Group decision making under hesitant fuzzy environment with application to personnel evaluation

Dejian Yu \(^a\), Wenyu Zhang \(^a\,*\), Yejun Xu \(^b\)

\(^a\) School of Information, Zhejiang University of Finance & Economics, 18 Xueyuan Street, Hangzhou 310018, China
\(^b\) Research Institute of Management Science, Business School, Hohai University, Nanjing 211100, China

**Abstract**

In many personnel evaluation scenarios, decision makers are asked to provide their preferences anonymously to both ensure privacy and avoid psychic contagion. The use of hesitant fuzzy sets is a powerful technique for representing this type of information and has been well studied. This paper explores aggregation methods for prioritized hesitant fuzzy elements and their application on personnel evaluation. First, the generalized hesitant fuzzy prioritized weighted average (GHFPWA) and generalized hesitant fuzzy prioritized weighted geometric (GHFPWG) operators are presented. Some desirable properties of the methods are discussed and special cases are investigated in detail. Previous research has indicated that many existing hesitant fuzzy aggregation operators are special cases of the proposed operators. Then, a procedure and algorithm for group decision making is provided using these proposed generalized hesitant fuzzy aggregation operators. Finally, the group decision making method is applied to a representative personnel evaluation problem that involves a prioritization relationship over the evaluation index.

**Keywords:** Hesitant fuzzy set, Group decision making, Generalized hesitant fuzzy aggregation operator, Prioritized aggregation operator

1. Introduction

Probation is an important component of the evaluation process. It allows an employee to demonstrate his/her suitability for a position. Probation gives the company an opportunity to observe an employee's work ethic and training status, assists the employee in adjusting to a new position, and removes an employee whose performance fails to meet expectations. To expand a company's potential market and maintain growth, it is important to provide sales engineers with professional and systematic trainings. Training is typically provided using a full scale development platform that allows engineers to learn a broad range of topics, from product knowledge to business skills, case studies to simulation, and classroom training to field practice. Effective training methods leverage the participants’ full potential, provide both opportunities and challenges, develop creativity, identify strengths and assist in career planning. After training, employees may receive systematic assessment and evaluation over the course of a probationary period.

Let us consider an interesting but realistic problem. Currently, five sales engineers' probation end dates are approaching, and we need to evaluate their performances to determine if their contracts should be renewed. Since sales engineers are responsible for establishing and maintaining productive working relationships with target customers in critical business areas, the company is very interested in evaluating their performance. A sales manager \((e_1)\), manufacturing manager \((e_2)\), engineering manager \((e_3)\) and human resource manager \((e_4)\) compose the panel of decision makers that will take responsibility for this evaluation. A strict evaluation of each of the five employees \(x_i (i = 1, 2, \ldots, 5)\) is performed from four aspects: work attitude \((C_1)\), communication skill \((C_2)\), problem solving skill \((C_3)\), and learning skill \((C_4)\). First, work attitude \((C_1)\) is considered a must for all professionals. This position also requires the sales engineer to work closely with internal colleagues and customers to develop effective sales proposals, thus communication skills \((C_2)\) represent another important competency area. In addition, sales engineers must be able to pair prospective customers with ongoing projects and follow through with appropriate sales activities to get the orders. This means that sales engineers must also possess solid problem solving skills \((C_3)\). Finally, learning skills \((C_4)\) help us to grow and adapt so that we may achieve more challenging targets. The prioritization of the criteria can be expressed as \(C_1 \succ C_2 \succ C_3 \succ C_4\), where “\(\succ\)” indicates “priority to”.

It should be noted that there are two key issues inherent to the above problems. The first key issue is anonymously depicting the decision makers’ preferences. The second key issue is mathematically expressing the criteria prioritization relationships. The focus of this paper is to develop a new decision making method that addresses both of the above problems.

To address these issues, the remainder of this paper is organized as follows. In Section 2, a literature review of decision making
methods for personnel evaluation is provided, and the concepts of hesitant fuzzy set and prioritized operators are introduced. Section 3 proposes the generalized hesitant fuzzy prioritized weighted average (GHFPWA) and generalized hesitant fuzzy prioritized weighted geometric (GHFPWG) operators to aggregate the hesitant fuzzy elements (HFEs), whose desirable properties and special cases are also studied in this section. Section 4 develops a multi-criteria group decision making method based on the proposed operators under the hesitant fuzzy environment, and applies this decision making method to the illustrative evaluation problem regarding sales engineers presented in this section. Section 5 provides some concluding remarks.

2. Literature and basic concept review

Personnel evaluation has been widely studied by a large number of research institutions, and is an important aspect of human resources management. In highly competitive markets, a company’s personnel play a crucial role in the future development of the company. In other words, talent can be a company’s biggest asset. Therefore, staffing is important for any company’s business development. Putting the right people in the right positions can produce many positive impacts on a company, such as lowering employee turnover rate, improving enthusiasm and productivity, and increasing customer satisfaction. Alternatively, waste and other negative impacts can result from poor hiring decisions.

Multi-criteria decision making (MCDM) has been identified as an essential technique in the personnel selection process [3,15,11]. Karsak [6] and Gil-Aluja [4] proposed an MCDM algorithm based on the concepts of ideal and anti-ideal solutions for personnel selection. Li [8] combined analytic network process (ANP) with fuzzy data envelopment analysis (DEA) and proposed an integrated method to deal with the personnel selection problem. An example is provided with an electric and machinery company in Taiwan. Chen and Cheng [2] proposed a fuzzy group decision support system based on metric distances to solve the personnel selection problem. Dursun and Karsak [3] developed a MCDM method based on the principles of fusion of fuzzy information and 2-tuple linguistic information. Zhang and Liu [37] used grey relational analysis (GRA) to solve the personnel selection problem under an intuitionistic fuzzy environment. Boran et al. [1] used the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) to solve the personnel selection problem under an intuitionistic fuzzy environment and illustrated the method with an example involving a sales manager position in a manufacturing company. Merigó and Gil-Lafuente [12,13] studied the human resource selection problem using the ordered weighted average (OWA) operator. The key differences between the above studies are illustrated in Table 1.

The above personnel selection methods operate under the assumption that the evaluation criteria are independent and have similar priority levels. This assumption is not always practical, however. Take the problem described in Section 1 as an example. If an alternative (i.e., a sales engineer) has a poor work attitude $C_1$, then the company will likely terminate employment regardless of the employee’s communication skills $C_2$, problem solving skills $C_3$ or learning skills $C_4$. In other words, work attitude $C_1$ has a higher priority than the other criteria. The prioritized average (PA) operator is a useful tool to deal with the above situation. To facilitate later study, the PA operator and hesitant fuzzy sets are introduced here.

A hesitant fuzzy set (HFS) is a generalized fuzzy set that is characterized by the membership degree of an element to a set presented as several possible values between 0 and 1. Torra [20,21] first proposed the concept of HFS, which has attracted significant attention from other researchers [23,36,38,27].

**Definition 1** ([20,21]). Let $X$ be a fixed set. A HFS on $X$ is defined in terms of a function that when applied to members of $X$ returns a subset with values in the range of $[0,1]$, which can be represented as the following mathematical symbol:

$$E = \{x \mid f(x) > \exists x \in X\}$$

where $f(x)$ is a set of some values in the range of $[0,1]$, that denotes the possible membership degrees of the element $x \in X$ to the set $E$. For convenience, Xia and Xu [25,26] call $f(x)$ a hesitant fuzzy element (HFE).

Take the problem illustrated in Section 1 as an example. A sales manager ($e_1$), manufacturing manager ($e_2$), engineering manager ($e_3$) and human resource manager ($e_4$) are each asked to estimate the degree to which an alternative (i.e., a sales engineer) presents an excellent work attitude. The sales manager gives the sales engineer a score of 0.6, the manufacturing manager rates the engineer with a 0.7, the engineering manager gives a 0.8 and the human resource manager ($e_4$) determines the score to be 0.9. Thus, the degree to which the alternative presents an excellent work attitude can be represented by a hesitant fuzzy element $[0.6,0.7,0.8,0.9]$. The HFS is utilized in this work because of its ability to efficiently express the uncertainty presented by this type of realistic situation. HFS therefore can address the first key issue mentioned in Section 1, i.e., how to anonymously describe the decision makers’ preferences.

For three HFEs $f_1$, $f_2$ and $f_3$, Xia [24] gave the following operational laws for the HFEs as follows:

**Definition 2.** Let $f_1$, $f_2$ and $f_3$ be three HFEs, then

1. $\bar{f}_i = \bigcup \{x \in X \mid f_i(x) = 1\}$, $i > 0$,
2. $\bar{f}_i = \bigcup \{x \in X \mid f_i(x) = 0\}$, $i > 0$,
3. $f_1 \otimes f_2 = \bigcup \{x \in X \mid f_i(x) = 1\}$, $i > 0$,
4. $f_1 \otimes f_2 = \bigcup \{x \in X \mid f_i(x) = 0\}$

where $l(t) = \{1 - t\}$. Klir et al. [7] pointed out that an additive generator of a continuous Archimedean $t$-norm is a strictly decreasing function, i.e., $k: [0,1] \rightarrow [0,\infty]$, such that $k(1) = 0$.

**Definition 3** [24]. For a HFE $f$, $S(f) = \frac{1}{n} \sum_{i=1}^{n} \xi_i$ is called the score function of $f$, where $#f$ is the number of the elements in $f$. For two HFEs $f_1$ and $f_2$, if $S(f_1) > S(f_2)$, then $f_1 > f_2$; if $S(f_1) = S(f_2)$, then $f_1 = f_2$.

Since $\xi \in [0,1]$, then $S(f) \in [0,1]$.

Yager [33] first proposed the prioritized average (PA) operator, which was defined as follows.
Definition 4. [33–35] Let $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criteria and a prioritization between the criteria can be expressed by the linear ordering $C_1 \succ C_2 \succ \ldots \succ C_n$, indicating criterion $C_j$ has a higher priority than $C_i$ if $j < k$. Then the value $C_j(x)$ is the performance of any alternative $x$ under criteria $C_j$ and satisfies $C_j(x) \in [0,1]$. If
\[
PA(C(x)) = \sum_{j=1}^{n} w_j C_j(x) \tag{2}
\]
where $w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}$, $T_j = \prod_{k=1}^{j-1} C_k(x) (j = 2, \ldots, n)$, $T_1 = 1$, then PA is called the prioritized average (PA) operator.

Example 1 (Paragraph 2 of Section 1). Consider four criteria: work attitude $C_1$, communication skill $C_2$, problem solving skill $C_3$, and learning skill $C_4$. The prioritization relationship for the criteria is $C_1 \succ C_2 \succ C_3 \succ C_4$. Suppose that the satisfaction of the four criteria are $C_1(x) = 0$, $C_2(x) = 1$, $C_3(x) = 1$, and $C_4(x) = 1$. Based on the PA operator, the calculation is performed as follows
\[
T_1 = 1, \quad T_2 = T_1 S_1 = 1 \times 0 = 0, \quad T_3 = T_2 S_2 = 0 \times 1 = 0, \quad \text{and} \quad T_4 = T_3 S_3 = 0 \times 1 = 0
\]
Therefore, the normalized weighting vectors are:
\[
\omega_1 = 1, \quad \omega_2 = 0, \quad \omega_3 = 0, \quad \text{and} \quad \omega_4 = 0
\]
The overall satisfaction of the four criteria can be obtained as:
\[
C(x) = 1 \times 0 + 0 \times 1 + 0 \times 1 + 0 \times 1 = 0
\]
Thus, the overall satisfaction of the sales engineer is zero, and the alternative (i.e., sales engineer) does not meet the stated needs of the company.

3. Generalized hesitant fuzzy prioritized aggregation operators

In this section, the generalized hesitant fuzzy prioritized weighted average (GHFPWA) operator and the generalized hesitant fuzzy prioritized weighted geometric (GHFPWG) operator are proposed to aggregate the HFEs. Some desirable properties and special cases of these operators are also studied in this section.

Definition 5. Let $f_j (j = 1, 2, \ldots, n)$ be a collection of HFEs, and let GHFPWA: $\mathbb{V}^n \rightarrow \mathbb{V}$. If
\[
\text{GHFPWA} (f_1, f_2, \ldots, f_n) = \frac{T_1 \oplus T_2 \oplus \cdots \oplus T_n}{\sum_{j=1}^{n} T_j} f_n
\]
then the function GHFPWA is called a GHFPWA operator, where $T_j = \prod_{k=1}^{j-1} S(f_k) (j = 2, \ldots, n)$, $T_1 = 1$ and $S(f_j)$ is the score of HFE $f_j$.

Theorem 1. Let $f_j (j = 1, 2, \ldots, n)$ be a collection of HFEs, then their aggregated value by using the GHFPWA operator is also an HFE, and
\[
\text{GHFPWA} (f_1, f_2, \ldots, f_n) = \bigcup_{i=1}^{n} \left\{ \bigoplus_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} I(\zeta_j) \right) \right\}
\]
where $T_j = \prod_{k=1}^{j-1} S(f_k) (j = 2, \ldots, n)$, $T_1 = 1$ and $S(x_j)$ is the score of HFE $f_j$.

Proof. The first result follows quickly from Definition 2 and Theorem 1. The equation
\[
\text{GHFPWA} (f_1, f_2, \ldots, f_n) = \bigcup_{i=1}^{n} \left\{ \bigoplus_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} I(\zeta_j) \right) \right\}
\]
can be proven by using mathematical induction on $n$:
(1) For $n = 2$, since
\[
\frac{T_1}{\sum_{j=1}^{n} T_j} f_1 = \bigcup_{i=1}^{n} \left\{ \bigoplus_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} I(\zeta_j) \right) \right\}
\]
then,
\[
\frac{T_1}{\sum_{j=1}^{n} T_j} f_1 = \bigcup_{i=1}^{n} \left\{ \bigoplus_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} I(\zeta_j) \right) \right\}
\]
That is, Eq. (5) holds when $n = 2$. Suppose that Eq. (5) also holds when $n = k$:
\[
\text{GHFPWA} (f_1, f_2, \ldots, f_k) = \bigcup_{i=1}^{n} \left\{ \bigoplus_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} I(\zeta_j) \right) \right\}
\]
then, when $n = k + 1$, the operational laws described in Definition 2 state that
\[
\text{GHFPWA} (f_1, f_2, \ldots, f_{k+1})
\]
GHFPWA \( (f_1, f_2, \ldots, f_n) = \bigcup_{\ell \subseteq \lambda} \left\{ I^{-1} \left( \sum_{j=1}^{n} T_j^{-1} J(\xi_j) \right) \right\} \). \hspace{1cm} (11)

It must be pointed out that the above proving method is based on Xu [29], Zhao et al. [40]. Now, consider some desirable properties of the GHFPWA operator. □

**Theorem 2.** Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, where \( T_j = \prod_{k=1}^{j-1} S(f_k) (j = 2, \ldots, n), T_1 = 1 \), and \( S(f_0) = \) the score of HFE \( f_0 \). If for all \( j, \xi_j = \zeta_j \), where \( \xi_j \) are elements of hesitant fuzzy set \( f_j \) and \( \zeta_j \) is an element of hesitant fuzzy set \( f \), then

GHFPWA \( (f_1, f_2, \ldots, f_n) = f. \hspace{1cm} (12) \)

**Proof.** By Theorem 1.

GHFPWA \( (f_1, f_2, \ldots, f_n) = \bigcup_{\ell \subseteq \lambda} \left\{ I^{-1} \left( \sum_{j=1}^{n} T_j^{-1} J(\xi_j) \right) \right\} \)

\[ = \bigcup_{\ell \subseteq \lambda} \left\{ I^{-1} \left( \sum_{j=1}^{n} T_j^{-1} J(\xi_j) \right) \right\} \]

\[ = \bigcup_{\ell \subseteq \lambda} \left\{ I^{-1} \left( \sum_{j=1}^{n} T_j^{-1} J(\xi_j) \right) \right\} \]

\[ = \bigcup_{\ell \subseteq \lambda} \left\{ I^{-1} (\ell(\xi_j)) \right\} = \bigcup_{\ell \subseteq \lambda} \{ \xi_j = f, \hspace{1cm} (13) \}

which completes the proof of Theorem 2. □

**Corollary 1.** If \( f_j (j = 1, 2, \ldots, n) \) is a collection of the largest HFEs, i.e., \( f_j = f^* = \{ 1 \} \) for all \( j \), then

GHFPWA \( (f_1, f_2, \ldots, f_n) = \) GHFPWA \( (f^*, f^*, \ldots, f^*) = \{ 1 \}, \hspace{1cm} (14) \)

which is also the largest HFE.

**Proof.** The proof for Corollary 1 may be obtained similar to the proof of Theorem 2. □

**Corollary 2** (Non-compensatory). If \( f_j \) is the smallest HFE, i.e., \( f_j = f = \{ 0 \} \), then

GHFPWA \( (f_1, f_2, \ldots, f_n) = \) GHFPWA \( (f, f, \ldots, f) = \{ 0 \}, \hspace{1cm} (15) \)

which is also the smallest HFE.

**Proof.** Since \( f_1 = \{ 0 \} \), then by the Definition 3,

\( S(f_1) = 0. \hspace{1cm} (16) \)

Since

\[ T_j = \prod_{k=1}^{j-1} S(f_k) (j = 2, \ldots, n) \text{ and } T_1 = 1, \hspace{1cm} (17) \]

then

\[ T_j = \prod_{k=1}^{j-1} S(f_k) = S(f_1) \times S(f_2) \times \ldots \times S(f_{j-1}) \]

\[ = 0 \times S(f_2) \times \ldots \times S(f_{j-1}) = 0 (j = 2, \ldots, n) \text{ and } \]

\[ \sum_{j=1}^{n} T_j = 1. \hspace{1cm} (19) \]

By Definition 5,

GHFPWA \( (f_1, f_2, \ldots, f_n) = \frac{T_1}{\sum_{j=1}^{n} T_j} f_1 \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} f_2 \oplus \cdots \]

\[ \oplus \frac{T_n}{\sum_{j=1}^{n} T_j} f_n \]

\[ = \frac{1}{\sum_{j=1}^{n} T_j} f_1 \oplus 0 \oplus \cdots \oplus 0 f_n = f_1 = \{ 0 \}. \hspace{1cm} (20) \]

Corollary 2 indicates that when the smallest HFE is associated with the highest priority criteria, the HFE for other criteria will be ignored. □

**Theorem 3.** Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, where \( T_j = \prod_{k=1}^{j-1} S(f_k) (j = 2, \ldots, n), T_1 = 1 \) and \( S(f_0) = \) the score of HFE \( f_0 \). If \( h \) is an HFE, \( \gamma = \) elements of hesitant fuzzy set \( h \) and \( \xi_j \) are elements of hesitant fuzzy set \( f_j \), then

GHFPWA \( (f_1, f_2, \ldots, f_n) \oplus h = \) GHFPWA \( (f_1, f_2, \ldots, f_n) \oplus h, \hspace{1cm} (21) \)

**Proof.** Since for any \( j, \)

\[ f_j \oplus h = \bigcup_{\ell \subseteq \lambda \cap \gamma} \{ I^{-1} (\ell(\xi_j) + l(\gamma)) \}, \hspace{1cm} (22) \]

then Theorem 1 states that

GHFPWA \( (f_1, f_2, \ldots, f_n) \oplus h = \bigcup_{\ell \subseteq \lambda \cap \gamma} \left\{ I^{-1} \left( \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} J(\xi_j) \right) \right\} \)

\[ = \bigcup_{\ell \subseteq \lambda \cap \gamma} \left\{ I^{-1} \left( \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} J(\xi_j) \right) \right\} \]

\[ = \bigcup_{\ell \subseteq \lambda \cap \gamma} \left\{ I^{-1} (l(\xi_j)) \right\} = \bigcup_{\ell \subseteq \lambda} \{ \xi_j = f, \hspace{1cm} (23) \}

which completes the proof of Theorem 3. □

**Theorem 4.** Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, where \( T_j = \prod_{k=1}^{j-1} S(f_k) (j = 2, \ldots, n), T_1 = 1 \) and \( S(f_0) = \) the score of HFE \( f_0 \). If \( r > 0 \), then

GHFPWA \( (rf_1, rf_2, \ldots, rf_n) = r \text{ GHFPWA}(f_1, f_2, \ldots, f_n). \hspace{1cm} (26) \)

**Proof.** According to Definition 2,

\[ x_r = \bigcup_{\ell \subseteq \lambda \cap \gamma} \left\{ I^{-1} (l(\xi_j)) \right\}, \hspace{1cm} \lambda > 0. \hspace{1cm} (27) \]

Theorem 1 states that

GHFPWA \( (rf_1, rf_2, \ldots, rf_n) = \bigcup_{\ell \subseteq \lambda \cap \gamma} \left\{ I^{-1} \left( \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} J(\xi_j) \right) \right\} \)

\[ = I^{-1} \left( \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} J(\xi_j) \right) \hspace{1cm} (28) \]
On the other hand,
\[
r \text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) = \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( T_j \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) \right) \right)^{1/r} \right\}
\]
\[
= \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) \right) \right) \right\}
\] (30)

Thus,
\[
\text{GHFPWA} \left( r f_1, r f_2, \ldots, r f_n \right) = r \text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right)
\]

\textbf{Theorem 5 follows from Theorems 3 and 4.} \qed

\textbf{Theorem 5.} Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFES, \( T_j = \prod_{k=1}^{n} S(f_k) (j = 2, \ldots, n), T_1 = 1, \) and \( S(f_k) \) is the score of HFE \( f_k. \) If \( r > 0 \) and \( h \) is an hesitant fuzzy element, then

\[
\text{GHFPWA} \left( r f_1 \oplus h, r f_2 \oplus h, \ldots, r f_n \oplus h \right)
\]
\[
= r \text{GHFPWA} \left( f_1, f_2, \ldots, f_n \oplus h \right)
\]

(31)

\textbf{Theorem 6.} Let \( f_j (j = 1, 2, \ldots, n) \) and \( h_j (j = 1, 2, \ldots, n) \) be two collections of HFES, \( T_j = \prod_{k=1}^{n} S(f_k) (j = 2, \ldots, n), T_1 = 1, \) and \( S(f_k) \) is the score of HFE \( f_k. \) then

\[
\text{GHFPWA} \left( f_1 \oplus h_1, f_2 \oplus h_2, \ldots, f_n \oplus h_n \right)
\]
\[
= \text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) \oplus \text{GHFPWA} \left( h_1, h_2, \ldots, h_n \right)
\]

(32)

\textbf{Proof.} According to \textbf{Definition 2},
\[
f_j \oplus h_j = \bigcup_{i, j} \left\{ l^{-1} \left( l(\tilde{z}_j) + l(y_j) \right) \right\}
\]

(33)

Combining the above equation with \textbf{Theorem 1},

\[
\text{GHFPWA} \left( f_1 \oplus h_1, f_2 \oplus h_2, \ldots, f_n \oplus h_n \right)
\]
\[
= \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) + \frac{T_j}{\sum_{j=1}^{n} T_j} l(y_j) \right) \right) \right\}
\]

(34)

On other hand,

\[
\text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) = \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) \right) \right) \right\}
\]

and

(35)

\[
\text{GHFPWA} \left( h_1, h_2, \ldots, h_n \right) = \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(y_j) \right) \right) \right\}
\]

(36)

which can be used to obtain,

\[
\text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) \oplus \text{GHFPWA} \left( h_1, h_2, \ldots, h_n \right)
\]
\[
= \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) \right) \right) \right\} \oplus \bigcup_{i, j} \left\{ 1 - \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(y_j) \right) \right) \right\}
\]

(37)

Thus,

\[
\text{GHFPWA} \left( f_1 \oplus h_1, f_2 \oplus h_2, \ldots, f_n \oplus h_n \right)
\]
\[
= \text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) \oplus \text{GHFPWA} \left( h_1, h_2, \ldots, h_n \right).
\]

(38)

It should be noted that Tan [19] studied serials of properties of generalized intuitionistic fuzzy geometric aggregation operator, which are the important reference material for the proving methods of \textbf{Theorems 2, 3, 4, 6}. Kliš [7] pointed out that an additive generator should satisfy \( k: [0, 1] \rightarrow [0, \infty) \) such that \( k(1) = 0. \) If the additive generator \( k \) is assigned with different forms, then some specific hesitant fuzzy aggregation operators can be obtained as follows:

\textbf{Case 1.} If \( k(t) = -\log(t) \), then the GHFPWA operator is reduced to the following:

\[
\text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) = \frac{T_1}{\sum_{j=1}^{n} T_j} \left[ 1 - \prod_{j=1}^{n} (1 - \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right]
\]

(39)

which is the hesitant fuzzy prioritized weighted agegor (HFPWA) operator studied by Yu [36] and Wei [23] in detail. \qed

\textbf{Proof.} Since \( k(t) = -\log(t) \), then \( l(t) = -\log(1 - t) \), \( k^{-1}(t) = e^{-t} \), and \( l^{-1}(t) = 1 - e^{-t}. \)

On the other hand,

\[
\frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) = -\frac{T_j}{\sum_{j=1}^{n} T_j} \log(1 - \xi_j)
\]

(40)

\[
\Rightarrow \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) \right) = -\log \prod_{j=1}^{n} (1 - \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j}
\]

(41)

\[
\Rightarrow l^{-1} \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} l(\tilde{z}_j) \right) \right) = 1 - e^{\log \prod_{j=1}^{n} (1 - \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j}} = 1 - \frac{n}{\prod_{j=1}^{n} (1 - \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j}},
\]

(42)

which completes the proof for Case 1. \qed

Furthermore, when the priority level of the aggregated arguments is reduced to the same level, then Eq. (39) is rewritten as

\[
\text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) = \bigcup_{i, j} \left\{ 1 - \prod_{j=1}^{n} (1 - \xi_j) \right\}
\]

(43)

which is the hesitant fuzzy weighted average operator proposed by Xia and Xu [25,26].

\textbf{Case 2.} If \( k(t) = \log \left( \frac{1}{1-t} \right) \), then the GHFPWA operator is reduced to the following:

\[
\text{GHFPWA} \left( f_1, f_2, \ldots, f_n \right) = \bigcup_{i, j} \left\{ \prod_{j=1}^{n} (1 + \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} - \prod_{j=1}^{n} (1 - \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right\}
\]

(44)

\[
\bigcup_{i, j} \left\{ \prod_{j=1}^{n} (1 + \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} + \prod_{j=1}^{n} (1 - \xi_j) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right\}
\]

which is the hesitant fuzzy Einstein prioritized weighted average operator.

\textbf{Proof.} Since \( k(t) = \log \left( \frac{1}{1-t} \right) \), then \( l(t) = \log \left( \frac{1}{1-t} \right) \), \( k^{-1}(t) = \frac{1}{1-e^{-t}} \), and \( l^{-1}(t) = \frac{1}{e^t-1} \), which can be used to show that
\[ \frac{T_j}{\sum_{j=1}^{n} T_j} \log \left( \frac{1 + \xi_j}{1 - \xi_j} \right) = \log \left( \frac{1 + \xi_j}{1 - \xi_j} \right) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j}, \] (45)

\[ \Rightarrow \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} \log \left( \frac{1 + \xi_j}{1 - \xi_j} \right) \right) = \sum_{j=1}^{n} \left( \log \left( \frac{1 + \xi_j}{1 - \xi_j} \right) \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right) \]

\[ = \log \prod_{j=1}^{n} \left( \frac{1 + \xi_j}{1 - \xi_j} \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right), \] (46)

\[ \Rightarrow \Gamma^{-1} \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} \log \left( \frac{1 + \xi_j}{1 - \xi_j} \right) \right) \right) \]

\[ = e^{\log \prod_{j=1}^{n} \left( \frac{1 + \xi_j}{1 - \xi_j} \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right) - 1} = \prod_{j=1}^{n} \left( \frac{1 + \xi_j}{1 - \xi_j} \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} - 1 \right) \]

\[ = \prod_{j=1}^{n} \left( 1 + \xi_j \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right) - \prod_{j=1}^{n} \left( 1 - \xi_j \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right). \] (47)

Here we define the generalized hesitant fuzzy prioritized weighted geometric (GHFPWG) operator based on the GHFPWA operator and the geometric mean. □

**Definition 6.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs and GHFPWG: \( V^n \rightarrow V \).

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = f_1^{T_1} \otimes f_2^{T_2} \otimes \cdots \otimes f_n^{T_n},
\] (48)

then the function GHFPWG is called a GHFPWG operator, where \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \).

**Theorem 7.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs, then their aggregated value obtained with the HFPWG operator is also an HFE, and

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{j=1}^{n} \left\{ k^{-1} \left( \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} k(\xi_j) \right) \right\},
\] (49)

where \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \).

**Proof.** Theorem 7 can be proven similar to Theorem 1. □

**Theorem 8.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs, where \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \). If for all \( j, \xi_j = \xi \), where \( \xi_j \) are elements of hesitant fuzzy set \( f_j \), and \( \xi \) is the element of hesitant fuzzy set \( f \), then

\[ \text{GHFPWG}(f_1, f_2, \ldots, f_n) = f. \] (50)

**Theorem 9.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs, where \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \). If \( h \) is an HFE, \( \gamma \) are elements of hesitant fuzzy set \( h \), and \( \xi_j \) are elements of hesitant fuzzy set \( f_j \), then

\[ \text{GHFPWG}(f_1 \otimes h, f_2 \otimes h, \ldots, f_n \otimes h) = \text{GHFPWG}(f_1, f_2, \ldots, f_n) \otimes h. \] (51)

**Theorem 10.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs, \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \). If \( r > 0 \), then

\[ \text{GHFPWG}(f_1^r, f_2^r, \ldots, f_n^r) = \text{GHFPWA}(f_1, f_2, \ldots, f_n)^r. \] (52)

**Theorem 11.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs, \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \). If \( h \) is an HFE, then

\[ \text{GHFPWG}(f_1 \otimes h, f_2 \otimes h, \ldots, f_n \otimes h) = r \text{GHFPWG}(f_1, f_2, \ldots, f_n) \otimes h. \] (53)

**Theorem 12.** Let \( f(j = 1, 2, \ldots, n) \) and \( h(j = 1, 2, \ldots, n) \) be two collections of HFEs, \( T_j = \prod_{n=1}^{j} S(f_n)(j = 2, \ldots, n) \), \( T_1 = 1 \), and \( S(f_n) \) is the score of HFE \( f_n \). Then

\[ \text{GHFPWG}(f_1 \otimes h, f_2 \otimes h, \ldots, f_n \otimes h_n) \]

\[ = \text{GHFPWG}(f_1, f_2, \ldots, f_n) \otimes \text{GHFPWG}(h_1, h_2, \ldots, h_n). \] (54)

**Case 1.** If \( k(t) = -\log(t) \), then the GHFPWG operator is reduced to the following:

\[ \text{GHFPWG}(f_1, f_2, \ldots, f_n) = f_1^{T_1} \otimes f_2^{T_2} \otimes \cdots \otimes f_n^{T_n}, \]

which is the hesitant fuzzy prioritized weighted geometric (HFPWG) operator studied by Wei [23], Yang [36] in detail.

**Proof.** Since \( k(t) = -\log(t) \), then \( l(t) = -\log(1 - t), k^{-1}(t) = e^{-t}, f^{-1}(-t) = 1 - e^{-t} \), and

\[ \frac{T_j}{\sum_{j=1}^{n} T_j} k(\xi_j) = -\frac{T_j}{\sum_{j=1}^{n} T_j} \log(\xi_j) \] (55)

\[ = \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} \right) \log(\xi_j) \]

\[ = -\log \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right), \] (56)

\[ \Rightarrow k^{-1} \left( \sum_{j=1}^{n} \left( \frac{T_j}{\sum_{j=1}^{n} T_j} \right) k(\xi_j) \right) = k^{-1} \left( -\log \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right) \right) \]

\[ \log \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \right) = \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j}, \] (57)

which completes the proof of the Case 1. □

Furthermore, when the priority level of the aggregated arguments is reduced to the same level, the Eq. (39) is transformed to

\[ \text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{j=1}^{n} \left\{ \prod_{j=1}^{n} \xi_j \right\}, \] (58)

which was the hesitant fuzzy weighted geometric operator proposed by Xia and Xu [25,26].
Case 2. If \( k(t) = \log \left( \frac{1}{2} \right) \), then the GHFPWG operator is reduced to the following:

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{i \in D} \left\{ 1 - k \left( \sum_{j=1}^{n} \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k^{-1}(\xi_j) \right) \right\}
\]

which we call the hesitant fuzzy Einstein prioritized weighted geometric operator.

**Proof.** Since \( k(t) = \log \left( \frac{1}{2} \right) \), then \( k(t) = \log \left( \frac{1}{2} \right) \), \( k^{-1}(t) = 2t \), and \( I^{-1}(t) = \frac{1}{2t} \), then

\[
\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k(\xi_j) = \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} \log \left( 2 - \frac{1}{2} \xi_j \right) = \log \left( 2 - \frac{1}{2} \xi_j \right) \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}
\]

\[
= \log \prod_{j=1}^{n} \left( 2 - \frac{1}{2} \xi_j \right) \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}
\]

\[
= k^{-1} \left( \sum_{j=1}^{n} \left( \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k(\xi_j) \right) \right) = \frac{2}{e^{\log \prod_{j=1}^{n} \left( \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} \right) + 1}}
\]

\[
= \frac{2}{\prod_{j=1}^{n} \left( \frac{1}{2} - \xi_j \right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + 1} \prod_{j=1}^{n} \left( 2 - \xi_j \right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} \prod_{j=1}^{n} \xi_j^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}
\]

which completes the proof of Case 2. \( \Box \)

Furthermore, when the priority of the aggregated arguments is reduced to the same level, then the Eq. (59) is transformed to

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{i \in D} \left\{ 1 - k \left( \sum_{j=1}^{n} \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k^{-1}(\xi_j) \right) \right\}
\]

which is referred to as the hesitant fuzzy Einstein weighted geometric operator.

In the following, we study the relationship between GHFPWA and GHFPWG operator.

**Theorem 13.** Let \( f(j = 1, 2, \ldots, n) \) be a collection of HFEs, \( T_j = \prod_{i=1}^{n} S(f_i)(j = 2, \ldots, n) \), \( T_1 = 1 \) and \( S(f_0) \) be the score of HFE \( f_0 \) then

(1) GHFPWA \((f_1, f_2, \ldots, f_n) = (\text{GHFPWG}(f_1, f_2, \ldots, f_n))^c\) and

(2) \( \text{GHFPWG}(f_1, f_2, \ldots, f_n) = (\text{GHFPWA}(f_1, f_2, \ldots, f_n))^c\).

**Proof.**

(1) According to **Definition 2** and the Definition of a GHFPWA operator,

\[
\text{GHFPWA}(f_1, f_2, \ldots, f_n) = \bigcup_{i \in D} \left\{ I^{-1} \left( \sum_{j=1}^{n} \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k^{-1}(\xi_j) \right) \right\}
\]

and

GHFPWA \((f_1, f_2, \ldots, f_n) = \bigcup_{i \in D} \left\{ 1 - k \left( \sum_{j=1}^{n} \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k^{-1}(\xi_j) \right) \right\}
\]

which is provided by each of the decision criterion \( C_1, C_2, \ldots, C_m \) under criteria \( C_1 \) evaluates the alternatives by the GHFPWA (or GHFPWG) operator.

(2) According to **Definition 2** and the definition of a GHFPWG operator,

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{i \in D} \left\{ 1 - I \left( \sum_{j=1}^{n} \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}} k^{-1}(\xi_j) \right) \right\}
\]

which completes the proof of **Theorem 13.** \( \Box \)

4. Personnel evaluation based on prioritized operators under hesitant fuzzy environment

This section introduces the methods to rank alternatives based on hesitant fuzzy information. In many group decision problems, a set of alternatives must be evaluated on the basis of criteria with prioritized relationships.

Consider a group decision making problem \([5,28,31,16,22,18,39,9,10,12,14,17,30,32,41]\). Let \( X = \{x_1, x_2, \ldots, x_m\} \) be the set of alternatives, \( C = \{C_1, C_2, \ldots, C_n\} \) be the set of criteria, and \( E = \{e_1, -e_2, \ldots, e_p\} \) be the set of decision makers. It is assumed that there is a prioritization between the criteria expressed by the linear ordering \( C_1 > C_2 > C_3 > \cdots > C_n \) indicating criterion \( C_j \) has a higher priority than \( C_i \) if \( j < i \). The decision maker \( e_k \) evaluates the alternative \( x_i \) under criterion \( C_j \) anonymously so as to protect his/her privacy or avoid psychic contagion. The evaluation of alternative \( x_i \) under criterion \( C_j \) is provided by each of the decision makers \( e_k \) \((k = 1, 2, \ldots, p)\) using several values. If two decision makers provide the same value, then that value will emerge only once, and the evaluation can be represented by HFEs. The hesitant fuzzy group decision matrix \( F = (f_{ij})_{m \times n} \) is constructed from these HFEs. Here, \( f_{ij} \) is the attribute value provided by the decision makers \( e_k \) \((k = 1, 2, \ldots, p)\), which is expressed in a HFE.

Based on the above analysis, the main steps of the multi-criteria group decision making method are as follows:

**Step 1.** Calculate the values of \( T_{ij} \) \((i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)\) based on the following equations.

\[
T_{ij} = \prod_{k=1}^{n} S(r_{ik}) \quad (i = 1, 2, \ldots, m, j = 2, \ldots, n)
\]

\[
T_{1j} = 1 \quad i = 1, 2, \ldots, m
\]

**Step 2.** Aggregate the hesitant fuzzy values \( f_{ij} \) for each alternative \( x_i \) by the GHFPWA (or GHFPWG) operator.
GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ f^{-1} \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} (i|j) \right) \right\}$

\[ i = 1, 2, \ldots, m \quad (70) \]

or

GHFPWG($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ k^{-1} \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} k(i|j) \right) \right\}$

\[ i = 1, 2, \ldots, m \quad (71) \]

**Step 3.** Rank all the alternatives by the score function defined by Definition 3.

\[ S(f_i) = \frac{1}{\#F} \sum_{x \in F} x_i, \quad i = 1, 2, \ldots, m \quad (72) \]

Using the above process, the bigger the value of $S(f_i)$, the larger is the overall HFE $f_i$. So is the alternative $x_i$ ($i = 1, 2, \ldots, m$).

**Example 2 (Paragraph 2 of Section 1).** Section 1 introduced a personnel evaluation problem that included a prioritization relationship between multiple criteria.

To draw a scientific conclusion that is free of psychic contagion, this evaluation will be made anonymously. If two decision makers provide the same value, then the value will emerge only once, and the evaluation can be represented by HFEs. The hesitant fuzzy group decision matrix $F = (f_{ij})_{m \times n}$ is constructed as follows (Table 2).

**Approach 1:** The ranking of alternatives can be obtained as follows using the GHFPWA operator and letting $k(t) = -\log(t)$:

**Step 1.** Calculate the values of $T_{ij}(i = 1, 2, \ldots, 5; j = 1, 2, \ldots, 4)$ based on Eqs. (68) and (69).

\[ T_{ij} = \begin{pmatrix}
1.0000 & 0.6500 & 0.2763 & 0.1842 \\
1.0000 & 0.4333 & 0.5033 & 0.2224 \\
1.0000 & 0.6000 & 0.3200 & 0.2240 \\
1.0000 & 0.4667 & 0.1400 & 0.0840 \\
1.0000 & 0.3333 & 0.1333 & 0.0533
\end{pmatrix} \]

**Step 2.** Since $k(t) = -\log(t)$, then

GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ f^{-1} \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} (i|j) \right) \right\}$

which is reduced to the

GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ 1 - \prod_{j=1}^{n} (1 - \frac{T_j}{\sum_{j=1}^{n} T_j}) \right\}$

\[ \quad (73) \]

**Step 3.** Calculate the scores of $f_i(i = 1, 2, 3, 4, 5)$.

\[ S_1 = 0.6040, \quad S_2 = 0.5923, \quad S_3 = 0.6360, \quad S_4 = 0.4797, \quad S_5 = 0.3671 \]

Using the above process, the bigger the value of $S(f_i)$, the larger is the overall HFE $f_i$. So is the alternative $x_i$ ($i = 1, 2, \ldots, m$).

**Example 2 (Paragraph 2 of Section 1).** Section 1 introduced a personnel evaluation problem that included a prioritization relationship between multiple criteria.

To draw a scientific conclusion that is free of psychic contagion, this evaluation will be made anonymously. If two decision makers provide the same value, then the value will emerge only once, and the evaluation can be represented by HFEs. The hesitant fuzzy group decision matrix $F = (f_{ij})_{m \times n}$ is constructed as follows (Table 2).

**Approach 1:** The ranking of alternatives can be obtained as follows using the GHFPWA operator and letting $k(t) = -\log(t)$:

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1.0000 & 0.6000 & 0.3200 & 0.2240 \\
1.0000 & 0.4667 & 0.1400 & 0.0840 \\
1.0000 & 0.3333 & 0.1333 & 0.0533
\end{pmatrix} \]

**Step 2.** Since $k(t) = -\log(t)$, then

GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ f^{-1} \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} (i|j) \right) \right\}$

which is reduced to the

GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ 1 - \prod_{j=1}^{n} (1 - \frac{T_j}{\sum_{j=1}^{n} T_j}) \right\}$

\[ \quad (73) \]

**Step 3.** Calculate the scores of $f_i(i = 1, 2, 3, 4, 5)$.

\[ S_1 = 0.5408, \quad S_2 = 0.5204, \quad S_3 = 0.5666, \quad S_4 = 0.4315, \quad S_5 = 0.3391 \]

Since

\[ S_1 > S_2 > S_4 > S_5 \]

the following results are obtained

\[ x_1 > x_2 > x_4 > x_5 \]

Clearly, the best option is alternative $x_2$.

**Approach 2:** The ranking of alternatives can be obtained as follows using the GHFPWA operator and letting $k(t) = \log(\frac{t+1}{2})$:

**Step 1’.** Same as the above Step 1.

**Step 2’.** Since $k(t) = \log(\frac{t+1}{2})$, then

GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ f^{-1} \left( \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} (i|j) \right) \right\}$

which is reduced to the

GHFPWA($f_1,f_2,\ldots,f_n$) = $\bigcup_{r\in R'} \left\{ 1 - \prod_{j=1}^{n} (1 - \frac{T_j}{\sum_{j=1}^{n} T_j}) \right\}$

\[ \quad (73) \]

**Step 3.** Calculate the scores of $f_i(i = 1, 2, 3, 4, 5)$.

\[ S_1 = 0.5408, \quad S_2 = 0.5204, \quad S_3 = 0.5666, \quad S_4 = 0.4315, \quad S_5 = 0.3391 \]

Since

\[ S_1 > S_2 > S_4 > S_5 \]

The final ranking can be obtained as

\[ x_1 > x_2 > x_4 > x_5 \]
The best option with this approach is also alternative $x_3$.

**Approach 3**: The ranking of alternatives can be obtained as follows, by using the GHFPWG operator and letting $k(t) = -\log(t)$:

**Step 1**. See Step 1.

**Step 2**. Since $k(t) = -\log(t)$, then

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{t \in F} \left\{ k^{-1} \left( \frac{1}{n} \sum_{j=1}^{n} t_i \text{GHFPWG}(\xi_j) \right) \right\}
\]

(77)

Which is reduced to

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{t \in F} \left\{ \frac{1}{n} \sum_{j=1}^{n} t_i \text{GHFPWG}(\xi_j) \right\}
\]

(78)

The overall preference values $f_i$ are obtained using the GHFPWG operator in Eq. (78) to aggregate the preference values $f_{ij}(i = 1, 2, 3, 4, 5)$ in the $i$th line of $F$. Due to the space limit, $f_1$ is provided as a representative example.

\[
f_1 = \{0.4111, 0.4233, 0.4210, 0.4336, 0.4440, 0.4572, 0.5089, 0.5241, 0.5212, 0.5367, 0.5496, 0.5660, 0.5451, 0.5614, 0.5583, 0.5749, 0.5887, 0.6063, 0.5766, 0.5938, 0.5905, 0.6081, 0.6227, 0.6413, 0.4422, 0.4554, 0.4529, 0.4664, 0.4776, 0.4918, 0.5475, 0.5638, 0.5607, 0.5774, 0.5913, 0.6089, 0.5864, 0.6039, 0.6006, 0.6185, 0.6333, 0.6522, 0.6203, 0.6388, 0.6353, 0.6542, 0.6699, 0.6899\}
\]

**Step 3**. Calculate the scores of $f_i (i = 1, 2, 3, 4, 5)$ respectively.

\[
S_1 = 0.5563, \quad S_2 = 0.5013, \quad S_3 = 0.5834, \quad S_4 = 0.4251, \quad S_5 = 0.3485
\]

Since

\[
S_3 > S_1 > S_2 > S_4 > S_5,
\]

the final ranking is obtained as

\[
x_3 > x_1 > x_2 > x_4 > x_5.
\]

Which again indicates the best option is alternative $x_3$.

**Approach 4**: The ranking of alternatives can be obtained as follows using the GHFPWG operator and letting $k(t) = \log \left( \frac{t}{2t-1} \right)$:

**Step 1**. Same as the above Step 1.

**Step 2**. Since $k(t) = \log \left( \frac{t}{2t-1} \right)$, then

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{t \in F} \left\{ k^{-1} \left( \frac{1}{n} \sum_{j=1}^{n} t_i \text{GHFPWG}(\xi_j) \right) \right\}
\]

(79)

which is reduced to

\[
\text{GHFPWG}(f_1, f_2, \ldots, f_n) = \bigcup_{t \in F} \left\{ \frac{1}{n} \sum_{j=1}^{n} t_i \text{GHFPWG}(\xi_j) \right\}
\]

(80)

The overall preference values $f_i$ are obtained using the GHFPWG operator in Eq. (80) to aggregate the preference values $f_{ij}(i = 1, 2, 3, 4, 5)$ in the $i$th line of $F$. Due to the space limit, $f_1$ is provided as a representative example.

\[
f_1 = \{0.4776, 0.4881, 0.4861, 0.4967, 0.5078, 0.5190, 0.5597, 0.5719, 0.5695, 0.5820, 0.5950, 0.6081, 0.5899, 0.6029, 0.6003, 0.6135, 0.6272, 0.6410, 0.6167, 0.6303, 0.6276, 0.6414, 0.6557, 0.6701, 0.5089, 0.5201, 0.5179, 0.5292, 0.5410, 0.5529, 0.5963, 0.6094, 0.6068, 0.6201, 0.6339, 0.6479, 0.6285, 0.6423, 0.6396, 0.6537, 0.6682, 0.6829, 0.6571, 0.6715, 0.6687, 0.6834, 0.6986, 0.7139\}
\]

**Step 3**. Calculate the scores of $f_i (i = 1, 2, 3, 4, 5)$.

\[
S_1 = 0.6015, \quad S_2 = 0.5537, \quad S_3 = 0.6232, \quad S_4 = 0.4763, \quad S_5 = 0.3938
\]

Since

\[
S_3 > S_1 > S_2 > S_4 > S_5,
\]

we can easily get

\[
x_3 > x_1 > x_2 > x_4 > x_5.
\]

The best option is therefore alternative $x_3$.

The above examples show that the same conclusion can be obtained with all four different approaches. This consistency demonstrates the stability of the proposed method.

5. Conclusions

In this age of increasingly competitive markets, talent has become essential for business success. Talent can be a critical factor in a company’s ability to achieve operational excellence. To address the disadvantages of traditional personnel evaluation methods, this paper proposes the use of a hesitant fuzzy group decision making method. The method is based on the GHFPWA and GHFPWG operators. The use of hesitant fuzzy sets is a powerful technique that captures the decision makers’ preferences. The ability to prioritize criteria not only addresses the actual needs of companies from a human resources perspective, but also opens the approach up to a wide range of applications. A realistic example is used to illustrate the effectiveness of the proposed method for personnel evaluation. It is worth noting that in addition to human resource issues, the proposed method is equally applicable to factory location, supplier selection and many other management decision problems.

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References


