

GEOMETRIC CLASS FIELD THEORY II

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1. PREVIEW OF THE ARGUMENT

This talk is just going to review some background ingredients that we'll use. We want to exhibit an isomorphism

$$\mathrm{Hom}_{\mathrm{cts}}(\pi_{1,\acute{\mathrm{e}}\mathrm{t}}(C), \overline{\mathbf{Q}}_\ell^\times) \cong \mathrm{Hom}(\mathrm{Pic}^0(k), \overline{\mathbf{Q}}_\ell^\times) + (\widehat{\mathbf{Z}} \xrightarrow{\mathrm{Frob}_c} \overline{\mathbf{Q}}_\ell^\times).$$

where $c \in C(\mathbf{F}_q)$ is a fixed rational point. Our proof will proceed by upgrading this equality to an equivalence of geometric objects. First, we'll interpret $\mathrm{Hom}_{\mathrm{cts}}(\pi_{1,\acute{\mathrm{e}}\mathrm{t}}(C), \overline{\mathbf{Q}}_\ell^\times)$ in terms of rank one ℓ -adic local systems on C . Similarly, we'll interpret the datum of $\mathrm{Hom}(\mathrm{Pic}^0(k), \overline{\mathbf{Q}}_\ell^\times) + (\widehat{\mathbf{Z}} \xrightarrow{\mathrm{Frob}_c} \overline{\mathbf{Q}}_\ell^\times)$ as a ‘‘character sheaf’’ on Pic_C . In broader context, this is a rank one ℓ -adic local system satisfying a ‘‘Hecke eigensheaf’’ condition.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{cts}}(\pi_{1,\acute{\mathrm{e}}\mathrm{t}}(C), \overline{\mathbf{Q}}_\ell^\times) & & \mathrm{Hom}(\mathrm{Pic}^0(k), \overline{\mathbf{Q}}_\ell^\times) + (\widehat{\mathbf{Z}} \xrightarrow{\mathrm{Frob}_c} \overline{\mathbf{Q}}_\ell^\times) \\ \parallel & & \parallel \text{function-sheaf} \\ \left\{ \begin{array}{c} \text{1-dim } \ell\text{-adic} \\ \text{local systems on } C \end{array} \right\} & & \left\{ \begin{array}{c} \text{1-dim } \ell\text{-adic local systems} \\ \text{on } \mathrm{Pic}(C), \text{ satisfying} \\ \text{Hecke eigensheaf property} \end{array} \right\}. \end{array}$$

Then, Deligne's Theorem will establish a correspondence along the bottom row.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{cts}}(\pi_{1,\acute{\mathrm{e}}\mathrm{t}}(C), \overline{\mathbf{Q}}_\ell^\times) & & \mathrm{Hom}(\mathrm{Pic}^0(k), \overline{\mathbf{Q}}_\ell^\times) + (\widehat{\mathbf{Z}} \xrightarrow{\mathrm{Frob}_c} \overline{\mathbf{Q}}_\ell^\times) \\ \parallel & & \parallel \text{function-sheaf} \\ \left\{ \begin{array}{c} \text{1-dim } \ell\text{-adic} \\ \text{local systems on } C \end{array} \right\} & \xlongequal{\text{Deligne's Theorem}} & \left\{ \begin{array}{c} \text{1-dim } \ell\text{-adic local systems} \\ \text{on } \mathrm{Pic}(C), \text{ satisfying} \\ \text{Hecke eigensheaf property} \end{array} \right\}. \end{array}$$

To motivate how the ingredients will fit together, we recall the outline of Deligne's argument. Given a $\overline{\mathbf{Q}}_\ell$ -local system \mathcal{L} on C , we want to produce a $\overline{\mathbf{Q}}_\ell$ -local system on $\mathrm{Pic}(C)$. How can we do this? The idea is to first pull \mathcal{L} back to a local system on the symmetric powers of C . We have d projection maps $p_i: C^d \rightarrow C$ for $i = 1, \dots, d$. We can form the local system

$$\mathcal{L}^{\boxtimes d} = p_1^* \mathcal{L} \boxtimes p_2^* \mathcal{L} \boxtimes \dots \boxtimes p_d^* \mathcal{L}$$

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on C . This is visibly preserved by the symmetric group S_d , so we can hope that it descends to the d th symmetric power of C ,

$$C^{(d)} = C^d/S_d.$$

This turns out to be a smooth projective variety.

The upshot is that at this point, we've gone from a $\overline{\mathbf{Q}}_\ell$ -local system on C to one on $C^{(d)}$. In turn, the symmetric power admits an *Abel-Jacobi map*

$$C^{(d)} \rightarrow \mathrm{Pic}^d(C),$$

sending a divisor to the corresponding line bundle, which is surjective. Moreover, if d is large enough then Riemann-Roch tells us that it is fibered in projective spaces. So we have an exact sequence of fundamental groups

$$\pi_1(\mathbf{P}^N) \rightarrow \pi_1(C^{(d)}) \rightarrow \pi_1(\mathrm{Pic}^d C) \rightarrow 1.$$

But since projective space is “simply-connected” this is telling us that

$$\pi_1(C^{(d)}) \cong \pi_1(\mathrm{Pic}^d C),$$

so a $\overline{\mathbf{Q}}_\ell$ -local system on $C^{(d)}$, which is the same as a representation of $\pi_1(C^{(d)}) \cong \pi_1(\mathrm{Pic}^d C)$, corresponds to a unique $\overline{\mathbf{Q}}_\ell$ -local system on $C^{(d)}$!

2. THE ÉTALE FUNDAMENTAL GROUP

2.1. Generalities. We're going to avoid defining the étale fundamental group carefully. Suffice it to say that you should think of it as being analogous to the profinite completion of the topological fundamental group. To help you remember this slogan, we will start forgetting the prefix “étale”. Most properties of the classical fundamental group that can be phrased algebraically also hold for the étale fundamental group. (One important exception is that the étale fundamental group of a product is not necessarily the product of étale fundamental groups.)

In particular, finite étale covers are the algebro-geometric analogues of finite covering spaces, and the étale fundamental group satisfies the same “universal” property with respect to connected finite étale covers as the usual fundamental group with respect to covering spaces: quotients

$$\pi_{1,\acute{\text{e}}\text{t}}(X) \twoheadrightarrow G$$

correspond to Galois étale covers with Galois group G . That is, there is a bijection

$$\left\{ \begin{array}{l} \text{open normal subgroups} \\ U \subset \pi_{1,\acute{\text{e}}\text{t}}(X) \end{array} \right\} \leftrightarrow \{\text{finite Galois covers of } X\}.$$

We have suppressed a detail, which is that we need to choose as basepoint a geometric point. Fortunately, for applications to the abelianized étale fundamental group the basepoint is largely irrelevant (since change of basepoint induces conjugation on the fundamental group).

As we discussed last time, for a geometrically connected scheme X there is a short exact sequence

$$0 \rightarrow \pi_1(\overline{X}, \overline{x}) \rightarrow \pi_1(X, \overline{x}) \rightarrow \mathrm{Gal}(k^s/k) \rightarrow 1.$$

The quotient $\pi_1(X, \bar{x}) \rightarrow \text{Gal}(k^s/k)$ corresponds to the étale cover of X obtained by base-changing from the étale cover k^s/k (under the above dictionary between open subgroups of the fundamental group and étale covers).

There is also a relative version: if $X \rightarrow S$ is a family of geometrically connected varieties, then we have an exact sequence

$$\pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \pi_1(S, \bar{s}) \rightarrow 1.$$

2.2. Projective space is simply-connected. We saw that a crucial part of Deligne’s argument is that $\pi_1(\mathbf{P}^n, \bar{x}) = 0$ for a geometric point $\bar{x} \in \mathbf{P}^n$. This is plausible by analogy from the classical case of complex projective space (which you should view as strong evidence), but we now prove it formally.

Theorem 2.1. *We have $\pi_1(\mathbf{P}^n, \bar{x}) = 0$ for any n .*

We begin with the case $n = 1$. For $Y \rightarrow \mathbf{P}^n$ an étale cover of degree d the Riemann-Hurwitz formula gives

$$2 - 2g(Y) = d(2) + 0.$$

We can assume that $d > 1$ because otherwise the cover is trivial, in which case clearly the equation cannot be solved.

The idea of the proof is to establish the Lefschetz hyperplane theorem for fundamental groups. This says that if $\dim X > 1$ and $S \subset X$ is a general hyperplane section of a projective variety, then the map

$$\pi_1(S, \bar{s}) \rightarrow \pi_1(X, \bar{x})$$

is a surjection. This will obviously imply the result we want by induction, since $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ is an ample divisor.

Now it is a little unclear how to interpret what it means for a map to induce a surjection of étale fundamental groups. One way to think about this is that a map $(S, \bar{s}) \rightarrow (X, \bar{x})$ induces a map of π_1 -sets

$$\{\pi_1(X, \bar{x}) - \text{sets}\} \rightarrow \{\pi_1(S, \bar{s}) - \text{sets}\}.$$

A little thought shows that the map $\pi_1(S, \bar{s}) \rightarrow \pi_1(X, \bar{x})$ is surjective if and only if the induced map of categories of π_1 -sets takes *transitive* π_1 -sets to transitive π_1 -sets. Under the interpretation of π_1 -sets as “covering spaces” (really finite étale map), the condition of transitivity translates into the *connectedness* of the covering space.

The upshot is that $\pi_1(S, \bar{s}) \rightarrow \pi_1(X, \bar{x})$ is surjective if and only if *pullback takes connected covers of X to connected covers of S* . Thus, we have reduced to:

Proposition 2.2. *Suppose X is a smooth connected variety of dimension at least 2. If $D \subset X$ is smooth subvariety representing an ample divisor, then D is connected.*

Proof. For the short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

we try to apply the associated long exact sequence in cohomology

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}_X(-D)) \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_D) \\ \rightarrow H^1(X, \mathcal{O}_X(-D)) \rightarrow \dots \end{aligned}$$

We want to show that $h^0(X, \mathcal{O}_D) = 1$. Since X is connected, $h^0(X, \mathcal{O}_X) = 1$. We don't know a priori that $H^1(X, \mathcal{O}_X(-D)) = 0$, but we *do* know that this is true after replacing D by nD for $n \gg 0$, by Serre duality plus Serre's theorem that twisting by sufficiently high powers of ample divisors kills cohomology. Since D and nD have the same underlying topological space, we can assume that $H^1(X, \mathcal{O}_X(-D)) = 0$, and then the result follows. \square

Since the pullback of an ample divisor across a finite map is ample, this shows the Lefschetz hyperplane theorem, and hence the desired result.

Remark 2.3. It isn't necessary to phrase the argument in terms of the Lefschetz hyperplane theorem. More directly, if $X \rightarrow \mathbf{P}^n$ is a connected finite étale cover, then its restriction to \mathbf{P}^{n-1} is still a connected finite étale cover. Then by induction we reduce to the case of \mathbf{P}^1 to see that the cover is trivial.

3. THE PICARD SCHEME

3.1. Geometry of the Picard scheme. For X/\mathbf{F}_q a smooth curve, Pic_X is the “moduli space of line bundles on X ”. We will ignore subtleties concerning the existence of this scheme, and attempt to handle it using our intuition. Its connected components are parametrized by the degree of the line bundle, and the identity component Pic_X^0 is an abelian variety of dimension equal to the genus of X . In particular, all the component are isomorphic, and any divisor D of degree d induces a map

$$\text{Pic}_X^e \xrightarrow{\otimes \mathcal{O}(D)} \text{Pic}_X^{d+e}$$

which is an isomorphism.

3.2. Symmetric powers. Let $C^{(d)} := C^d/S_d$ be the d th symmetric power of the curve C .

Proposition 3.1. *The scheme $C^{(d)}$ is smooth.*

Example 3.2. We first check the statement for $C = \mathbf{P}^1$. Since it is a local assertion, we can even reduce to $C = \mathbf{A}^1 = \text{Spec } k[x]$. Then $C^d = \text{Spec } k[x_1, \dots, x_d]$. Then $C^d/S_d = \text{Spec } k[x_1, \dots, x_d]^{S_d}$, where by the classical theory of symmetric invariants we know that

$$k[x_1, \dots, x_d]^{S_d} = k[\sigma_1, \dots, \sigma_d]$$

where $\sigma_1, \dots, \sigma_d$ are the elementary symmetric polynomials in d variables. But this is just \mathbf{A}^d again.

Proof. The idea is to check the claim “étale-locally”. In fact, since smoothness is equivalent to the completed local rings being power series rings, it suffices to study

$\mathcal{O}_{C,c} \cong k[[x]]$. Then $\mathcal{O}_{C,c}^{\otimes d} \cong k[[x_1, \dots, x_d]]$. By the same calculation as above, we find that

$$(\mathcal{O}_{C,c}^{\otimes d})^{S_d} \cong k[[x_1, \dots, x_d]]^{S_d} \cong k[[\sigma_1, \dots, \sigma_d]].$$

□

Remark 3.3. Here is a more sophisticated point of view on the matter. We can view $C^{(d)}$ as the Hilbert scheme of 0-dimensional subvarieties of C of length d . Then the tangent space to $C^{(d)}$ at a divisor D is the space of deformations of D , which is $\text{Ext}^2(\mathcal{O}_D, \mathcal{O}_D)$. This vanishes because it is H^2 of a sheaf on C , which is a curve of dimension *one*.

3.3. The Abel-Jacobi map. There is an *Abel-Jacobi* map $C^{(d)} \rightarrow \text{Pic}_C^d$ which on rational points sends a divisor D to its underlying line bundle $\mathcal{O}(D)$. The fiber is the *linear system* $|D| = \mathbf{P}(H^0(X, \mathcal{O}(D)))$, which on rational points is the set of divisors linearly equivalent to D .

By Riemann-Roch, if d is sufficiently large then $h^0(X, \mathcal{O}(D)) = d + 1 - g$, so the Abel-Jacobi map is a fibration with fibers being projective spaces of constant dimension (so this is in fact locally trivial).

$$\begin{array}{ccc} \mathbf{P}^{d+1-g} & \longrightarrow & C^{(d)} \ni D \\ & & \downarrow AJ \\ & & \text{Pic}_C^d \ni \mathcal{O}(D) \end{array}$$

4. CHARACTER SHEAVES

4.1. Local systems. Local systems are locally constant sheaves for the étale topology satisfying finiteness conditions. In particular, we'll be interested in $\overline{\mathbf{Q}}_\ell$ -local systems, and we'll demand that their stalks be finite-dimensional. As is familiar from topology, there is an equivalence of categories

$$\{\overline{\mathbf{Q}}_\ell\text{-local systems on } X\} \iff \left\{ \begin{array}{l} \pi_{1,\text{ét}}(X)\text{-representations} \\ \text{on finite } \overline{\mathbf{Q}}_\ell\text{-vector spaces} \end{array} \right\}.$$

We will consider only local systems of rank *one*, in which case the right hand side becomes simply homomorphisms $\pi_{1,\text{ét}}(X) \rightarrow \overline{\mathbf{Q}}_\ell^\times$.

4.2. The function-sheaf correspondence. Given an ℓ -adic local system \mathcal{L} on X , we can form a function $X(k) \rightarrow \overline{\mathbf{Q}}_\ell$ by taking the trace of Frobenius at x evaluated at the stalk \mathcal{L}_x :

$$x \mapsto \text{Tr}(\text{Frob}_x, \mathcal{L}_x).$$

This is the *function-sheaf correspondence*. The philosophy is that sufficiently nice functions should be realized geometrically in this way from some sheaf, although it can be difficult to do so in practice. We'll focus on one particular class of examples called *character sheaves*.

4.3. **Character sheaves.** Let G be a smooth commutative algebraic group over \mathbf{F}_q .

Definition 4.1. A *character sheaf* on G is an ℓ -adic local system \mathcal{L} of rank 1 on G such that

$$m^* \mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$$

for the multiplication map $G \times G \xrightarrow{m} G$.

The main result of this section is to prove:

Theorem 4.2. *There is a natural bijection*

$$\{\text{character sheaves on } G\} \iff \{\text{homomorphisms } G(\mathbf{F}_q) \rightarrow \overline{\mathbf{Q}}_\ell^\times\}.$$

The construction a homomorphism $G(\mathbf{F}_q) \rightarrow \overline{\mathbf{Q}}_\ell^\times$ from a character sheaf \mathcal{L} is simply the function-sheaf correspondence $\mathcal{L} \mapsto f_{\mathcal{L}}$. We just need to check that the function produced in this way is indeed a homomorphism. We are given that $m^* \mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$. Evidently

$$f_{\mathcal{L} \boxtimes \mathcal{L}}(x, y) = f_{\mathcal{L}}(x) \cdot f_{\mathcal{L}}(y).$$

What is $f_{m^* \mathcal{L}}$? You can think of pullback on character sheaves as corresponding, at the level of π_1 -representations, with “restriction”. In other words, $m^* \mathcal{L}$ corresponds to the homomorphism $\pi_1(G \times G) \rightarrow \overline{\mathbf{Q}}_\ell^\times$ obtained by composing with the map induced by multiplication $\pi_1(G \times G) \xrightarrow{m_*} \pi_1(G)$, and then with the homomorphism $\pi_1(G) \rightarrow \overline{\mathbf{Q}}_\ell^\times$ defining the local system \mathcal{L} . From the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \uparrow & & \uparrow \\ (x, y) & \longrightarrow & x + y \end{array}$$

(writing the group law on g additively) we see that $\text{Frob}_{(x,y)} \in \pi_1(G \times G)$ maps to $\text{Frob}_{x+y} \in \pi_1(G)$ under m_* , so

$$f_{m^* \mathcal{L}}(x, y) = f_{\mathcal{L}}(m(x, y)) = f_{\mathcal{L}}(x + y)$$

hence

$$f_{\mathcal{L}}(x + y) = f_{\mathcal{L}}(x) \cdot f_{\mathcal{L}}(y).$$

To construct a character sheaf from a homomorphism, we consider the *Lang isogeny* $L: G \rightarrow G$ defined by $\varphi := \text{Frob}_q^{-1} \cdot \text{Id}$. This is an isogeny (because G is commutative!) with kernel $G(\mathbf{F}_q)$;

$$0 \rightarrow G(\mathbf{F}_q) \rightarrow G \xrightarrow{L} G \rightarrow 0$$

and in fact defines an étale cover $G \rightarrow G$ with Galois group $G(\mathbf{F}_q)$, because the differential of the Lang isogeny is $-d(\text{Frob}_q) + d(\text{Id}) = 0 + \text{Id}$. The existence of such a Galois cover implies, by the dictionary between covers and subgroups of π_1 , that there is a surjection ma

$$\pi_1(G) \rightarrow G(\mathbf{F}_q)$$

and so from a homomorphism $G(\mathbf{F}_q) \rightarrow \overline{\mathbf{Q}}_\ell^\times$ we obtain by composition a homomorphism from $\pi_1(G)$, i.e. a one-dimensional ℓ -adic local system \mathcal{L} on G .

To check the character sheaf property, you can view the local system thus constructed as a representation of $G(\mathbf{F}_q)$. That is, \mathcal{L} is trivialized after pulling back via the Lang isogeny L , so we know that both $m^*\mathcal{L}$ and $\mathcal{L} \boxtimes \mathcal{L}$ are trivial after pulling back by $L \times L$ on $G \times G$:

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ L \times L \downarrow & & \downarrow L \\ G \times G & \xrightarrow{m} & G \end{array}$$

Thus it suffices to check that actions of the Galois group of the cover $G \times G \xrightarrow{L \times L} G \times G$ are the same. That Galois group is $G(\mathbf{F}_q) \times G(\mathbf{F}_q)$, and unwinding the action of $(x, y) \in G(\mathbf{F}_q) \times G(\mathbf{F}_q)$ boils down to exactly the homomorphism property. We leave the details to be checked by the reader.

REFERENCES

[1] Toth, P. Geometric Abelian Class Field Theory. <http://dspace.library.uu.nl/handle/1874/206061>