Geometric phase and chiral anomaly; their basic differences[1]

Kazuo Fujikawa

Institute of Quantum Science, College of Science and Technology
Nihon University, Chiyoda-ku, Tokyo 101-8308, Japan

Abstract

All the geometric phases are shown to be topologically trivial by using the second quantized formulation. The exact hidden local symmetry in the Schrödinger equation, which was hitherto unrecognized, controls the holonomy associated with both of the adiabatic and non-adiabatic geometric phases. The second quantized formulation is located in between the first quantized formulation and the field theory, and thus it is convenient to compare the geometric phase with the chiral anomaly in field theory. It is shown that these two notions are completely different.

1 Introduction

Phases are intriguing notions, as was emphasized by C.N. Yang on various occasions. Here we discuss two phases, and the first phase is the geometric phase in quantum mechanics [1, 2, 3, 4, 5, 6, 7, 8, 9] for which we present the recent developments on the basis of the second quantized formulation of all the geometric phases [10, 11, 12, 13]. The second phase is the chiral anomaly in field theory [14, 15, 16, 17], which is by now well understood [18]. The second quantized formulation is located in between the first quantized formulation and the field theory, and thus it is convenient to compare the geometric phase with the chiral anomaly in field theory [19, 20, 21].

We then show

1. A unified treatment of adiabatic and non-adiabatic geometric phases is possible in the second quantized formulation by using the exact hidden local (*i.e.*,time-dependent) symmetry in the Schrödinger equation.

2. The topology of all the geometric phases is trivial by using an exactly

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solvable example.
3. Geometric phases in the Schrödinger problem and the chiral anomaly in field theory are completely different.

2 Second quantized formulation

We start with defining an *arbitrary* complete basis set

$$\int d^3x v_n^\star(t,\vec{x}) v_m(t,\vec{x}) = \delta_{nm} \tag{2.1}$$

and expand the field operator $\hat{\psi}(t,\vec{x})$ as

$$\hat{\psi}(t,\vec{x}) = \sum_n \hat{b}_n(t) v(t,\vec{x}). \tag{2.2}$$

The action

$$S = \int_0^T dt d^3x [\hat{\psi}^\star(t,\vec{x}) i\hbar \frac{\partial}{\partial t} \hat{\psi}(t,\vec{x}) - \hat{\psi}^\star(t,\vec{x}) \hat{H}(t) \hat{\psi}(t,\vec{x})] \tag{2.3}$$

which gives rise to the field equation

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(t,\vec{x}) = \hat{H}(t) \hat{\psi}(t,\vec{x}) \tag{2.4}$$

then becomes

$$S = \int_0^T dt \{ \sum_n \hat{b}_n^\dagger(t) i\hbar \partial_t \hat{b}_n(t) - \hat{H}_{eff} \}. \tag{2.5}$$

The effective Hamiltonian is given by

$$\begin{aligned} \hat{H}_{eff}(t) &= \sum_{n,m} \hat{b}_n^\dagger(t) [\int d^3x v_n^\star(t,\vec{x}) \hat{H}(t) v_m(t,\vec{x}) \\ &\quad - \int d^3x v_n^\star(t,\vec{x}) i\hbar \frac{\partial}{\partial t} v_m(t,\vec{x})] \hat{b}_m(t) \end{aligned} \tag{2.6}$$

and the canonical commutation relations $[\hat{b}_n(t), \hat{b}_m^\dagger(t)]_\mp = \delta_{n,m}$, but statistics (fermions or bosons) is not important in our application.

The Schrödinger picture $\hat{\mathcal{H}}_{eff}(t)$ is obtained by replacing $\hat{b}_n(t)$ with $\hat{b}_n(0)$ in $\hat{H}_{eff}(t)$ (2.6). Then the evolution operator is given by[11]

$$\begin{aligned}&\langle m|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{\mathcal{H}}_{eff}(t)dt\}|n\rangle \\ &= \langle m(t)|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{H}(\hat{\vec{p}},\hat{\vec{x}},X(t))dt\}|n(0)\rangle \end{aligned} \tag{2.7}$$

with time ordering symbol $T^\star$. In the second quantized formulation on the left-hand side we have $|n\rangle = \hat{b}_n^\dagger(0)|0\rangle$, and in the first quantized formulation on the right-hand side we have $\langle \vec{x}|n(t)\rangle = v_n(t,\vec{x})$.

The exact Schrödinger probability amplitude which satisfies $i\hbar\partial_t\psi_n(t,\vec{x}) = \hat{H}(t)\psi_n(t,\vec{x})$ with $\psi_n(0,\vec{x}) = v_n(0,\vec{x})$ is given by

$$\begin{aligned}\psi_n(t,\vec{x}) &= \langle 0|\hat{\psi}(t,\vec{x})\hat{b}_n^\dagger(0)|0\rangle \\ &= \sum_m v_m(t,\vec{x})\langle 0|\hat{b}_m(t)\hat{b}_n^\dagger(0)|0\rangle \\ &= \sum_m v_m(t,\vec{x})\langle m|T^\star \exp\{-\frac{i}{\hbar}\int_0^t \hat{\mathcal{H}}_{eff}(t)dt\}|n\rangle \end{aligned} \tag{2.8}$$

which is confirmed by using the relation $i\hbar\partial_t\hat{\psi}(t,\vec{x}) = \hat{H}(t)\hat{\psi}(t,\vec{x})$ in (2.4). We note that the general geometric terms automatically appear as the second terms in the *exact* $\hat{\mathcal{H}}_{eff}(t)$ in (2.8). See $\hat{H}_{eff}(t)$ in (2.6).

2.1 Hidden local symmetry

Since the basic field variable is written as $\hat{\psi}(t,\vec{x}) = \sum_n \hat{b}_n(t)v_n(t,\vec{x})$, we have an exact *hidden* local (i.e., time dependent) symmetry [11]

$$\begin{aligned}&v_n(t,\vec{x}) \to v_n'(t,\vec{x}) = e^{i\alpha_n(t)}v_n(t,\vec{x}), \\ &\hat{b}_n(t) \to \hat{b}_n'(t) = e^{-i\alpha_n(t)}\hat{b}_n(t), \qquad n = 1,2,3,... \end{aligned} \tag{2.9}$$

which keeps $\hat{\psi}(t,\vec{x})$ invariant. This symmetry means arbitrariness in the choice of the coordinates in the functional space. The Schrödinger amplitude $\psi_n(t,\vec{x}) = \langle 0|\hat{\psi}(t,\vec{x})\hat{b}_n^\dagger(0)|0\rangle$ is then transformed as

$$\psi_n'(t,\vec{x}) = e^{i\alpha_n(0)}\psi_n(t,\vec{x}) \tag{2.10}$$

under the hidden symmetry for any t. Namely, it gives the ray representation with a constant phase. We thus have the enormous hidden local symmetry behind the ray representation, which was not recognized in the past. The product $\psi_n(0,\vec{x})^\star\psi_n(T,\vec{x})$ is then manifestly gauge invariant for a periodic system.

If one chooses a specific basis

$$\hat{H}(X(t))v(\vec{x};X(t)) = \mathcal{E}_n(X(t))v(\vec{x};X(t)) \tag{2.11}$$

in (2.1) for a periodic Hamiltonian $\hat{H}(X(0)) = \hat{H}(X(T))$ and assumes "diagonal dominance" in the effective Hamiltonian, we have from (2.8)

$$\psi_n(t,\vec{x}) \simeq v_n(\vec{x};X(t))\exp\{-\frac{i}{\hbar}\int_0^t [\mathcal{E}_n(X(t)) - \langle n|i\hbar\frac{\partial}{\partial t}|n\rangle]dt\} \tag{2.12}$$

which reproduces the result of the conventional adiabatic approximation.

This shows that

Adiabatic approximation = Approximate diagonalization of H_{eff}

and thus the geometric phases are *dynamical*, i.e., a part of the Hamiltonian. In fact, it has been recently shown that the second quantized formulation nicely resolves some of the subtle problems in the conventional adiabatic approximation [22].

In the adiabatic approximation (2.12), we have a gauge invariant quantity (for a general choice of the hidden local symmetry)

$$\begin{aligned}\psi_n(0,\vec{x})^\star\psi_n(T,\vec{x}) &= v_n(0,\vec{x};X(0))^\star v_n(T,\vec{x};X(T)) \\ &\quad\times\exp\{-\frac{i}{\hbar}\int_0^T [\mathcal{E}_n(X(t)) - \langle n|i\hbar\frac{\partial}{\partial t}|n\rangle]dt\}. \end{aligned}\tag{2.13}$$

If one chooses a specific hidden local gauge such that $v_n(T,\vec{x};X(T)) = v_n(0,\vec{x};X(0))$, the pre-factor $v_n(0,\vec{x};X(0))^\star v_n(T,\vec{x};X(T))$ becomes real and positive and thus the factor on the exponential in (2.13) represents the entire gauge invariant phase. This unique gauge invariant quantity reproduces the conventional adiabatic phase[3, 4].

2.2 Parallel transport and holonomy

The parallel transport of $v_n(t,\vec{x})$ is defined by

$$\int d^3x v_n^\dagger(t,\vec{x})\frac{\partial}{\partial t}v_n(t,\vec{x}) = 0 \tag{2.14}$$

which is derived from the conditions

$$\int d^3x v_n^\dagger(t,\vec{x}) v_n(t+\delta t,\vec{x}) = \text{real and positive} \tag{2.15}$$

and

$$\int d^3x v_n^\dagger(t+\delta t,\vec{x}) v_n(t+\delta t,\vec{x}) = \int d^3x v_n^\dagger(t,\vec{x}) v_n(t,\vec{x}). \tag{2.16}$$

By using the hidden local gauge $\bar{v}_n(t,\vec{x}) = e^{i\alpha_n(t)} v_n(t,\vec{x})$ for a general $v_n(t,\vec{x})$, which may not satisfy the condition (2.14), the parallel transport condition

$$\int d^3x \bar{v}_n^\dagger(t,\vec{x}) \frac{\partial}{\partial t} \bar{v}_n(t,\vec{x}) = 0 \tag{2.17}$$

gives

$$\bar{v}_n(t,\vec{x}) = \exp[i \int_0^t dt' \int d^3x v_n^\dagger(t',\vec{x}) i\partial_{t'} v_n(t',\vec{x})] v_n(t,\vec{x}). \tag{2.18}$$

Since $\bar{v}_n(t,\vec{x})$ satisfies the parallel transport condition, the **holonomy**, i.e., the phase change after one cycle, is given by [13]

$$\begin{aligned} &\bar{v}_n^\dagger(0,\vec{x})\bar{v}_n(T,\vec{x}) \\ &= v_n^\dagger(0,\vec{x}) v_n(T,\vec{x}) \exp[i \int_0^T dt' \int d^3x v_n^\dagger(t',\vec{x}) i\partial_{t'} v_n(t',\vec{x})]. \end{aligned} \tag{2.19}$$

This holonomy of *basis vectors*, not of the Schrödinger amplitude, associated with the hidden local symmetry determines *all* the geometric phases. In fact, the adiabatic phase in (2.13) is an example.

2.3 Non-adiabatic phase: Cyclic evolution

The cyclic evolution is defined by [6]

$$\begin{aligned} &\int d^3x \psi^\dagger(t,\vec{x})\psi(t,\vec{x}) = 1, \\ &\psi(t,\vec{x}) = e^{i\phi(t)}\tilde{\psi}(t,\vec{x}), \quad \tilde{\psi}(T,\vec{x}) = \tilde{\psi}(0,\vec{x}). \end{aligned} \tag{2.20}$$

namely, $\psi(T,\vec{x}) = e^{i\phi}\psi(0,\vec{x})$ with $\phi(T) = \phi, \ \ \phi(0) = 0$.

If one chooses the first element of the arbitrary basis set $\{v_n(t,\vec{x})\}$ in (2.1) such that $v_1(t,\vec{x}) = \tilde{\psi}(t,\vec{x})$, one can confirm that the exact Schrödinger amplitude (2.8) is written as

$$\begin{aligned}\psi(t,\vec{x}) &= v_1(t,\vec{x})\exp\{-\frac{i}{\hbar}[\int_0^t dt\int d^3x v_1^\star(t,\vec{x})\hat{H}v_1(t,\vec{x}) \\ &\quad -\int_0^t dt\int d^3x v_1^\star(t,\vec{x})i\hbar\partial_t v_1(t,\vec{x})]\}. \qquad (2.21)\end{aligned}$$

Under the hidden local symmetry of basis vectors, we have

$$\psi(t,\vec{x}) \to e^{i\alpha_1(0)}\psi(t,\vec{x}) \qquad (2.22)$$

and the gauge invariant quantity is given by

$$\begin{aligned}&\psi^\dagger(0,\vec{x})\psi(T,\vec{x}) \\ &= v_1^\star(0,\vec{x})v_1(T,\vec{x})\exp\{-\frac{i}{\hbar}\int_0^T dt\int d^3x[v_1^\star(t,\vec{x})\hat{H}v_1(t,\vec{x}) \\ &\qquad -v_1^\star(t,\vec{x})i\hbar\partial_t v_1(t,\vec{x})]\}. \qquad (2.23)\end{aligned}$$

If one chooses the specific hidden local symmetry $v_1(0,\vec{x}) = v_1(T,\vec{x})$, $v_1^\star(0,\vec{x})v_1(T,\vec{x})$ becomes real and positive, and the factor

$$\beta = \oint dt\int d^3x v_1^\star(t,\vec{x})i\frac{\partial}{\partial t}v_1(t,\vec{x}) \qquad (2.24)$$

gives the unique *non-adiabatic phase* [6]. Eq.(2.23) gives another example of the holonomy (2.19), namely, the holonomy of the basis vector, not of the Schrödinger amplitude, determines the non-adiabatic phase in our formulation [12].

Note that the so-called "projective Hilbert space" and the transformation of the Schrödinger amplitude [6]

$$\psi(t,\vec{x}) \to e^{i\omega(t)}\psi(t,\vec{x}), \qquad (2.25)$$

which is not the symmetry of the Schrödinger equation, is not used in our formulation. We note that the consistency of the "projective Hilbert space" (2.25) with the superposition principle is not obvious [12]. More about this will be discussed later.

2.4 Non-adiabatic phase: Non-cyclic evolution

Any exact Schrödinger amplitude is written in the form

$$\psi_k(\vec{x},t) = v_k(\vec{x},t)\exp\{-\frac{i}{\hbar}\int_0^t\int d^3x[v_k^\dagger(\vec{x},t)\hat{H}(t)v_k(\vec{x},t) -v_k^\dagger(\vec{x},t)i\hbar\frac{\partial}{\partial t}v_k(\vec{x},t)]\} \tag{2.26}$$

if one chooses $\{v_k(\vec{x},t)\}$ suitably [13]. Note, however, the periodicity

$$v_k(T,\vec{x}) = v_k(0,\vec{x}) \tag{2.27}$$

is lost in general, and thus *non-cyclic.*

In this case, the quantity

$$\int d^3x\psi_k^\dagger(0,\vec{x})\psi_k(T,\vec{x}) = \int d^3xv_k^\dagger(0,\vec{x})v_k(T,\vec{x}) \times\exp\{\frac{-i}{\hbar}\int_0^T dtd^3x[v_k^\dagger(t,\vec{x})\hat{H}(t)v_k(t,\vec{x}) -v_k^\dagger(t,\vec{x})i\hbar\partial_t v_k(t,\vec{x})]\} \tag{2.28}$$

is manifestly invariant under the hidden local symmetry (2.9). By choosing a suitable hidden symmetry $v_k(t,\vec{x}) \rightarrow e^{i\alpha_k(t)}v_k(t,\vec{x})$, one can make the prefactor

$$\int d^3xv_k^\dagger(0,\vec{x})v_k(T,\vec{x}) \tag{2.29}$$

real and positive. It is important that we can make only the integrated prefactor (2.29) real and positive in the present non-cyclic case, since one cannot make $v_k^\dagger(0,\vec{x})v_k(T,\vec{x})$ real and positive by a time dependent gauge transformation for all $\vec{x}$ for the non-cyclic case [11]. Then the exponential factor in (2.28) defines the unique non-cyclic and non-adiabatic phase [7]. We have a structure similar to (2.19) in the present non-cyclic case also, though it may not be called holonomy in a rigorous sense. We emphasize that we do not use the projective Hilbert space defined by (2.25) in the present formulation of non-adiabatic and non-cyclic geometric phase [13].

2.5 Geometric phase for mixed states

We start with a given hermitian Hamiltonian $\hat{H}(t)$ and given $\mathcal{U}(t) = T^\star\exp[-\frac{i}{\hbar}\int_0^t\hat{H}(t)dt]$. We employ a diagonal form of the density matrix

$$\rho(0) = \sum_k \omega_k\psi_k(0,\vec{x})\psi_k^\dagger(0,\vec{x}), \tag{2.30}$$

where the exact Schrödinger amplitudes are defined by

$$\psi_k(t,\vec{x}) = \langle\vec{x}|\mathcal{U}(t)|k\rangle = \int d^3y\langle\vec{x}|\mathcal{U}(t)|\vec{y}\rangle v_k(0,\vec{y}). \tag{2.31}$$

We define the total phases for pure states $\psi_k(t,\vec{x})$ by

$$\phi_k(t) = \arg\int d^3x\psi_k^\dagger(0,\vec{x})\psi_k(t,\vec{x}) \tag{2.32}$$

and the complete set of basis vectors in (2.1) by

$$v_k(t,\vec{x}) = e^{-i\phi_k(t)}\psi_k(t,\vec{x}), \quad \int d^3x v_k^\dagger(t,\vec{x})v_l(t,\vec{x}) = \delta_{k,l}. \tag{2.33}$$

One can then confirm that the exact Schrödinger amplitudes are written as

$$\begin{aligned}\psi_k(\vec{x},t) &= v_k(\vec{x},t) \\ &\quad\times\exp\{-\frac{i}{\hbar}\int_0^t[\int d^3x v_k^\dagger(\vec{x},t)\hat{H}(t)v_k(\vec{x},t) - \langle k|i\hbar\frac{\partial}{\partial t}|k\rangle]\}\end{aligned} \tag{2.34}$$

with

$$\langle k|i\hbar\frac{\partial}{\partial t}|k\rangle \equiv \int d^3x v_k^\dagger(\vec{x},t)i\hbar\frac{\partial}{\partial t}v_k(\vec{x},t). \tag{2.35}$$

The Schrödinger amplitude $\psi_k(t,\vec{x})$ is transformed under the hidden local symmetry as $\psi_k(t,\vec{x}) \rightarrow e^{i\alpha_k(0)}\psi_k(t,\vec{x})$ independently of t and thus the Schrödinger equation is invariant under the hidden local symmetry.

The quantity $\mathrm{Tr}\mathcal{U}(T)\rho(0)$ is then written as

$$\begin{aligned}\mathrm{Tr}\mathcal{U}(T)\rho(0) &= \sum_k \omega_k\psi_k^\dagger(0,\vec{x})\psi_k(T,\vec{x}) \\ &= \sum_k \omega_k v_k^\dagger(0,\vec{x})v_k(T,\vec{x})\exp\{\frac{i}{\hbar}\int_0^T dtd^3x[v_k^\dagger(t,\vec{x})i\hbar\partial_t v_k(t,\vec{x}) \\ &\quad -v_k^\dagger(t,\vec{x})\hat{H}(t)v_k(t,\vec{x})]\}\end{aligned} \tag{2.36}$$

without integration over $\vec{x}$. If all the pure states perform cyclic evolution with the same period T, one can choose the hidden local gauge such that

$$v_k^\dagger(0,\vec{x})v_k(T,\vec{x}) = \text{real and positive} \tag{2.37}$$

for all k, and the exponential factor in (2.36) exhibits the entire geometrical phase together with the "dynamical phase" $(1/\hbar)\int_0^T dtd^3xv_k^\dagger(t,\vec{x})\hat{H}(t)v_k(t,\vec{x})$ of each pure state. In practice, the cyclic evolution of all the pure states $\psi_k(t)$ with a period T may be rather exceptional. For a generic case, we need to define the phase for non-cyclic evolution [7] as the phase of (see (2.28))

$$
\begin{aligned}
\mathrm{Tr}\mathcal{U}(T)\rho(0) &= \sum_k \omega_k \int d^3x\psi_k^\dagger(0,\vec{x})\psi_k(T,\vec{x}) \qquad (2.38)\\
&= \sum_k \omega_k \int d^3xv_k^\dagger(0,\vec{x})v_k(T,\vec{x})\\
&\quad\times \exp\{\frac{i}{\hbar}\int_0^T dtd^3x[v_k^\dagger(t,\vec{x})i\hbar\partial_t v_k(t,\vec{x}) - v_k^\dagger(t,\vec{x})\hat{H}(t)v_k(t,\vec{x})]\}
\end{aligned}
$$

These quantities (2.36) and (2.38) are manifestly invariant under the hidden local symmetry [13], and thus not only the total phase $\arg\mathrm{Tr}\mathcal{U}(T)\rho(0)$ but also the visibility $|\mathrm{Tr}\mathcal{U}(T)\rho(0)|$ in the interference pattern [8]

$$
I \propto 1 + |\mathrm{Tr}\mathcal{U}(T)\rho(0)|\cos[\chi - \arg\mathrm{Tr}\mathcal{U}(T)\rho(0)] \qquad (2.39)
$$

which are experimentally observable are manifestly gauge invariant. Here χ stands for the variable $U(1)$ phase (difference) in the interference beams. We note that the gauge invariance of the interference pattern (2.39) does not hold in the sense of the projective Hilbert space (2.25) in the conventional formulation [8, 9], which is related to the fact that the projective Hilbert space defined by (2.25) is not consistent with the superposition principle to describe interference [12].

3 Exactly solvable example

We discuss the model

$$
\begin{aligned}
&\hat{H} = -\mu\hbar\vec{B}(t)\vec{\sigma},\\
&\vec{B}(t) = B(\sin\theta\cos\varphi(t), \sin\theta\sin\varphi(t), \cos\theta) \qquad (3.1)
\end{aligned}
$$

with $\varphi(t) = \omega t$ and constant ω, B and θ. This model has been analyzed in the past by various authors by using the adiabatic approximation [3]. It has been recently shown that this model is exactly treated in the framework of the second quantized formulation [13, 22].

The exact effective Hamiltonian (2.6) is given by

$$\hat{H}_{eff}(t) = [-\mu\hbar B - \frac{(1+\cos\theta)}{2}\hbar\omega]\hat{b}^{\dagger}_{+}\hat{b}_{+}$$
$$+[\mu\hbar B - \frac{1-\cos\theta}{2}\hbar\omega]\hat{b}^{\dagger}_{-}\hat{b}_{-} - \frac{\sin\theta}{2}\hbar\omega[\hat{b}^{\dagger}_{+}\hat{b}_{-} + \hat{b}^{\dagger}_{-}\hat{b}_{+}] \tag{3.2}$$

if one uses the instantaneous eigenstates

$$\hat{H}(t)v_{\pm}(t) = \mp\mu\hbar B v_{\pm}(t) \tag{3.3}$$

as the complete basis set in (2.1) and the expansion $\hat{\psi}(t) = \sum \hat{b}_n v_n(t)$. This effective H_{eff} is not diagonal, but it is *diagonalized* if one performs a unitary transformation

$$\begin{pmatrix} \hat{b}_{+}(t) \\ \hat{b}_{-}(t) \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\alpha & -\sin\frac{1}{2}\alpha \\ \sin\frac{1}{2}\alpha & \cos\frac{1}{2}\alpha \end{pmatrix} \begin{pmatrix} \hat{c}_{+}(t) \\ \hat{c}_{-}(t) \end{pmatrix} \tag{3.4}$$

with a constant α satisfying the parameter equation

$$\tan\alpha = \frac{\hbar\omega\sin\theta}{2\mu\hbar B + \hbar\omega\cos\theta}. \tag{3.5}$$

The corresponding new basis vectors are then explicitly given by

$$w_{+}(t) = \begin{pmatrix} \cos\frac{1}{2}(\theta-\alpha)e^{-i\varphi(t)} \\ \sin\frac{1}{2}(\theta-\alpha) \end{pmatrix}, w_{-}(t) = \begin{pmatrix} \sin\frac{1}{2}(\theta-\alpha)e^{-i\varphi(t)} \\ -\cos\frac{1}{2}(\theta-\alpha) \end{pmatrix} \tag{3.6}$$

which satisfies $\hat{\psi}(t) = \sum \hat{b}_n v_n(t) = \sum \hat{c}_n w_n(t)$. These new basis vectors are periodic $w_{\pm}(0) = w_{\pm}(T)$ with $T = \frac{2\pi}{\omega}$, and one can confirm

$$w^{\dagger}_{\pm}(t)\hat{H}w_{\pm}(t) = \mp\mu\hbar B\cos\alpha$$
$$w^{\dagger}_{\pm}(t)i\hbar\partial_t w_{\pm}(t) = \frac{\hbar\omega}{2}(1 \pm \cos(\theta-\alpha)). \tag{3.7}$$

The effective Hamiltonian H_{eff} (3.2) is now diagonalized in terms of $w_{\pm}(t)$, and thus the *exact* solution of the Schrödinger eq., $i\hbar\partial_t\psi(t) = \hat{H}\psi(t)$, is given by

$$\psi_{\pm}(t) = w_{\pm}(t)\exp\{-\frac{i}{\hbar}\int_0^t dt'[w^{\dagger}_{\pm}(t')\hat{H}w_{\pm}(t')$$
$$-w^{\dagger}_{\pm}(t')i\hbar\partial_{t'}w_{\pm}(t')]\} \tag{3.8}$$

if one uses the formula (2.8). This amplitude may be regarded either as an exact version of the adiabatic phase or as a non-adiabatic cyclic phase in our formulation in (2.21).

We examine the two extreme limits of this formula:
(i) For the *adiabatic limit* $\hbar\omega/(\hbar\mu B) \ll 1$, the parameter equation (3.5) gives

$$\alpha \simeq [\hbar\omega/2\hbar\mu B]\sin\theta, \tag{3.9}$$

and if one sets $\alpha = 0$ in the exact solution (3.8), one recovers the ordinary Berry phase [3, 4]

$$\begin{aligned}\psi_\pm(T) &\simeq \exp\{i\pi(1\pm\cos\theta)\} \\ &\qquad \times\exp\{\pm\frac{i}{\hbar}\int_0^T dt\mu\hbar B\}v_\pm(T)\end{aligned} \tag{3.10}$$

where the first exponential factor stands for the "monopole-like phase" and

$$v_+(t) = \begin{pmatrix}\cos\frac{1}{2}\theta e^{-i\varphi(t)} \\ \sin\frac{1}{2}\theta\end{pmatrix}, v_-(t) = \begin{pmatrix}\sin\frac{1}{2}\theta e^{-i\varphi(t)} \\ -\cos\frac{1}{2}\theta\end{pmatrix}. \tag{3.11}$$

(ii) For the *non-adiabatic* limit $\hbar\mu B/(\hbar\omega) \ll 1$, the parameter equation (3.5) gives

$$\theta - \alpha \simeq [2\hbar\mu B/\hbar\omega]\sin\theta \tag{3.12}$$

and if one sets $\alpha = \theta$ in the exact solution (3.8), one obtains the trivial phase

$$\psi_\pm(T) \simeq w_\pm(T)\exp\{\pm\frac{i}{\hbar}\int_0^T dt[\mu\hbar B\cos\theta]\} \tag{3.13}$$

with

$$\begin{aligned}w_+(t) &= \begin{pmatrix}e^{-i\varphi(t)} \\ 0\end{pmatrix}, \\ w_-(t) &= \begin{pmatrix}0 \\ -1\end{pmatrix}.\end{aligned} \tag{3.14}$$

This shows that the "monopole-like singularity" is smoothly connected to a trivial phase in the exact solution, and thus the geometric phase is *topologically trivial* [22].

The adiabatic and non-adiabatic phases are treated in a unified manner in the present second quantized formulation, and thus this example shows that all the geometric phases are topologically trivial.

4 Chiral anomaly

We consider the evolution operator

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{i\int d^4x[\bar{\psi}i\gamma^\mu(\partial_\mu - igA_\mu)\psi]\} \tag{4.1}$$

for the Dirac fermion $\psi(t,\vec{x})$ inside the background gauge field $A_\mu(t,\vec{x})$. The chiral anomaly in gauge field theory is understood in path integrals as arising from the non-trivial Jacobian under the chiral transformation. For an infinitesimal chiral transformation of field variables

$$\psi(x) \to e^{i\omega(x)\gamma_5}\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{i\omega(x)\gamma_5} \tag{4.2}$$

we have a non-trivial Jacobian

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \exp\{-i\int d^4x\omega(x)\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\}\mathcal{D}\bar{\psi}\mathcal{D}\psi \tag{4.3}$$

which is valid for a general class of regularization including the lattice gauge theory [18]. The Jacobian factor is identified with the chiral anomaly, and the integrated or summed Jacobian is called the Wess-Zumino term [16].

Some of the known essential and general properties of the quantum anomalies are [18]:
1. The anomalies are not recognized by a naive manipulation of the classical Lagrangian or action (or by a naive canonical manipulation in operator formulation), which leads to the naive Nöther's theorem.
2. The quantum anomaly is related to the quantum breaking of classical symmetries (and the failure of the naive Nöther's theorem). For example, the Gauss law operator (or BRST charge) becomes time-dependent and thus it cannot be used to specify physical states in anomalous gauge theory.
3. The quantum anomalies are generally associated with an infinite number of degrees of freedom. The anomalies in the practical calculation are thus closely related to the regularization, though the anomalies by themselves are perfectly finite.
4. In the path integral formulation, the anomalies are recognized as non-trivial Jacobians for the change of path integral variables associated with classical symmetries, as is explained above.

None of these essential properties are shared with the geometric phases discussed in Sections 2 and 3. One rather recognizes the following basic differences between the geometric phases and chiral anomaly [21]:

1.The Wess-Zumino term, which is obtained by a sum of the infinitesimal Jacobian such as in (4.3), is added to the classical action in path integrals, whereas the geometric term appears *inside* the classical action sandwiched by field variables as in (2.6)

$$\hat{H}_{eff}(t) = \sum_{n,m} \hat{b}_n^\dagger(t)[\int d^3x v_n^\star(t,\vec{x})\hat{H}(t)v_m(t,\vec{x}) - \int d^3x v_n^\star(t,\vec{x}) i\hbar\frac{\partial}{\partial t}v_m(t,\vec{x})]\hat{b}_m(t). \tag{4.4}$$

The geometric phase thus depends on each state in the Fock space generated by $\hat{b}_n^\dagger$, whereas the chiral anomaly is state-independent.
2. The topology of chiral anomaly, which is provided by given gauge field, is exact, whereas the topology of the adiabatic geometric phase, which is valid only approximately in the adiabatic limit, is trivial as we have shown in Section 3.
3. The geometric phases are basically different from the topologically exact objects such as the Aharonov-Bohm phase or chiral anomaly. For example, the Aharonov-Bohm phase is identical for adiabatic or non-adiabatic motion of the electron.
4. *Similarity* between the geometric phase and a special class of chiral anomaly was noted by M. Stone on the basis of a model [20]

$$\mathcal{H}(t) = \frac{\vec{L}^2}{2I} - \psi^\dagger\mu\mathbf{n}(t)\cdot\vec{\sigma}\psi \tag{4.5}$$

where $\mathbf{n}(t)$ plays a role of the magnetic field in (3.1) which acts on the spin $\vec{\sigma}$, and $\vec{L}$ induces the rotation of $\mathbf{n}(t)$. But it is obvious from our analysis of topological properties in Section 3 that these two notions are fundamentally different.
5. The topology of Berry's phase is valid only when the adiabatic approximation is strictly valid, whereas the anomaly appears in field theory only when the adiabatic approximation *fails* in a version of the Hamiltonian analysis [19]. Thus these notions cannot be compatible.

5 Conclusion

We have illustrate the advantages of the second quantized formulation of all the geometric phases. The second quantized formulation is located in between the first quantization and field theory, and thus it is convenient to compare the geometric phase with other phases such as chiral anomaly. We clarified the basic differences between these two notions.

In the early literature on the geometric phase, the similarity between the geometric phase and other phases such as the chiral anomaly and the Aharonov-Bohm phase, was often emphasized. But in view of the wide use of the loosely defined terminology "geometric phase" in various fields in physics today, it is our opinion that a more precise distinction of "identical phenomena" from "similar phenomena" is important. To be precise, what we are suggesting is to call chiral anomaly as chiral anomaly, Wess-Zumino term as Wess-Zumino term, and Aharovov-Bohm phase as Aharonov-Bohm phase, etc., since those terminologies convey very clear messages and well-defined physical contents which the majority in physics community can readily recognize. Even in this sharp definition of terminology, one should still be able to clearly identify the geometric phase and its physical characteristics, which are intrinsic to the geometric phase and cannot be described by other notions.

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