General Possibility Theorems for Group Decisions [1, 2]

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INTRODUCTION

Kenneth Arrow [1], in his classic work on social choice, investigated the extent to which it is possible to construct a mechanism for aggregating arbitrary sets of individual preference relations. An analogous subject of inquiry is the extent to which a rule exists for passing from arbitrary sets of individual preferences to (complete) group relations (not necessarily transitive) which generate a choice from every finite set of alternatives. Sen [5] calls such rules Social Decision Functions (SDF's). He observes (p. 391) " Arrow's General Possibility Theorem rests squarely on the requirement of full transitivity ", and provides counter-examples to the General Possibility Theorem in the context of SDF's. In this paper, following a programme suggested by Sen ([5], p. 391), we investigate to what extent the impossibility result can be retrieved by strengthening the conditions imposed on SDF's. The theorems we prove stand in the same relation to SDF's as Arrow's does to Social Welfare Functions.

We assume that all n-tuples of individual preference orderings are admissible and that the number of individuals is at least three. A SDF-Q is a SDF which has only quasi-transitive range elements, and a weak dictator is an individual whose preference for any x over any y guarantees that y is not socially preferred to x. The first two theorems establish that there exists no SDF-Q which satisfies Independence of Irrelevant Alternatives, The Pareto Principle, and alternatively the non-existence of a weak dictator or May's Positive Responsiveness [2]. The third theorem (valid only for at least four individuals) establishes that there does not exist a SDF which satisfies Independence, the Pareto Principle, the non-existence of a weak dictator and Positive Responsiveness. Examples demonstrate that we cannot omit either of the final two conditions; however, impossibility still obtains with a weaker version of Positive Responsiveness if the non-existence of a weak dictator condition is strengthened (Theorem 4).

I. NOTATION, DEFINITIONS, AND CONDITIONS

A preference ordering on S is a complete (xRy or yRx for all x and y in S) and transitive binary relation on S.[3] We are concerned with the problem of aggregating the individual

[1] *First version received October 1970; final version received August 1971 (Eds.).*

[2] We thank Y. Murakami and A. K. Sen for a helpful discussion concerning the results obtained in this paper, and A. Gibbard and B. Guha for providing us with manuscripts of their work. H. Sonnenschein's research was supported in part by The Ford Foundation and The National Science Foundation; he appreciates very much their aid.

[3] Unless otherwise stated we follow the notation and terminology of Arrow [1].

preference orderings (abbreviated IPO's) of at least three individuals over a set S of at least three alternatives. M denotes both the number of individuals and the set $\{1, 2, ..., M\}$. Following standard notation, the IPO of the ith individual is denoted by R_i. A *Rule* is a function which associates with every M-tuple of IPO's $(R_1, R_2, ..., R_M)$ a complete binary relation R. A *Social Decision Function* (abbreviated SDF) is a *Rule* which has the property that every finite subset of S has an R-greatest element. Strict group preference P, group indifference I, strict individual preference P_i, and individual indifference I_i are derived from R and R_i in the customary way. A *Social Welfare Function* (abbreviated SWF) is a SDF with the property that every R in the range of the SDF is transitive.

The following conditions play an important role in the analysis. The first is Arrow's third condition; the now classic and much discussed independence of irrelevant alternatives.

Condition A.3. If $(R_1, R_2, ..., R_M)$ and $(R'_1, R'_2, ..., R'_M)$ are any two M-tuples of IPO's, and if for all $i \in M$ xR_iy if and only if xR'_iy, then xRy if and only if $xR'y$. (This condition is used extensively throughout the paper, often without mentioning that it is the justification for a particular step.)

Our next condition is the well known Pareto Principle.

Condition P. For any $x, y \in S$, xP_iy for all i implies xPy.

Positive responsiveness was first used by May in his characterization of majority voting between pairs [2].

Positive Responsiveness. Let $(R'_1, R'_2, ..., R'_M)$ and i be given and x and y be arbitrary. If $(R_1, R_2, ..., R_M)$ results in xRy, $R_j = R'_j$ for all $j \neq i$, and

$$yP_ix \quad \text{and} \quad xI'_iy$$

or

$$xI_iy \quad \text{and} \quad xP'_iy,$$

then $xP'y$.[1]

The notion of a decisive set of individuals, also due to Arrow [1], will be of use to us.

Definition. A set of individuals $J \subset M$ is said to be *decisive for x over y* (written xD_Jy) if xP_iy for all $i \in J$ and yP_ix for all $i \in M-J$ implies xPy.

Next we give the definition of a dictator.

Definition. Individual i is called a *dictator* if for all x and y xP_iy implies xPy.

Condition D. There does not exist a dictator.

Two types of weak dictators are considered.

Definition. Individual i is called a *weak dictator-D* if for all x and y, xP_iy and yP_jx for all $j \neq i$ implies xRy.

Definition. Individual i is called a *weak dictator* if for all x and y, xP_iy implies xRy.

A stronger non-dictatorship condition than considered by Arrow [1] is

Condition WD. There does not exist a weak dictator.

Let R be a complete relation on the set of basic alternatives S, and P be the associated strict relation. It is well known that the following condition is both necessary and sufficient for R to have a greatest element in every finite set (Sen [4], p. 16).

Acyclicity. For any finite set $\{x_1, x_2, ..., x_n\} \subset S$, if x_1Px_2, x_2Px_3, ..., and $x_{n-1}Px_n$, then x_1Rx_n.

The next condition on R is easily seen to imply acyclicity.

[1] As formulated here condition *PR* (together with *A*.3) implies the positive association condition of Arrow, *A*.2 [1]. This would not be the case, and still all of the theorems we prove would remain valid, if the statement of *PR* was reformulated by replacing xRy with xIy. The reformulation would yield slightly stronger theorems at the cost of more cumbersome proofs.

Quasi-Transitivity. For all x, y, and z in S, xPy and yPz, imply xPz.

Section II begins by establishing two theorems for *Rules* whose range elements are required to be quasi-transitive. Such *Rules* are special kinds of SDF's, since quasi-transitivity implies acyclicity, which we have noted is equivalent to the requirement of greatest elements in all finite subsets of S. Observe that *not* every SDF must have quasi-transitive range elements, so that these theorems are not possibility theorems for SFD's, but rather possibility theorems for a special kind of SDF. They deserve attention in their own right because quasi-transitivity is equivalent to an invariance condition (Plott [3]) which has been considered important (Arrow [1], p. 120).[1]

Before beginning our study of general possibility theorems for SDF's, it is useful to consider an example similar to one given by Sen ([5], Theorem V) which proves the existence of an SDF satisfying Conditions $A.3$, P, and D.

Example 1. For all x and y define xRy if and only if xR_iy for some i.

It is easily seen that Example 1 defines such a SDF; however, the example does not provide a convincing demonstration of the general existence of a satisfactory SDF because it resolves all conflict by ranking alternatives equally well. It is about as sluggish as a SDF can be, responding only to satisfy Condition P. Note that Condition PR fails in a very dramatic manner. Note also that the inability of the rule to resolve conflict is embodied in the fact that everybody is a weak dictator. These observations suggest the theorems of the next section.

II. GENERAL POSSIBILITY THEOREMS FOR SOCIAL DECISION FUNCTIONS WITH QUASI-TRANSITIVE RANGE RELATIONS

If all of the range relations of a SDF are quasi-transitive, the SDF will be called a SDF-Q.

Theorem 1. *There does not exist a SDF-Q satisfying conditions A.3, P, and WD.*

The proof follows after we establish two Lemmas.

Lemma 1. *If f is a SDF-Q satisfying Conditions A.3 and P, and $xD_{\{i\}}y$ for some $x, y \in S$, then i is a dictator.*

Proof. Note that Arrow's proof of the General Possibility Theorem ([1], pp. 98-99) establishes this result for the case of SWF's; however, since only the transitivity of P is used in his analysis, it is also established for SDF-Q's.

Definition. Let $a, b \in S$ and assume aD_Vb for some $V \subset M$. If xD_Wy for some $x, y \in S$ and $W \subset M$ implies that the number of individuals in W is at least as great as the number of individuals in V, then V is called a *smallest decisive set*. (If Condition P is satisfied, then the existence of such a set is guaranteed.)

Lemma 2. *Let f be a SDF-Q satisfying Conditions A.3, P, and D, and assume that V is a smallest decisive set with respect to a and b, then*

V contains at least two individuals, ...(2.1)

and

every person in V is a *weak dictator*.[2] ...(2.2)

Proof. (2.1) follows directly from Lemma 1 and D. To prove (2.2) we show first that if $i \in V$ then

$$xP_iy \Rightarrow xRy \text{ for some } x \text{ and } y \text{ in S.} \quad ...(2.3)$$

[1] Invariance in the sense that the R-greatest elements of S will be independent of how the elements of S are grouped and compared throughout the use of a tree.

[2] Lemma 2 is a version of an unpublished result due to A. Gibbard. A similar result was proved independently by B. Guha.

Suppose not, then for a and an arbitrary $z \in S$ there exist a set of IPO's of the form:

i	$M-\{i\}$	
a	Some $(M-1)$-tuple of IPO's between z and a	...(2.4)
z	(not necessarily the same for each individual).	

with the result zPa.

Let $W \subset V$ and $V-W = \{i\}$.

i	W	$M-V$	
a	The same orderings among	b	
b	z and a as in (2.4).	The same orderings among	...(2.5)
z	b	z and a as in (2.4).	

yields aPb since aD_Vb and zPa by (2.4); therefore, zPb since P is transitive and $A.3$ holds. But this implies that zD_Wb, which contradicts the minimality of V. Therefore (2.3) is established.

We now prove (2.2); i.e., if $i \in V$ then

$$sP_it \Rightarrow sRt \text{ for all } s \text{ and } t \text{ in } S. \quad ...(2.6)$$

It is now enough to establish:

$$(xP_iy \Rightarrow xRy) \Rightarrow (\text{for all } s \in S\ sP_iy \Rightarrow sRy), \quad ...(2.7)$$

and

$$(xP_iy \Rightarrow xRy) \Rightarrow (\text{for all } t \in S\ xP_it \Rightarrow xRt), \quad ...(2.8)$$

since repeated use of (2.7) and (2.8) together with (2.3) yield (2.6).

If (2.7) does not hold, we will obtain a contradiction. For in this case there exists some set of IPO's satisfying: xP_iy implies xRy, sP_iy, and not sRy, for some $s \in S$.

i	$M-\{i\}$	
s	Some $(M-1)$-tuple of possibly different	...(2.9)
x	orderings among y and s.	
y	x	

For this set of IPO's we have xRy and yPs by assumption, and sPx by P. Therefore, by quasi-transitivity, yPx and xRy; which is a contradiction.

The proof of (2.8) is similar. This completes the proof of Lemma 2.

We are now ready for the proof of Theorem 1.

Proof of Theorem 1. Since a *dictator* is a *weak dictator*, Condition *WD* implies Condition *D*. Thus if a SDF-*Q* exists satisfying *A*.3, *P*, and *WD* (and hence *D*) Lemma 2 yields the conclusion that there must exist a *weak dictator*, and this contradicts *WD*. (Note that the proof applies for the case $M > 2$ as well as for the case $M = 2$.)

Theorem 2. *There does not exist a SDF-Q satisfying A.3, P, D, and PR, for* $M > 2$.

Proof. By Lemma 2 there must exist two *weak dictators*: we will label them 1 and 2. For any x and y, if xP_1y and yP_2x, then xIy independent of the other individuals' rankings. Since in fact $M > 2$ this contradicts *PR*.

Note. If $M = 2$ the theorem is false. For two individuals majority voting is a SDF-*Q* satisfying *A*.3, *P*, *D*, and *PR*.

III. GENERAL POSSIBILITY THEOREMS FOR SOCIAL DECISION FUNCTIONS

The following example demonstrates that Theorems 1 and 2, which have been proved for SDF-*Q*'s, are invalid for SDF's. Example 2 is a SDF which satisfies *A*.3, *P*, and *WD*, and

Example 3 is a SDF which satisfies $A.3$, P, D, and PR. It is not difficult to show that they possess the characteristics we have claimed.

Example 2. Let a and b be distinct elements of S.

If $(x, y) \neq (a, b)$ we write xPy if xP_iy for all $i \in M$, and yRx otherwise.

If $(x, y) = (a, b)$ we write aPb if aP_ib for $M-1$ integers $i \in M$, and bRa otherwise.

Example 3. The M individuals are ordered $(1, 2, 3, \ldots, M)$. For any $\{x, y\}$ in S we write

xIy if either xI_iy for all $i \in M$, or xP_1y and yP_jx for all $j \neq 1$,

xPy if xP_iy and xR_jy for some $j \neq i$, where i is the least integer k such that xI_ky does not hold, and

yPx otherwise.

The main result of this section is that if the number of individuals exceeds three, then there exists no SDF which satisfies the conditions of both Theorems 1 and 2.

Formally we state

Theorem 3. *If $M > 3$, then there does not exist a SDF satisfying Conditions $A.3$, P, PR, and WD.*

The proof of Theorem 3 is obtained as a consequence of Lemmas 3 and 4. Before proving these lemmas we make note of the following fact.

A *weak dictator-D* for a SDF satisfying Conditions $A.3$, P, and PR, is a *weak dictator*. ...(3.1)

Suppose not. Let i be a weak dictator. If (3.1) is violated then there exist a pair $x, y \in S$ and some $(M-1)$-tuple of IPO's such that

i	$M-\{i\}$
x	Some $(M-1)$-tuple of possibly different
y	orderings among x and y.

yields yPx, and *a fortiori*, yRx.

Repeated application of PR generates the conclusion that (xP_iy and yP_jx for all $j \neq i$) implies yPx, which contradicts the fact that i is a *weak dictator-D*.

Recall that if R is a relation in the range of a SDF then:

$$(xPy \text{ and } yPz) \Rightarrow xRz, \text{ for any } x, y, z \in S. \quad \ldots(3.2)$$

Lemma 3 is an analogue of (2.3) for the case of SDF's.

Lemma 3. *If a SDF satisfies $A.3$, P, and PR, then there exists an individual $i \in M$ such that:*

$$(xP_iy \text{ and } yP_jx \text{ for all } j \neq i) \Rightarrow xRy \text{ for some } x \text{ and } y \text{ in } S. \quad \ldots(3.3)$$

Proof. Suppose not. Then for all $x, y \in S$ and all $i \in M$

$$(xP_iy \text{ and } yP_jx \text{ for all } j \neq i) \Rightarrow yPx. \quad \ldots(3.4)$$

Let V be a *smallest decisive set* (see the paragraph preceeding Lemma 2). By (3.4) it has at least two elements. Assume V is decisive for x over y. Let $i \in V$ and $V-\{i\} = W$.

Observe that

i	W	$M-V$	
x	z	y	
y	x	z	...(3.5)
z	y	x	

yields xPy since xD_Vy and zPx by (3.4); therefore, zRy by (3.2).

Let $U = W-\{j\}$.

i	j	U	$M-V$	
x	y	x	y	...(3.6)
y	x	y	x	

yields yRx since V is minimal.

i	j	U	$M-V$	
zxy	z	x	y	
	y	z	x	...(3.7)
	x	y	z	

yields yPx by the previous table and PR, and zPy by (3.5) and PR; therefore, zRx by (3.2)

i	j	U	$M-V$	
z	z	x	x	...(3.8)
x	x	z	z	

yields zPx by the previous table and PR.

We have shown $V = \{i, j\}$.

i	j	$M-V$	
x	z	y	
y	x	z	...(3.9)
z	y	x	

yields xPy since xD_Vy and yPz by (3.4); therefore, xRz by (3.2), but this contradicts (3.4). This finishes the proof of Lemma 3.

Lemma 4. *If $M > 3$ and a SDF satisfies Conditions A.3, P, and PR, then there exists an individual $i \in M$ who is a weak dictator-D.*

Proof. It is enough to establish

$$(xP_iy \text{ and } yP_jx \text{ for all } j \neq i) \Rightarrow xRy$$
$$\Rightarrow \qquad \ldots(3.10)$$
$$\text{for all } s \in S\ (sP_iy \text{ and } yP_is \text{ for all } j \neq i) \Rightarrow sRy,$$

and

$$(xP_iy \text{ and } yP_jx \text{ for all } j \neq i) \Rightarrow xRy$$
$$\Rightarrow \qquad \ldots(3.11)$$
$$\text{for all } t \in X\ (xP_it \text{ and } tP_jx \text{ for all } j \neq i) \Rightarrow xRt,$$

since repeated use of (3.10) and (3.11) together with Lemma 3 yields the conclusion that i is a *weak dictator-D*.

We will prove (3.11). The proof of (3.10) is similar.

Assume the hypothesis of (3.11) and let 2, 3, and 4 be three other distinct elements of M.

For concreteness let $i = 1$.

1	2	3	4	$M-\{1, 2, 3, 4\}$	
x	xy	y	y	y	
y	t	t	t	t	...(3.12)
t		x	x	x	

yields xPy by hypothesis and PR, and yPt by P; therefore, xRt by (3.2).

1	2	3	4	$M-\{1, 2, 3, 4\}$	
y	y	t	y	t	
x	x	y	xt	y	...(3.13)
t	t	x		x	

yields xPt by the previous table and PR, and yPx by P; therefore, yRt by (3.2).

1	2	3	4	$M-\{1, 2, 3, 4\}$	
x	y	xyt	y	t	
y	t		t	y	...(3.14)
t	x		x	x	

yield xPy by hypothesis and PR, and yPt by the previous table and PR; therefore, xRt by (3.2).

1	2	3	4	$M-\{1, 2, 3, 4\}$	
y	t	y	t	t	
x	y	x	y	y	...(3.15)
t	x	t	x	x	

yield xPt by the previous table and PR, and yPx by P; therefore, yRt by (3.2).

Finally,

1	2	3	4	$M-\{1, 2, 3, 4\}$	
x	t	y	yt	t	
y	xy	t	x	y	...(3.16)
t		x		x	

yields xPy by hypothesis and PR, and yPt by the previous table and PR; therefore, xRt and this proves (3.11).

The proof of Theorem 3 is now immediate.

Proof of Theorem 3. Under the hypothesis of the theorem a *weak dictator-D* is a *weak dictator* (see 3.1). Thus Lemma 4 guarantees the result.

For $M = 3$ the theorem is false.

Example 4. Consider a situation with three alternatives x, y, and z, and three individuals 1, 2, 3. There are two decision rules for choices among pairs. Majority voting and

Rule B. Set aPb if for some $i, j \in I = \{1, 2, 3\}$, $i \neq j$, aP_ib and aR_jb. Set bPa if bR_ia for all i and bP_ia for some i. Otherwise set aIb. The decision between x and z and y and z is made according to rule B, and the decision between x and y is made by majority voting.

It is easily shown that this example satisfies *A*.3, *P*, *PR*, and *WD*. In order to prove that the example generates a SDF it is sufficient, because of symmetry, to prove that xPy, zPx, and yPz cannot hold simultaneously. This may be achieved by listing the cases in which xPy and zPx hold, and observing that yPz cannot be generated by any combination of these cases.

We conclude with a discussion of Condition *PR*. As stated the condition is very strong and perhaps somewhat unappealing; yet its importance is illustrated effectively in our first two examples. Without *PR*, there is no guarantee that a SDF will be responsive to changes in preferences and thus degenerate SDF's may be admissible.

The reader should note, however, that it is the interplay between conditions *PR* and *WD* rather than the Condition *PR* by itself that is important in the proof of Theorem 3. Condition *PR* can be weakened to require a change in preferences by several individuals if we are willing to strengthen Condition *WD* to rule out groups of several individuals who collectively may act as a weak dictator. The following definition and theorem makes this observation precise. The theorem can be proved by reinterpreting the proof of Theorem 3.

Condition PR_N. Let $(R'_1, \ldots, R'_M)$ be given and x and y be arbitrary. Let $J \subset M$ have at least N elements.

If $(R_1, \ldots, R_M)$ results in xRy, $R_j = R'_j$ for all $j \notin J$, and

$$yP_ix \quad \text{and} \quad xI'_iy$$

$$\text{or}$$

$$xI_iy \quad \text{and} \quad xP'_iy$$

for all $i \in J$, then $xP'y$.

Theorem 4. *Let A.3, P, and* PR_N *hold for some* $N \leqq M/4$. *Then there is* $K \subset M$ *composed of at most N individuals, such that for any* $x, y \in S$, *if* xP_iy *for all* $i \in K$ *then* xRy.

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