2 + \lambda \bar{\lambda} + (\gamma + \bar{\gamma}) \mu \quad \longrightarrow \Delta A_2 = 0. \end{aligned} \quad (4.7.b)$$

Finally from the last Kerr-Schild equation

$$\begin{aligned} \delta(\bar{\tau} - \pi) + \bar{\delta}(\tau - \bar{\pi}) + \frac{1}{2}(\rho + \bar{\rho}) \Delta(\ln V) &= 2Re \Psi_2 \\ -(\gamma + \bar{\gamma})(\rho + \bar{\rho}) + \frac{1}{2}(\mu + \bar{\mu}) \left(\frac{\Psi_0}{\sigma} + \rho - \bar{\rho} \right) - (\mu - \bar{\mu})(\rho - \bar{\rho}) \\ + \tau(\bar{\tau} + 2\alpha - 3\pi) + \bar{\tau}(\tau + 2\bar{\alpha} - 3\bar{\pi}) + 4\pi\bar{\pi} - \pi(2\bar{\alpha} - \bar{\pi}) - \bar{\pi}(2\alpha - \pi) \end{aligned} \quad (4.8)$$

we get $n^j \partial(\ln V_0)/\partial x^j = 0$. As a consequence, we have:

$$A_0 = A_1 = \Psi_3 = \nu = 0 \quad \eta, B, V_0, A_2 \text{ are real numbers} \quad (4.9)$$

Let us denote $A_2 = M$.

Ψ_4 can be determined algebraically from

$$(NP4.2.j) \quad \Delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu - \Psi_4 \quad \longrightarrow \quad \Psi_4 = -M^2 \sin 2\eta \left[\frac{r^{\cos\eta-1}}{(r^{\cos\eta} + iB)^2} \right]^2 \quad (4.10)$$

The rest of Eqs. (NP 4.2) and (NP 4.5) are identities.

The commutators $[\delta, \Delta]$ and $[\bar{\delta}, \delta]$ when applied to x^j give the relations:

$$[N, Q_1] = [N, Q_2] = 0 \quad [Q_1, Q_2] = -B \cos\eta N \quad (4.11)$$

where the three-vector N is defined $N = \{n^j\}$. By Eqs. (4.11), we can adapt the coordinates x and y to the vectors N and Q_1 as follows,

$$N = \partial/\partial x \quad Q_1 = \partial/\partial y \quad Q_2 = B \cos\eta y \partial/\partial x + \partial/\partial z. \quad (4.12)$$

We now employ the completeness relation $g^{ab} = 2l^{(a}n^{b)} - 2m^{(a}\bar{m}^{b)}$ to assemble the inverse metric:

$$g^{ab} = \frac{1}{H} \begin{pmatrix} 4M & H & 0 & 0 \\ H & -2B^2 \cos^2\eta y^2 r^{\sin\eta-1} & 0 & -2B \cos\eta y r^{\sin\eta-1} \\ 0 & 0 & -2r^{-\sin\eta-1} & 0 \\ 0 & -2B \cos\eta y r^{\sin\eta-1} & 0 & -2r^{\sin\eta-1} \end{pmatrix}. \quad (4.13)$$

Here $H = r^{\cos\eta} + B^2 r^{-\cos\eta}$.

Notice that (4.13) is just the Kóta-Perjés⁷ metric (44). The metric \tilde{g}^{ab} differs from g^{ab} only by the value of the parameter $4\tilde{M} = 4M - V_0$. The curvature components are:

$$\begin{aligned} \Psi_0 &= -\frac{\sin 2\eta}{4r^2} \\ \Psi_1 &= \Psi_3 = 0 \\ \Psi_2 &= M \cos\eta r^{\cos\eta-2} \frac{\cos\eta(r^{\cos\eta} - iB) + (r^{\cos\eta} + iB)}{(r^{\cos\eta} + iB)^3} \\ \Psi_4 &= -M^2 \sin 2\eta \left[\frac{r^{\cos\eta-1}}{(r^{\cos\eta} + iB)^2} \right]^2, \end{aligned} \quad (4.14)$$

These expressions are well-behaved in the limit $r \rightarrow \infty$, and singular only in the limit $r \rightarrow 0$. The space-time is Type I in the Petrov classification.

To summarize, the Kerr-Schild map in **I** generates a Kóta-Perjés metric from a type N vacuum metric. The image of the map becomes here the parent space-time for the second Kerr-Schild map to a Kóta-Perjés space-time with another value of the parameter M .

5. THE VANISHING OF THE INHOMOGENEOUS TERMS

In this section we show that the presence of source terms in the Ψ_2 equations is incompatible with the nonradial field equations. To this effect, we prove that $\alpha = \beta = \tau = \pi = \Psi_1 = 0$. Our procedure consists in proving the following lemmas:

- (i) In the generic case, $\rho - \bar{\rho} \neq 0$, for any real function ϕ with $D\phi = \delta\phi = 0$, it follows also $\Delta\phi = 0$.
- (ii) The integration functions B, η and V_0 are constants.
- (iii) The vector n has the radial component $n^0 = -(\mu - \bar{\mu})/(\rho - \bar{\rho})$.
- (iv) The Δ operator, when it acts on any of the quantities ρ, σ or Ψ_0 , can be written as $\Delta = n^0 D$.
- (v) There exists a linear equation among the source terms of the nonradial equations which cannot be satisfied unless all source terms vanish.

We can establish (i) easily by use of the last commutator (NP 4.4):

$$(\bar{\delta}\delta - \delta\bar{\delta})\phi = [(\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta]\phi. \quad (5.1)$$

When $B = 0$, the vector l is curl-free. These fields will be separately discussed at the end of the section. The main Theorem in **I** has already yielded that the δ derivatives of the

functions B, η and V_0 vanish. Hence and from (i) it follows that B, η and V_0 are constant. Thus we have proven (ii) and that ρ, σ and Ψ_0 depend only on the affine parameter r .

Choosing $\phi = r$ in the commutator (5.1), together with the form (4.3) of the vector n , (iii) follows at once. Next, (iv) is a straightforward consequence of the fact seen [under (ii)] that the functions ρ, σ and Ψ_0 depend only on r . With n^0 given in (iii), we may substitute

$$\Delta \rightarrow -\frac{\mu - \bar{\mu}}{\rho - \bar{\rho}} D$$

when acting either on ρ, σ or Ψ_0 or any of their complex conjugate functions.

Turning to (v), we observe that the real Kerr-Schild Eq. (4.8), the complex Ricci Eqs. (NP 4.2 l,p,q) and the fifth Bianchi identity (NP 4.5), form a set of linear inhomogeneous algebraic equations for the spin coefficient quantities λ, μ, γ and Ψ_2 ,

$$\rho\mu - \sigma\lambda - \Psi_2 + (\rho - \bar{\rho})\gamma = a_1 \quad (5.2a)$$

$$\sigma\mu - \frac{D\sigma}{\bar{\rho} - \rho}(\bar{\mu} - \mu) + \rho\bar{\lambda} - \sigma(3\gamma - \bar{\gamma}) = a_2 \quad (5.2b)$$

$$\rho\bar{\mu} - \frac{D\rho}{\bar{\rho} - \rho}(\bar{\mu} - \mu) + \sigma\lambda + \Psi_2 - \rho(\gamma + \bar{\gamma}) = a_3 \quad (5.2c)$$

$$\Psi_0\mu - \frac{D\Psi_0}{\bar{\rho} - \rho}(\bar{\mu} - \mu) - 3\sigma\Psi_2 - 4\Psi_0\gamma = a_4 \quad (5.2d)$$

$$\left[\frac{1}{2} \frac{\bar{\rho} + \rho}{\bar{\rho} - \rho} \left(\frac{1}{r} + \bar{\rho} + \rho \right) \right] (\bar{\mu} - \mu) + \frac{1}{2} \left(\frac{\Psi_0}{\sigma} + \rho - \bar{\rho} \right) (\bar{\mu} + \mu) + \Psi_2 + \bar{\Psi}_2 - (\rho + \bar{\rho})(\gamma + \bar{\gamma}) = a_5 \quad (5.2e)$$

The source terms on the right-hand sides are

$$\begin{aligned} a_1 &= \delta\alpha - \bar{\delta}\beta - \alpha\bar{\alpha} - \beta\bar{\beta} + 2\alpha\beta \\ a_2 &= \delta\tau - \tau(\tau + \beta - \bar{\alpha}) \\ a_3 &= \bar{\delta}\tau + \tau(\bar{\beta} - \alpha - \bar{\tau}) \\ a_4 &= \delta\Psi_1 - (4\tau + 2\beta)\Psi_1 \\ a_5 &= \delta(\bar{\tau} - \pi) + \bar{\delta}(\tau - \bar{\pi}) - 6\pi\bar{\pi} + 2\pi\bar{\alpha} + 2\bar{\pi}\alpha - 2\tau\bar{\tau} - 2\tau\alpha - 2\bar{\tau}\bar{\alpha} + 3\tau\pi + 3\bar{\tau}\bar{\pi} . \end{aligned} \quad (5.3)$$

These and the complex conjugate equations are nine real conditions on eight real unknown functions. Therefore, there exists a linear combination of the nine equations

such that the left hand sides cancel. This is just the sought-for condition on the source terms:

$$a_1 + \frac{\sigma}{\bar{\rho}} \bar{a}_2 - \frac{a_4}{3\sigma} + \left(\frac{\rho\bar{\rho} + \sigma\bar{\sigma}}{\bar{\rho}} - \frac{\Psi_0}{3\sigma} \right) b_1 + \left(\bar{\rho} - \rho - \frac{4\Psi_0}{3\sigma} - \frac{\sigma\bar{\sigma}}{\bar{\rho}} \right) b_2 + \frac{3\sigma\bar{\sigma}}{\bar{\rho}} \bar{b}_2 = 0 \quad (5.4)$$

where

$$\begin{aligned} b_1 &= \frac{a_1 + a_3 - \bar{a}_1 - \bar{a}_3}{2(\bar{\rho} - \rho)} \\ b_2 &= \frac{1}{\bar{\rho}^2 - \rho^2} \left\{ \rho \frac{\rho\bar{\rho}}{\sigma\bar{\sigma}} (-a_5 + a_3 + \bar{a}_3) + (3\rho - \bar{\rho})(a_1 + a_3) - \frac{\rho^2}{\bar{\sigma}} \bar{a}_2 - \frac{\rho\bar{\rho}}{\sigma} a_2 \right. \\ &\quad \left. + \rho \left[\frac{\rho\bar{\rho}}{\sigma\bar{\sigma}} \left(\frac{1}{r} + 2\rho + 2\bar{\rho} \right) + 5\rho - 3\bar{\rho} \right] b_1 \right\}. \end{aligned} \quad (5.5)$$

All the spin coefficient quantities may be expressed, by use of Eqs. (2.1) and (2.2), in terms of the general solution

$$\pi = \sum_{k=1}^4 c_k \pi^{(k)} \quad \tau = \sum_{k=1}^4 c_k \tau^{(k)} \quad (5.6)$$

and the corresponding complex conjugate quantities. We can take out the fractional multipliers of the fundamental solutions:

$$\begin{aligned} \pi^{(1)} &= r^{\frac{-\sin\eta-1}{2}} \pi_1 & \pi^{(2)} &= r^{\frac{\sin\eta-1}{2}} \pi_2 & \pi^{(3)} &= r^{\frac{-\sin\eta-3}{2}} \pi_3 & \pi^{(4)} &= r^{\frac{\sin\eta-3}{2}} \pi_4 \\ \tau^{(1)} &= r^{\frac{-\sin\eta-1}{2}} \tau_1 & \tau^{(2)} &= r^{\frac{\sin\eta-1}{2}} \tau_2 & \tau^{(3)} &= r^{\frac{-\sin\eta-3}{2}} \tau_3 & \tau^{(4)} &= r^{\frac{\sin\eta-3}{2}} \tau_4. \end{aligned} \quad (5.7)$$

Thus the affine-parameter dependence of the entries in Eq. (5.4) is explicitly known, but the detailed form is extremely lengthy. Since the value of η is arbitrary, one can simplify these computations by choosing some Pythagorean values, *e.g.*, $\sin\eta = 3/5$.

We may break up Eq. (5.4) by noting that each coefficient of the linearly independent functions of r must vanish. There are two kinds of terms in (5.4); the derivative terms are linear in π, τ and the complex conjugates and the algebraic terms are quadratic. By inspection of (5.7) and the form (1.10) of the δ operator, we find that the independent powers of r occurring in the derivative terms are: $\pm\sin\eta - 2, \pm\sin\eta - 1, -2$ and -1 . The algebraic terms may have the following powers: $\pm\sin\eta - 3, \pm\sin\eta - 2, \pm\sin\eta - 1, -3, -2$ and -1 . The powers $r^{\sin\eta-3}$ occur only multiplied with the factor c_4^2 . Thus, when collecting these terms in (5.4), we can put $c_1 = c_2 = c_3 = 0$. This yields $c_4 = 0$. Next,

the terms with $r^{-\sin\eta-3}$, arising with a factor c_3^2 , can be computed in a similar fashion, with the result that $c_3 = 0$. The surviving terms in (5.4) contain the factors $\pm\sin\eta - 1$ and -1 , with derivative and the algebraic terms mixed. The terms with $\sin\eta - 1$ do not contain any c_1 factor, hence we can put for these $c_1 = 0$. As expected, we get $c_2 = 0$. Finally, from the terms $-\sin\eta - 1$ we find $c_1 = 0$.

We have thus shown that only the trivial solution (Sec. 4) of the Ψ_1 equations satisfies Eq. (5.4).

The *curl-free fields* with $B = 0$ need to be considered separately. The computations follow the pattern of the generic case, and once again only the trivial solution of the Ψ_1 system survives. In the class with arbitrary values of η , we obtain the Kasner metric⁹ for $V_0 < 0$, and a sign-flipped version of the Kasner metric as described by McIntosh¹⁰, for $V_0 > 0$. Both are special cases of the Kóta-Perjés metric (I 5.11) for $B = 0$.

6. THE SPACE-TIMES WITH SPECIAL VALUES OF η

By our main theorem of I on the vacuum Kerr-Schild space-times, there exist such exceptional values of the deformation parameter η for which no restriction follows from Eq. (I.4.7) for the δ derivatives. To complete our investigation of vacuum Kerr-Schild space-times, we now consider in turn the metrics with either of the values $\sin\eta = 0, \pm 1, \pm 1/\sqrt{2}$.

(a) When $\sin\eta = 0$, both σ and Ψ_0 vanish, and l is a principal null vector of the curvature. By the Goldberg-Sachs theorem³, these parent space-times are algebraically special, and $\Psi_1 = 0$. It then follows from Thompson's Theorem 3.2 that the ensuing space-time is also algebraically special, with the Kerr-Schild congruence a principal null congruence⁵. All the vacuum Kerr-Schild spacetimes generated from the flat space-time are in this class.

(b) The case with $\cos\eta = 0$ contains automorphisms of the Minkowski space-time. With a spatial rotation of the vector m , the r -independent phase factor in σ can be

removed: $\rho = \sigma = -1/2r$. Using these and $\Psi_0 = 0$ in the Ψ_1 equations of **I** we get $\Psi_1 = 0$. Then we get from the Bianchi identities $\Psi_2 = \Psi_3 = \Psi_4 = 0$. Calculation shows that the ensuing space-time is also flat, as expected from the Goldberg-Sachs theorem for $\sigma \neq 0$.

(c) Case $\sin\eta = 1/\sqrt{2} = k$ contains the following Kóta-Perjés metrics:

$$ds^2 = -\frac{f^0}{f}(r^{1-k}dx^2 + r^{1+k}dy^2) + 2dr(l_a dx^a) + f(l_a dx^a)^2 \quad (6.1)$$

with $l = \partial/\partial r$ tangent to the Kerr-Schild congruence. Metric (53) of Ref. 8 is given by

$$f = \Lambda \operatorname{Re} \left\{ \frac{x + ir^k y}{r^k + iB} \right\}, \quad f^0 = \Lambda(x + By) \quad (6.2)$$

with Λ the pencil parameter and B a real constant. For metric (66), $B = x/y$ and

$$f = \Lambda \frac{x + by}{x^2 r^{-k} + y^2 r^k}, \quad f^0 = \Lambda(x + by)/y^2. \quad (6.3)$$

Though metric (6.3) is of the Kerr-Schild type, it is *not* a solution of the vacuum Einstein equations⁸, and the contrary claim in Ref. 7 is invalid.

Notice that above we have enlisted all the metrics of Ref. 7.

Corollary: The Kóta-Perjés metrics all belong to the Kerr-Schild class.

7. CONCLUDING REMARKS

Our solution of the vacuum Kerr-Schild problem has been made possible by integration of the Ψ_1 system. However, these lengthy integrals do not satisfy all the field equations.

The vacuum Kerr-Schild pencils generated from a *flat* space-time are associated with complex surfaces in three-dimensional homogeneous spaces. This relationship follows from the Kerr theorem¹¹. It has been known for some time that shear-free null geodesic congruences do not coexist with the Weyl curvature. Our main theorem in **I** has already

indicated what we find in this paper that the variety of *shearing Kerr-Schild* congruences is too small to allow for the complex-analytic structures of the Kerr theorem.

REFERENCES

- [1] Kerr, R. P. and Schild, A., *Atti Del Convegno Sulla Relativit Generale: Problemi Dell' Energia e Onde Gravitazionali (Anniversary Volume, Fourth Centenary of Galileo's Birth)*, G. Barbra, Ed. (Firenze, 1965), p. 173
- [2] Gergely, L. Á. and Perjés, Z.: Kerr-Schild metrics revisited I. The ground state (Paper I)
- [3] Newman, E. and Penrose, R., *J. Math. Phys.* **3**, 566 (1962)
- [4] Gergely, L. Á. and Perjés, Z.: Kerr-Schild metrics revisited II. The homogeneous integrals, KFKI-1993-09/B preprint
- [5] Thompson, A.H., *Tensor* **17**, 92 (1966)
- [6] Kamke, E.: *Differentialgleichungen I., Sec. 8.2.* Teubner, Stuttgart, 1977
- [7] Kóta, J. and Perjés, Z., *J. Math. Phys.* **13**, 1695 (1972)
- [8] Evans, B., Perjés, Z. and Scott, S., in preparation
- [9] Kasner, E., *Amer. J. Math.* **43**, 217 (1921)
- [10] McIntosh, C. B. G., in *Relativity Today*, Z. Perjés, Ed., Nova Science Publishers, New York (1992), p. 147
- [11] Debney, G. C., Kerr, R. P. and Schild, A., *J. Math. Phys.* **10**, 1842 (1969)

p	P_{-3}	p	P_{-3}	p	P_{-3}
$\cos\eta + \sin\eta + 1$	1	$\cos\eta + \sin\eta + 3$	i	$3\cos\eta + \sin\eta + 3$	1
$\cos\eta - \sin\eta + 1$	i	$\cos\eta - \sin\eta + 3$	1	$3\cos\eta - \sin\eta + 3$	i
$-\cos\eta + \sin\eta + 1$	i	$-\cos\eta + \sin\eta + 3$	1	$-3\cos\eta + \sin\eta + 3$	i
$-\cos\eta - \sin\eta + 1$	1	$-\cos\eta - \sin\eta + 3$	i	$-3\cos\eta - \sin\eta + 3$	1

Table 1. *The twelve solutions for τ^o and π^o*

$$\begin{aligned}
P_{-3} &= P, & P_{-2} &= \frac{p-2}{\cos\eta} P, & P_{-1} &= \frac{p^2-4p-2\cos^2\eta+4}{2\cos^2\eta} P, & P_0 &= \frac{\sin\eta}{2\cos^2\eta} R\bar{P} \\
&+ \frac{p^3+(2p^3-12p^2+32p-94)\cos^2\eta-6p^2-8p\cos^4\eta+44\cos^4\eta+32}{2\cos\eta(p^2\cos^2\eta-p^2-8p\cos^2\eta+8p+25\cos^2\eta-16)} P, \\
P_1 &= \frac{3}{4\cos^2\eta} (p-2)RP + \frac{\sin\eta}{4\cos^3\eta(p^2\cos^2\eta-p^2-8p\cos^2\eta+8p+25\cos^2\eta-16)} \\
&\quad \times (p^6-16p^5-6p^4\cos^2\eta+100p^4+42p^3\cos^2\eta-304p^3+12p^2\cos^4\eta \\
&\quad -72p^2\cos^2\eta+448p^2-84p\cos^4\eta-48p\cos^2\eta-256p-36\cos^4\eta+192\cos^2\eta)\bar{P}, \\
P_2 &= \frac{3}{4\cos\eta} RP + \frac{\sin\eta}{4\cos^2\eta(p^2\cos^2\eta-p^2-8p\cos^2\eta+8p+25\cos^2\eta-16)} \\
&\quad \times (p^5-14p^4-2p^3\cos^2\eta+68p^3-10p^2\cos^2\eta-120p^2+24p\cos^4\eta \\
&\quad +140p\cos^2\eta-132\cos^4\eta-200\cos^2\eta+128)\bar{P}, \\
P_3 &= 2\frac{\sin\eta}{\cos\eta} \bar{P} \\
R &= \frac{p^4-12p^3+52p^2-96p+64+(2p^3-26p^2+84)\cos^2\eta+(4p+8)\cos^4\eta}{(8-p)\sin^2\eta p+25\cos^2\eta-16}
\end{aligned}$$

Table 2. The universal coefficients of π . The fundamental solution is given by the four independent values $p = \cos\eta \pm \sin\eta + 1$ and $p = \cos\eta \pm \sin\eta + 3$

$$\begin{aligned}
T_{-3} &= \frac{\sin\eta}{\cos\eta} P, & T_{-2} &= \frac{\sin\eta(p-2)}{\cos^2\eta} P, & T_{-1} &= \frac{\sin\eta(p-4)P+3\bar{P}}{\cos^2\eta} R, \\
T_0 &= \frac{\sin\eta}{4\cos^2\eta} RP + \frac{1}{4\cos^3\eta(p^2\cos^2\eta-p^2-8p\cos^2\eta+8p+25\cos^2\eta-16)} \\
&\quad \times (p^5\cos^2\eta-p^5-16p^4\cos^2\eta+16p^4-6p^3\cos^4\eta+118p^3\cos^2\eta \\
&\quad -100p^3+74p^2\cos^4\eta-468p^2\cos^2\eta+304p^2-8p\cos^6\eta-344p\cos^4\eta \\
&\quad +932p\cos^2\eta-448p+44\cos^6\eta+456\cos^4\eta-720\cos^2\eta+256)\bar{P}, \\
T_1 &= \frac{p^2-5p-3\cos^2\eta+6}{\cos^2\eta} \bar{P}, & T_2 &= \frac{2p-5}{\cos\eta} \bar{P}, & T_3 &= 2\bar{P}
\end{aligned}$$

Table 3. The universal coefficients of τ