Node-Disjoint Paths Algorithm in a Transposition Graph

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SUMMARY In this paper, we give an algorithm for the node-to-set disjoint paths problem in a transposition graph. The algorithm is of polynomial order of \( n \) instead of the number of nodes, \( n! \).

The rest of this paper is organized as follows. Section 2 introduces some preliminary definitions and a simple routing algorithm. Section 3 explains our algorithm to obtain node-to-set disjoint paths in detail. In Sect. 4, we give a proof of validity of our algorithm and estimations of its complexities. We conduct computer experiments in Sect. 5. Section 6 describes the conclusion.

1. Introduction

In practical use of parallel and distributed computing systems, finding disjoint paths in interconnection networks is one of the fundamental issues [3], [5]–[7], [9]–[11]. Amongst them is the node-to-set disjoint paths problem: Given a source node \( s \) and a set \( D = \{d_1, d_2, \ldots, d_k\} (s \notin D) \) of \( k \) destination nodes in a \( k \)-connected graph \( G \), find \( k \) paths from \( s \) to \( d_i \) \((1 \leq i \leq k)\) that are node-disjoint except for \( s \). Once these \( k \) paths are obtained, they achieve fault tolerance; that is, at least one path can survive with \( k - 1 \) faulty components. This problem can be solved by using the maximum flow technique, which takes polynomial time of the number of nodes \([4]\). However, if graph \( G \) has many nodes, this approach is far from practical. For an \( n \)-hypercube, an \( n \)-star graph and an \( n \)-rotator graph, the algorithms of polynomial time of \( n \) for this problem have already been proposed [2], [5], [10].

An \( n \)-transposition graph [8] is a Cayley graph [1]. It is an \( n(n - 1)/2 \)-connected undirected graph with \( n! \) nodes and \( n(n - 1)/2 \) edges. Its diameter is \( n - 1 \). As an interconnection network, this graph attracts some attention because it can include other topologies as its subgraphs, such as meshes, hypercubes, star graphs and bubble-sort graphs. In addition, the fault diameter of an \( n \)-transposition graph is \( n \), and the graph has a wide container between any pair of nodes with length of the distance of them plus two at most.

In this paper, we take an \( n \)-transposition graph as a target and propose an algorithm that solves the node-to-set disjoint paths problem in the time complexity of polynomial order of \( n \) instead of the number of nodes, \( n! \).

2. Preliminaries

In this section, we introduce definitions of the transposition operation, transposition graphs, and the shortest-path routing algorithm in a transposition graph.

**Definition 1:** For an arbitrary permutation \( u = u_1u_2\cdots u_n \) of \( n \) symbols \( 1, 2, \ldots, n \), the transposition operation \( t_{i,j}(u) \) \((1 \leq i < j \leq n)\) is defined as follows:

\[
t_{i,j}(u) = u_1 \cdots u_{i-1}u_ju_{i+1} \cdots u_{j-1}u_iu_{j+1} \cdots u_n.
\]

**Definition 2:** An \( n \)-transposition graph, \( T_n \), has \( n! \) nodes. Each node has a unique address which is a permutation of \( n \) symbols \( 1, 2, \ldots, n \). A node which has an address \( u = u_1u_2\cdots u_n \) is adjacent to \( n(n - 1)/2 \) nodes whose addresses are elements of the set \{\( t_{i,j}(u)\) | \( 1 \leq i < j \leq n \}\).

In an \( n \)-transposition graph \( T_n \), a subgraph induced by nodes that have a common symbol \( k \) at the \( i \)-th position of their addresses constitutes an \((n - 1)\)-transposition graph. In this paper, we denote the subgraph induced by nodes whose last symbols are \( k \) as \( T_{n-k} \). Figure 1 shows some examples of transposition graphs.

For given nodes \( s = s_1s_2\cdots s_n \) and \( d = d_1d_2\cdots d_n \) in \( T_n \), we use the routing algorithm *route* shown in Fig. 2 to obtain one of the shortest paths between \( s \) and \( d \). We assume that the address of a node is represented by using a linear array and each element of the array consists of a word that can store the value \( n \). Then its time complexity is \( O(n^2) \) and its path length is \( O(n) \).

For an arbitrary node \( u \), let \( N(u) \) denote the set of neighbor nodes of \( u \).

3. The Algorithm

In this section, we propose an algorithm for the node-to-set disjoint paths problem in an \( n \)-transposition graph.
Fig. 1 Examples of transposition graphs.

Fig. 2 A shortest-path routing algorithm route.

3.1 Classification

If \( n \leq 2 \), the problem is trivial. That is, a 2-transposition graph consists of two nodes and an edge between them. Hence, if one node is the source, then the other one is the destination, and the path is the edge itself. Therefore, we assume \( n \geq 3 \) in the following. We can fix the source node as \( s = 1, 2, ..., n \), taking advantage of the symmetric property of \( T_n \).

Let \( D = \{ d_1, d_2, ..., d_{n(1)-1/2} \} \) be the set of destination nodes. The algorithm has recursive structure. Here let us consider the following two cases.

Case 1 \( |D \setminus V(T_{n-1}n)| \leq n - 1 \)

Case 2 \( |D \setminus V(T_{n-1}n)| \geq n \)

where \( V(G) \) represents the node set of \( G \), and \( |D \setminus V(T_{n-1}n)| \) represents the number of destination nodes that are not included in \( T_{n-1}n \).

3.2 Case 1: \( |D \setminus V(T_{n-1}n)| \leq n - 1 \)

This subsection presents the procedure in the case that \( |D \setminus V(T_{n-1}n)| \leq n - 1 \). Note that the number of destination nodes that are included in \( T_{n-1}n \) is at least \( (n-1)(n-2)/2 \) in this case.

Step 1 In \( T_{n-1}n \), by calling the algorithm recursively, construct node-disjoint paths from \( s \) to \((n-1)(n-2)/2\) arbitrary destination nodes in \( T_{n-1}n \).

Step 2 If a destination node, say \( d_x \), other than these \((n-1)(n-2)/2\) destination nodes is on one of the constructed path from \( s \) to, say \( d_y \), then discard the subpath from \( d_x \) to \( d_y \) and exchange the indices \( x \) and \( y \). Repeat this step until no destination node is on the paths except for the \((n-1)(n-2)/2\) nodes. See Fig. 3.

Step 3 Select the edges \((s, t_{i(n)}(s)) (1 \leq i \leq n - 1)\). Note that \( t_{i(n)}(s) \in V(T_{n-1}i) \).

Step 4 For each \( T_{n-1}i \) (1 \( \leq i \leq n - 1 \)), if there exist some destination nodes in \( T_{n-1}i \), choose one of the nearest nodes among them from \( t_{i(n)}(s) \). Construct the shortest path between these two nodes by route in Fig. 2. See Fig. 4.

Step 5 For each \( T_{n-1}i \) (1 \( \leq i \leq n - 1 \)), if there exists no destination node, choose one of the destination nodes to which the path is not yet constructed from \( s \). Let the chosen node be \( d_z \). Select the edge \((N(d_z) \cap V(T_{n-1}i), d_z)\) and construct the shortest path

procedure route(s, D);
begin
  c := s; P := [c];
  for i := 1 to n - 1 do
    if \( c_i \neq d_i \) then begin
      find \( j \) such that \( c_j = d_i \);
      \( c := t(i,j)(c) \);
      \( P := P ++ [c] \);
    end;
end;

Fig. 3 Construction of node-disjoint paths from \( s \) to \((n-1)(n-2)/2\) nodes in \( T_{n-1}n \).

Fig. 4 Construction of paths to the nearest destinations.
Step 2

For each destination node $d_i$ outside $T_{n-1}n$, select two nodes $u_i$ and $c_i$ satisfying the following conditions if possible.
- $c_i = d_i$,
- $u_i = (N(c_i) \cap V(T_{n-1}n)) \setminus D_i$,
- $u_i = s$ or $u_i \neq u_j$ if $i \neq j$.

Step 3

For each destination node $d_i$ outside $T_{n-1}n$, if $c_i$ for $d_i$ was not selected in Step 1, select two nodes $u_i$ and $c_i$ satisfying the following conditions.
- $c_i \in N(d_i) \setminus D_i$,
- $u_i = (N(c_i) \cap V(T_{n-1}n)) \setminus D_i$,
- $u_i = s$ or $u_i \neq u_j$ if $i \neq j$.

Step 4

Let $M$ and $U$ be a set $\{d_i \mid d_i \notin V(T_{n-1}n)\} \cup \{c_i \mid c_i \neq d_i\} \cup \{b_i\}$ and a set $\{u_i\}$, respectively.

Step 5

Select the edges $(s, t_{i(n)}(s))$ $(1 \leq i \leq n - 1)$. Note that $t_{i(n)}(s) \in V(T_{n-1}n)$.

3.3 Case 2: $|D \setminus V(T_{n-1}n)| \geq n$

This subsection presents the procedure in the case that $|D \setminus V(T_{n-1}n)| \geq n$.

Step 6

For each $T_{n-1}i$ $(1 \leq i \leq n - 1)$, if there exist some nodes in $M \cap V(T_{n-1}i)$ and a path from $t_{i(n)}(s)$ is not yet constructed, choose one node $v_i$ among the nodes in $M \cap V(T_{n-1}i)$ such that $v_i$ is one of the nearest nodes from $t_{i(n)}(s)$ in $M \cap V(T_{n-1}i)$.

Step 7

For each $v_j$ $(1 \leq j \leq n - 1)$, if $v_j$ is a destination, say $d_x$, construct the shortest path from $t_{i(n)}(s)$ to $d_x$ by route in Fig. 2, and update $M$ and $U$ by $M \setminus \{b_x, c_x, d_x\}$ and $U \setminus \{u_x\}$, respectively. See Fig. 7. In this step, if $M$ is updated, go back to Step 6.

Step 8

For each $v_j$ $(1 \leq j \leq n - 1)$, if $v_j$ is one of $c_x$’s, say $c_y$, construct the shortest path from $t_{i(n)}(s)$ to $c_y$ by route in Fig. 2 and select the edge $(c_x, d_x)$ as shown in Fig. 8, and update $M$ and $U$ by $M \setminus \{b_x, c_x, d_x\}$ and $U \setminus \{u_x\}$, respectively. In this step, if $M$ is updated, go back to Step 6.

Step 9

For each $v_j$ $(1 \leq j \leq n - 1)$, $v_j$ is one of $b_x$’s, say $b_z$. Construct the shortest path from $t_{i(n)}(s)$ to $b_z$ by route in Fig. 2. Update $M$ and $U$ by $M \setminus \{b_x, c_x, d_x\}$ and $U \setminus \{u_x\}$, respectively.

Step 10

For each $T_{n-1}i$ $(1 \leq i \leq n - 1)$, if there exists no node in $M \cap V(T_{n-1}i)$ and a path from $t_{i(n)}(s)$ is not constructed, choose one destination node from $M$, say $d_z$, select the edge $(N(d_z) \cap V(T_{n-1}i), d_z)$, construct the shortest path from $t_{i(n)}(s)$ to $N(d_z) \cap V(T_{n-1}i)$ by route...
In this section, we give a proof of validity of our algorithm and estimate the time complexity $T(n)$ and the maximum length of each path $L(n)$ for an $n$-transposition graph. The proof is based on the mathematical induction on $n$.

**Lemma 1:** Paths constructed by the procedure for Case 1 are node-disjoint. For this case, the maximum length of each path is $\max\{L(n-1), n\}$ and the time complexity of the procedure is $T(n-1) + L(n-1) \times O(n^4)$.

**Proof:** In Steps 1 and 2, the obtained $(n-1)(n-2)/2$ paths are node-disjoint except for $s$ by the induction hypothesis. The pair of the edges $(s, t_{(n)}(s))$ and $(s, t_{(j)}(s))$ selected in Step 3 are node-disjoint except for $s$ if $i \neq j$. The paths in $T_{n-1}i$ and $T_{n-1}j$ constructed in Step 4 are node-disjoint because the selected destination node in each subgraph is the one of the nearest nodes from $t_{(i)}(s)$ or $t_{(j)}(s)$. The paths constructed in Step 5 are also node-disjoint. Concatenations of all paths constructed in Steps 3 and 4 or Steps 3 and 5 are node-disjoint except for $s$. The $(n-1)(n-2)/2$ paths obtained in Steps 1 and 2 are all inside $T_{n-1}n$ and $n-1$ paths obtained in Steps 3, 4 and 5 are all outside $T_{n-1}n$ except for $s$ if $|D \setminus V(T_{n-1}n)| = n-1$, or except for $s$ and some destinations otherwise. Hence, the $n(n-1)/2$ paths obtained by the procedure are node-disjoint except for $s$.

The maximum lengths of Steps 1, 3, 4 and 5 are $L(n-1)$, $1$, $n-2$ and $n-1$, respectively. Hence, for Case 1, we obtain $L(n) = \max\{L(n-1), n\}$.

The time complexities of Steps 1 and 2 are of $T(n-1)$ and $L(n-1) \times O(n^4)$, respectively. Considering that distance between two nodes in $T_n$ can be calculated in $O(n)$ time, the time complexities of Step 3, 4 and 5 are $O(n)$, $O(n^2)$, $O(n^3)$, respectively. From above, we obtain $T(n) = T(n-1) + L(n-1) \times O(n^3)$ for Case 1.

**Lemma 2:** Paths constructed by the procedure for Case 2 are node-disjoint. The maximum length of each path is $\max\{L(n-1) + 3, n+1\}$ and the time complexity of the procedure for Case 2 is $T(n-1) + O(n^6)$.

**Proof:** In $T_n$, consider the destination node $d_i \notin T_{n-1}n$. Let $N_0(d_i)$, $N_1(d_i)$, and $N_2(d_i)$ represent the sets of the nodes whose distances from $d_i$ are 0, 1, and 2, respectively. Then, $|N_0(d_i)| = 1$, $|N_1(d_i)| = n(n-1)/2$, and $|N_2(d_i)| = n(n-1) - 2(3n-1)/24$. In addition, let $\tilde{N}_0(d_i) \subseteq N_0(d_i)$, $\tilde{N}_1(d_i) \subseteq N_1(d_i)$, and $\tilde{N}_2(d_i) \subseteq N_2(d_i)$ represent the sets of the nodes of which the positions of $n$ are same as $d_i$. Then, $|\tilde{N}_0(d_i)| = 1$, $|\tilde{N}_1(d_i)| = n(n-1)(n-2)/2$, and $|\tilde{N}_2(d_i)| = (n-1)(n-2)(n-3)(3n-4)/24$.

For any two distinct nodes $a, b \in \tilde{N}_0(d_i) \cup \tilde{N}_1(d_i) \cup \tilde{N}_2(d_i))$, since the positions of $n$ of $a$ and $b$ are same, $N(a) \cap V(T_{n-1}n)$ and $N(b) \cap V(T_{n-1}n)$ are also distinct. Hence, after Steps 1 and 2, at least $(n-1)(n-2)/2 + 1$ destination nodes $d'_j$’s have their corresponding $u'_j$’s, and the remaining destination nodes are at most $n-2$.

Consider the case where a destination node $d_i = \ldots$

![Fig. 8](image_url)  Construction of paths to the nearest $c'_i$’s.

![Fig. 9](image_url)  Construction of paths to destinations outside $T_{n-1}n$ to which any path is not yet constructed from $s$.

![Fig. 10](image_url)  Recursive application of the algorithm.
(d_1, d_2, \ldots, d_n) is processed in Step 3. Assume that d_h = n. Then, for each node in N_0(d_i) \cup N_1(d_i), the node itself is selected as c_i, or its neighbor node in T_{n-1}n is selected as u_j for another destination node. If there is an available node in N_0(d_i), then let c_i be the node itself that is obtained by exchanging the element d_h = n and the element d_k where k \neq h, n. Let b_i be the node obtained by exchanging the element d_i and the element d_i where l \neq h, k. There are n - 2 candidates for b_i. Then, the path d_i \rightarrow c_i \rightarrow b_i does not have any common node with N_0(d_i) \cup N_1(d_i), and the node in N_1(b_i) \cap V(T_{n-1}n) is adjacent to the node in N_2(d_i). Hence, there are paths whose terminal nodes are distinct n - 2 nodes in T_{n-1}n.

The above discussion ensures the existence of a path from d_i to u_j via c_i and b_i for each d_i of the n - 2 remaining destination nodes in Step 3 of Case 2. That is, the algorithm can always select c_i, b_i, and u_i for each d_i. Note that if there are a node in N_0(d_i) and a node in N_2(d_i) whose corresponding neighbor nodes in T_{n-1}n are not selected as u_i's, they are not used simultaneously for a single destination node.

If i's are different, the edges selected in Step 5 are disjoint except for s. Paths constructed in Steps from 7 to 10 are disjoint because of the conditions described in Steps 1, 2 and 3. The (n - 1)(n - 2)/2 paths constructed in Step 11 are node-disjoint by the induction hypothesis. Hence, the n(n - 1)/2 paths which include a collection of (n - 1)(n - 2)/2 concatenations of paths selected in Step 11, and n - 1 paths constructed in Steps from 5 to 10, are node-disjoint except for s.

The maximum lengths of Step 5 and Steps from 7 to 12 are 1, n - 2, n - 1, n - 1, L(n - 1) and 3, respectively. Hence, for Case 2, we obtain L(n) = \max\{L(n - 1) + 3, n + 1\}.

It takes O(n^6) time to select u_i's, c_i's and b_i's for d_i's in Steps 1, 2 and 3. Considering that a distance between two nodes in T_n can be calculated in O(n) time, the time complexities of Steps from 5 to 12 are O(n), O(n^3), O(n^3), O(n^3), O(n^3), O(n^3), T(n - 1) and O(n^2) in this order. Steps from 6 to 8 repeat at most O(n^2) times. From discussions above, we obtain T(n) = T(n - 1) + O(n^6) for Case 2. □

Lemmas 1 and 2 give the following theorem.

Theorem 1: For an n-transposition graph, n(n - 1)/2 paths constructed by our algorithm are node-disjoint except for s. The time complexity and the maximum length of each path are O(n^7) and 3n - 5, respectively.

5. Computer Experiment

To evaluate the average performance of the algorithm, we conducted the following computer experiment for an n-transposition graph. The algorithm is implemented in the programming language C. The program is compiled by gcc with -O2 option and executed on a target machine with an Intel Celeron 400 MHz CPU and a 128 MB memory unit.

1. Fix a source node to be 12 \cdots n and select destination nodes randomly other than the source.
2. Apply the algorithm and measure the length of each path and execution time.

Experiment is performed 1,000 times for each n from 2 to 50. Results are shown in Figs. 11 and 12. From these figures we can observe that the average length of each path and the average time of paths construction are of polynomial order and approximately O(n) and O(n^{3.5}), respectively, in their ranges.

6. Conclusions

In this paper, we proposed a polynomial algorithm for the node-to-set disjoint paths problem in an n-transposition graph whose time complexity and the maximum length of each path are O(n^7) and 3n - 5, respectively. We also conducted computer experiments to show that the average length of each path and the average time are O(n) and O(n^{3.5}), respectively.

References


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