Abstract—In a multiple-input multiple-output (MIMO) beamforming system with finite rate feedback, the receiver sends back quantized channel information to the transmitter via a feedback channel. The overall system performance is degraded due to feedback errors. In this paper, we treat the feedback of the beamforming vector as a generalized vector quantizer (VQ), and adopt index assignment (IA) technique to cope with feedback errors. A lower bound to the average symbol error rate (SER) is derived, and an IA design criterion is proposed to minimize this bound, which is in accord with the pseudo-Gray coding principle in conventional IA design literature. Numerical results show that well-designed IA schemes improve the SER considerably.

I. INTRODUCTION

Transmit beamforming with receive combining is an attractive method to utilize the benefit of multiple-input multiple-output (MIMO) wireless systems because it provides large diversity and array gains without sophisticated signal processing at the transmitter and/or receiver. However, transmit beamforming requires channel state information at the transmitter, which is not usually available. This motivates recent researches on beamforming with finite rate feedback.

In transmit beamforming systems with finite rate feedback, the beamforming vector is restricted to lie in a finite set (codebook) that is known to both the transmitter and receiver. The receiver chooses the optimal codeword from codebook and conveys its index back to the transmitter. This beamforming technique has been extensively studied, e.g. [1]-[6]. In particular, [1] and [5] treated the beamformer design as a generalized vector quantization (VQ) problem, and employed a generalized Lloyd algorithm to construct codebooks. [2] and [3] derived bounds on outage probability and average receive signal-to-noise ratio (SNR), and designed good beamformers to approach these bounds. [4] and [6] analyzed the symbol error rate (SER), the SNR loss, and the outage probability of beamforming systems with finite rate feedback.

In [1]-[6], the feedback is assumed to be ideal (error-free and delay-free). But feedback error and delay are always present in practice, causing degradation in overall system performance [7] [8]. Based on Markov chain theory, [7] studied the effect of feedback delay on transmit beamforming with finite rate feedback. [8] analyzed the impact of feedback errors on MISO systems with quantized equal gain beamforming.

In this paper, we study MIMO beamforming systems with finite rate feedback and feedback errors. In such a case, the effect of feedback errors on the system performance depends on the index assignment (IA) scheme of the codewords. In conventional VQ designs, judiciously selected IA yields a significant reduction in the mean square error (MSE) [9]-[12]. We extend this method to beamforming systems with finite rate feedback. Based on the analysis of the average SER, we propose an IA design criterion for beamforming systems, and simplify the IA design problem to a minimization problem. Interestingly, the proposed criterion has a similar geometrical explanation to the pseudo-Gray coding principle [9].

The main advantage of IA technique is that it requires neither additional feedback bits nor additional signal processing, i.e. it is redundancy-free. Moreover, IA technique can be easily combined with other error-protection methods, e.g. error-control coding.

The rest of the paper is organized as follows. In Section II, the system model is described, and the IA design problem is formulated. Section III derives the proposed IA design criterion. Simulation results are presented in Section IV. Finally, Section V draws conclusions.

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. Forward Part

The frequency-flat MIMO beamforming system is illustrated in Figure 1. The forward channel obeys the independent and identically distributed (i.i.d.) block fading model: it is modeled as a \(N_r \times N_t\) random matrix \(H\) with independent \(\mathcal{CN}(0, 1)\) entries\(^1\), remains static within a block, and changes independently between blocks.

The forward part adopts transmit beamforming with receive combining. With \(s \in \mathbb{C}\) denoting the transmitted symbol, the signal \(r \in \mathbb{C}\) after receive combining is given by

\[
r = z^H H w s + z^H \eta,
\]

where \(\eta \in \mathbb{C}^{N_r}\) is the noise vector with i.i.d. \(\mathcal{CN}(0, N_0)\) entries; \(w \in \mathbb{C}^{N_t}\) is the unit-norm beamforming vector, which is determined by feedback information and discussed later;

\(^1\)In this paper, \(\mathcal{CN}(\mu, \sigma^2)\) stands for the complex Gaussian distribution with mean \(\mu\) and variance \(\sigma^2\). Moreover, we use \((\cdot)^H\) to denote conjugate transposition; \(||\cdot||\) and \(||\cdot||_F\) to denote vector 2-norm and matrix Frobenius norm, respectively; \(\mathbb{E}\) and \(\mathbb{P}\) to denote the expectation and probability operators. The unit complex hypersphere is \(\Omega^N = \{x \in \mathbb{C}^N : x^H x = 1\}\).
z \in \mathbb{C}^{N_r}$ is the unit-norm combining vector. Assuming that the receiver knows the channel $\mathbf{H}$ and the beamforming vector $\mathbf{w}$, the optimal combining vector is $z = \mathbf{H}\mathbf{w} / \|\mathbf{H}\mathbf{w}\|$ [3]. The corresponding instantaneous receive SNR is

$$\gamma = \gamma_0 \|\mathbf{H}\mathbf{w}\|^2,$$

(2)

where $\gamma_0 \triangleq \mathbb{E}(|s|^2) / N_0$ is the SNR.

**B. Feedback Part**

It is assumed that the receiver has perfect channel knowledge, and is able to send back $B$ bits per block to the transmitter. A $B$-bit codebook $\mathcal{C} = \{\mathbf{c}_k\}_{k=1}^K$, where $\mathbf{c}_k$’s are unit-norm codewords and $K = 2^B$, is designed in advance and known to both the transmitter and the receiver. This codebook can be constructed using the methods in [1]-[5].

The feedback part in Figure 1 consists of five modules. The VQ encoder finds out the optimal codeword by maximizing the instantaneous receive SNR, i.e.

$$\mathbf{c}_{op} = \underset{\mathbf{c} \in \{\mathbf{c}_1, \ldots, \mathbf{c}_K\}}{\arg \max} \|\mathbf{H}\mathbf{c}\|^2. \quad (3)$$

The index of $\mathbf{c}_{op}$ is sent to the IA module, which performs permutation $\Pi$ on this index and outputs the permuted index to the feedback channel. So the following events are equivalent:

$$\mathbf{c}_{op} = \mathbf{c}_k \iff \text{the IA module’s input is } k \iff \text{the feedback channel’s input is } \Pi(k) \quad (4)$$

In this paper, the feedback channel is modeled as a binary symmetric channel (BSC) with transition probability

$$t_{i,j} \triangleq P[\text{BSC output } j \mid \text{BSC input } i] = p^{d_h(i,j)}(1-p)^{B-d_h(i,j)}, \quad i, j = 1, \ldots, K, \quad (5)$$

where $d_h(i,j)$ denotes the Hamming distance between the binary representations of $i - 1$ and $j - 1$; $p$ is a parameter of the BSC; $B$ is the number of feedback bits ($K = 2^B$).

After the feedback channel, the inverse IA module performs the inverse-permutation $\Pi^{-1}$. The VQ decoder performs a lookup table operation and selects the codeword according to the output of inverse IA. Then, the transmitter uses the selected codeword as beamforming vector $\mathbf{w}$. These operations are described by the following equivalent events:

$$\mathbf{w} = \mathbf{c}_l \iff \text{the inverse IA module’s output is } l \iff \text{the feedback channel’s output is } \Pi(l) \quad (6)$$

Due to feedback errors, the beamforming vector may not be the desired one.

**C. Problem Statement**

The IA and inverse IA modules affect the overall behavior of the feedback part. Therefore, the adopted IA scheme also influences the performance of the forward part. In this paper, our goal is to design good IA schemes using the average SER as the design metric. For notation brevity, we assume coherent reception of binary signals in the derivation. However, the result is applicable to other constellations, as shown in Section III-C.

Conditioned on the instantaneous receive SNR $\gamma$, the SER for binary modulation is given by [15, Eq.(4.2)]

$$P_e(\gamma) = Q(\sqrt{2g\gamma}) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp \left( -\frac{g\gamma}{\sin^2 \theta} \right) d\theta, \quad (7)$$

where $g$ is a constellation-dependent constant. For example $g = 1$ for binary phase-shift keying (BPSK), $g = \frac{1}{2}$ for binary frequency-shift keying (BFSK). Then, the average SER is expressed as

$$\overline{P}_e = \mathbb{E} P_e(\gamma) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \mathbb{E} \exp \left( -\frac{g\gamma}{\sin^2 \theta} \right) d\theta, \quad (8)$$

The dependence of $\overline{P}_e$ on the IA scheme $\Pi$ will be made clear in the next section. Our ultimate objective is to find out the IA scheme that minimizes the average SER, i.e.

$$\text{minimize: } \overline{P}_e \quad (9)$$

subject to: $\Pi$ is a permutation.
III. PSEUDO-GRAY CODING FOR BEAMFORMING SYSTEMS

The solution of the optimization problem (9) is, in general, hard to obtain. In this section, we resort to a simplified, albeit suboptimal, problem formulation. A lower bound to the average SER is derived in Section III-A.B. An IA design criterion is proposed in Section III-C to minimize this bound, which consists with the pseudo-Gray coding principle in [9].

A. SER Analysis in MISO Systems

In this subsection, the discussion is restricted within MISO systems, where a geometrical framework has been presented in [2] and [4]. In MISO cases, the channel matrix reduces to a $N_t \times 1$ vector $\mathbf{h} \sim \mathcal{CN}(0, \mathbf{I}_{N_t})$ ($\mathbf{h}^H$ corresponds to $\mathbf{H}$ in the system model). The useful information for the VQ encoder is the channel direction $\hat{\mathbf{h}} \triangleq \mathbf{h}/||\mathbf{h}||_2$, which is uniformly distributed on the unit hypersphere $\Omega_{N_t}$. An equivalent operation to (3) is that the VQ encoder divides the unit hypersphere into $K$ Voronoi regions $V_1, \ldots, V_k, \ldots, V_K$, with

$$V_k \triangleq \{ \mathbf{h} \in \Omega_{N_t} : ||\mathbf{h}^H \mathbf{c}_k^*||^2 > ||\mathbf{h}^H \mathbf{c}_l||^2, \forall l \neq k \}.$$  

(10)

The optimal codeword is $\mathbf{c}_k$, if and only if $\hat{\mathbf{h}} \in V_k$. Taking into account (4)–(6), one obtains

$$\mathbb{P}(w = c_l | \hat{\mathbf{h}} \in V_k) = \mathbb{P}(w = c_l | c_{op} = \mathbf{c}_k) = t_{(l)} \cdot t_{(k)}, \quad k, l = 1, \ldots, K.$$  

(11)

Moreover, the instantaneous receive SNR in (2) reduces to

$$\gamma = \gamma_0 ||\mathbf{h}^H \mathbf{w}||^2 = \gamma_0 ||\mathbf{h}||_2^2 ||\mathbf{h}^H \mathbf{w}||^2,$$  

(12)

So the expectation in (8) is with respect to $||\mathbf{h}||_2^2$, $\hat{\mathbf{h}}$, and $\mathbf{w}$, since $\gamma$ depends on them. As in [4], we take the expectation with respect to $||\mathbf{h}||_2^2$ at first, and get

$$\overline{P}_c = \frac{1}{\pi} \int_0^{\pi/N_t} \mathbb{E} \left[ \left( 1 + \frac{\gamma_0 \mathbf{h}^H \mathbf{w}^H \mathbf{w}^H}{\sin \theta} \right)^{-N_t} \right] d\theta = \frac{1}{\pi} \int_0^{\pi} \sum_{k=1}^K \sum_{l=1}^K \mathbb{E} \left[ \left( 1 + \frac{\gamma_0 \mathbf{h}^H \mathbf{c}_l^*||^2}{\sin \theta} \right)^{-N_t} \right] t_{(l)} \cdot t_{(k)} \mathbb{P}(h \in V_k) d\theta,$$  

(13)

where the last equality follows from the law of total expectation and Eq.(11).

Because the actual value of (13) is hard to obtain, we consider

$$\overline{P}_c^{lb} = \frac{1}{\pi} \sum_{k=1}^K \sum_{l=1}^K \int_0^{\pi/N_t} \left[ 1 + \frac{\gamma_0 \mathbf{h}^H \mathbf{c}_l^*||^2}{\sin \theta} \right]^{-N_t} t_{(l)} \mathbb{P}(\hat{\mathbf{h}} \in V_k) d\theta,$$  

(14)

which is a lower bound to $\overline{P}_c$, due to the convexity of the function $(1 + x)^{-N_t}, x \geq 0$. When a beamforming system relies on finite rate feedback, its behavior is complicated, and a commonly used technique is to find its performance bound, e.g. [2]-[6].

In the geometrical framework presented in [2] and [4], a key idea is the spherical cap approximation for Voronoi regions,

$$V_k \approx \mathcal{S}(\mathbf{c}_k) \triangleq \{ \mathbf{x} \in \Omega_{N_t} : ||\mathbf{x}^H \mathbf{c}_k||^2 > \rho \}, \quad k = 1, \ldots, K,$$  

(15)

where

$$\rho \triangleq 1 - K^{-\frac{1}{N_t-1}}$$  

(16)

is a constant. This approximation is accurate for well-designed codebooks. Substituting (15) into (14), the SER bound is approximated by

$$\overline{P}_c^{lb} \approx \frac{1}{\pi} \sum_{k=1}^K \sum_{l=1}^K \int_0^{\pi} \left[ 1 + \frac{\gamma_0 \rho}{\sin \theta} \right]^{-N_t} t_{(l)} \mathbb{P}(\hat{\mathbf{h}} \in \mathcal{S}(\mathbf{c}_k)) d\theta \times \sum_{k=1}^K \sum_{l=1}^K t_{(l)} \cdot \left\{ \frac{1 - \rho}{N_t} + \frac{||\mathbf{c}_k^H \mathbf{c}_l||^2}{N_t} \right\}^{-N_t} d\theta$$  

(17)

where the equality follows from Eq.(35)(36) in Appendix. At high SNR, we omit the ‘1’ in the integrant and simplify (17) to the desired result

$$\overline{P}_c^{lb} \approx \left\{ \frac{1}{\pi K} \int_0^{\pi} \left( \frac{\gamma_0 \rho}{\sin \theta} \right)^{-N_t} d\theta \right\} \times \sum_{k=1}^K \sum_{l=1}^K t_{(l)} \cdot \left\{ \frac{1 - \rho}{N_t} + ||\mathbf{c}_k^H \mathbf{c}_l||^2 \right\}^{-N_t}.$$  

(18)

B. SER Analysis in MIMO Systems

In MIMO cases, the Voronoi region of the codeword $\mathbf{c}_k$ is

$$V_k' \triangleq \{ \mathbf{H} \in \mathbb{C}^{N_r \times N_t} : ||\mathbf{H} \mathbf{c}_k||^2 > ||\mathbf{H} \mathbf{c}_l||^2, \forall l \neq k \}.$$  

(19)

The channel matrix $\mathbf{H}$ has the following eigen-decomposition

$$\mathbf{H}^H \mathbf{H} = \mathbf{U} \Lambda \mathbf{U}^H = \sum_{n=1}^{N_t} \lambda_n \mathbf{u}_n \mathbf{u}_n^H,$$  

(20)

where $\mathbf{A} = \text{diag}\{\lambda_1, \ldots, \lambda_{N_t}\}$ and $\mathbf{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_{N_t}\}$; $\lambda_1 \geq \cdots \geq \lambda_{N_t} \geq 0$ are eigenvalues of $\mathbf{H}^H \mathbf{H}$; $\mathbf{u}_n, n = 1, \ldots, N_t$, is the eigenvector corresponding to $\lambda_n$. The eigen matrix $\mathbf{U}$ is uniformly distributed over $\mathbf{U}$, and independent of the eigenvalues [3, Lemma 1].

In [6], the spherical cap approximation for Voronoi regions is generalized to MIMO cases, i.e.

$$V_k' \approx \{ \mathbf{H} : \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k) \}, \quad k = 1, \ldots, K.$$  

(21)

where the spherical cap $\mathcal{S}(\mathbf{c}_k)$ has been defined in (15). The intuition behind (21) is that the contribution of $\lambda_1$ to the receive SNR is much larger than the other eigenvalues, so the eigenvector $\mathbf{u}_1$ plays a similar role to the channel direction $\hat{\mathbf{h}}$ in MISO cases.

To analyze the SER, we consider the moment generating function of the instantaneous receive SNR in (2). Similar to the
derivations in Section III-A, the moment generating function is bounded as

\[
E(e^{s\gamma}), \ s \leq 0 = \mathbb{E}\exp(s\gamma_{0}\|Hw\|^{2})
\]

\[
= \sum_{k=1}^{K} \sum_{l=1}^{K} E\bigg[\exp(s\gamma_{0}\|Hc_{k}\|^{2})\bigg|H \in \mathcal{V}_{k}\bigg] \ t_{I(k),I(l)} \mathbb{P}(H \in \mathcal{V}_{k})
\]

\[
\approx \sum_{k=1}^{K} \sum_{l=1}^{K} \mathbb{E}\bigg[\exp(s\gamma_{0}\sum_{n=1}^{N_{t}} \lambda_{n}\|u_{n}^{H}c_{n}\|^{2})\bigg|u_{1} \in \mathcal{S}(c_{k})\bigg] \ t_{I(k),I(l)} \mathbb{P}(u_{1} \in \mathcal{S}(c_{k}))
\]

\[
= \sum_{k=1}^{K} \sum_{l=1}^{K} \mathbb{E}_{U|u_{1} \in \mathcal{S}(c_{k})}\bigg[\exp(s\gamma_{0}\sum_{n=1}^{N_{t}} \lambda_{n}\|u_{n}^{H}c_{n}\|^{2})\bigg] \ t_{I(k),I(l)} \mathbb{P}(u_{1} \in \mathcal{S}(c_{k}))
\]

\[
\geq \sum_{k=1}^{K} \sum_{l=1}^{K} \mathbb{E}\bigg[\exp(s\gamma_{0}\sum_{n=1}^{N_{t}} \lambda_{n}E\bigg[\|u_{n}^{H}c_{n}\|^{2}\bigg|u_{1} \in \mathcal{S}(c_{k})\bigg])\bigg] \ t_{I(k),I(l)} \mathbb{P}(u_{1} \in \mathcal{S}(c_{k}))
\]

\[
\geq \sum_{k=1}^{K} \sum_{l=1}^{K} \mathbb{E}\bigg[\exp\bigg(s\gamma_{0}\frac{1-\rho}{N_{t}} + \rho\|c_{k}^{H}c_{l}\|\bigg)\bigg]\ t_{I(k),I(l)} \frac{1}{K}
\]

\[
= \sum_{k=1}^{K} \sum_{l=1}^{K} \bigg[1 - s\gamma_{0}\rho\frac{1-\rho}{N_{t}} + \rho\|c_{k}^{H}c_{l}\|\bigg]^{-N_{t}N_{r}} \ t_{I(k),I(l)} \frac{1}{K}.
\]

The derivation is explained as follows

- (22) results from the law of total expectation;
- (23) follows from (20) (21);
- (24) is due to the fact that the eigen matrix is independent of the eigenvalues [3];
- (25) follows from Jensen’s inequality;
- (26) relies on (37) (38) in Appendix (Note that \(s \leq 0\));
- (27) is because \(\|H\|_{F}^{2} = \sum_{m} \sum_{n} |h_{m,n}|^{2}\) is chi-square distributed with \(2N_{t}N_{r}\) degrees of freedom. Its moment generating function is \(E\exp(\tau\|H\|_{F}^{2}) = (1 - \tau)^{-N_{t}N_{r}}\).

Substituting (27) into (8), we get a lower bound to the average SER

\[
\overline{P}_{e}^{lb} = \frac{1}{\pi} \sum_{k=1}^{K} \sum_{l=1}^{K} \int_{0}^{\pi} \bigg[1 + \frac{g_{QAM}}{\sin^{2}\theta}(\frac{1-\rho}{N_{t}} + \rho\|c_{k}^{H}c_{l}\|)\bigg]^{-N_{t}N_{r}} \ t_{I(k),I(l)} \frac{1}{K} \ d\theta.
\]

At high SNR, we omit the ‘1’ in the integrant and simplify (28) to the desired result

\[
\overline{P}_{e}^{lb} \approx \frac{1}{K} \int_{0}^{\pi} \bigg(\frac{g_{QAM}}{\sin^{2}\theta} - N_{t}N_{r}\bigg) \ t_{I(k),I(l)} \frac{1}{K} \ d\theta.
\]

Note that the results in (17) (18) are special cases of (28) (29).

C. The Proposed IA Design Criterion

Eq.(29) has a simple structure, and clearly shows the dependence of SER on the adopted IA scheme. Replacing \(\overline{P}_{e}^{lb}\) in the original problem (9) by \(\overline{P}_{e}^{lb}\) in (29), we end up with a simplified, albeit suboptimal, formulation as follows. (Note that in (29), the term in \(\{\}\) can be omitted, because it doesn’t depend on the IA scheme \(II\).)

**IA Design Criterion for Beamforming Systems:** Given the codewbook \(\mathcal{C} = \{c_{k}\}_{k=1}^{K}\) and the number of antennas \((N_{t}, N_{r})\), design an IA scheme \(II\) by solving

minimize: \[
\sum_{k=1}^{K} \sum_{l=1}^{K} t_{I(k),I(l)} \left(\frac{1-\rho}{N_{t}N_{r}} + \|c_{k}^{H}c_{l}\|\right)^{-N_{t}N_{r}}
\]

subject to: \(II\) is a permutation. (30)

where \(t\) and \(\rho\) are defined in (5) and (16), respectively.

Minimization of the cost function in (30) over all possible permutations is a special case of the quadratic assignment problem (QAP), and is known to be NP-complete. If \(K\) is small, we can solve (13) by brute-force search. If \(K\) is large, brute-force search is impractical, and suboptimal algorithms have been suggested in the literature, e.g. [9]–[12].

In practice, these algorithms yield satisfactory results. Note that the IA schemes can be designed offline, so the computation complexity is acceptable.

At last, we give some discussion on the proposed design criterion.

**Remark 1 (Pseudo-Gray coding principle):** In a beamforming system, the codewords \(\{c_{k}\}_{k=1}^{K}\) are unit-norm vectors, and the Euclidean distance is not appropriate. A suitable metric of codeword pairs is the chordal distance [3]

\[
d_{c}(c_{k}, c_{l}) = \sqrt{1 - \|c_{k}^{H}c_{l}\|^{2}}.
\]

In the cost function of (30), the term

\[
\left(\frac{1-\rho}{N_{t}N_{r}} + \|c_{k}^{H}c_{l}\|\right)^{-N_{t}N_{r}} = \left(\frac{1-\rho}{N_{t}N_{r}} + 1 - d_{c}(c_{k}, c_{l})\right)^{-N_{t}N_{r}}
\]

is an increasing function of the chordal distance \(d_{c}(c_{k}, c_{l})\). Intuitively, this IA design problem would accord with the pseudo-Gray coding principle [9], which implies that a good IA scheme should make the distance between codeword pairs correspond closely to the Hamming distance between their indexes, just like Gray coding for quadrature amplitude modulation (QAM) constellations. This intuition is confirmed by the numerical results in next section.

**Remark 2 (Extension to arbitrary constellations):** For notation brevity, the IA design criterion (30) is derived for binary signals. However, it is applicable to arbitrary one- or two-dimensional signal constellations, because they all have similar SER formulas to (7). For example, the average SER of a square QAM constellation can be written as [15]

\[
\overline{P}_{QAM} = \frac{1}{\pi} \sqrt{\frac{3}{M-1}} \int_{0}^{\pi} \mathbb{E}\exp\bigg(-\frac{g_{QAM}}{\sin^{2}\theta}\bigg) \ d\theta
\]

\[
+ \frac{1}{\pi} \sqrt{\frac{3}{M-1}} \int_{0}^{\pi} \mathbb{E}\exp\bigg(-\frac{g_{QAM}}{\sin^{2}\theta}\bigg) \ d\theta
\]
Good IA, Bad IA. We design IA schemes for these codebooks using the system. These codebooks are downloaded from channel (5) is set to 0.02 in the design. The IA schemes for error-free case)

Therefore, (30) is applicable to square QAM constellations.

**IV. Numerical Results**

Some numerical results are presented in this section. We consider a $2 \times 1$ transmit beamforming system and a $4 \times 2$ MIMO beamforming system.

**IA design.** A 3-bit codebook (8 codewords) and a 6-bit codebook (64 codewords) are used in the $2 \times 1$ and $4 \times 2$ system, respectively. These codebooks are downloaded from [14]. We design IA schemes for these codebooks using the proposed design criterion. The parameter $p$ of the feedback channel (5) is set to 0.02 in the design. The IA schemes for the $2 \times 1$ system are shown in TABLE I, where the good IA is the brute-force search solution of (30) and the bad IA is obtained by maximizing the cost function of (30). For the $4 \times 2$ system, (30) is solved by the binary switching algorithm [9].

From the IA results, we can gain an insight into how the good IA improves the system performance. For example, in TABLE I, the first codeword is closest (with respect to the chordal distance) to the last codeword. The Hamming distance between their good indexes (7 and 5 respectively) is 1. Similarly, the second codeword is closest to the fifth codeword. The Hamming distance between their good indexes (1 and 2) is 1. It is shown from these examples that the good IA scheme has assigned close index pairs (in term of Hamming distance) to close codeword pairs (with respect to chordal distance).

**SER performance.** The SER performance of the $2 \times 1$ system is shown in Fig. 2 and 3. QPSK constellation and ideal coherent detection is assumed. Fig. 2 illustrates that the average SER of ‘good IA’ is much lower than that of ‘bad IA’, and approaches that of ‘error-free feedback’. In the case of ‘original IA’, the output of VQ encoder is directly sent to the feedback channel without index assignment, and the output of the feedback channel is directly sent to VQ decoder.

**TABLE I**

<table>
<thead>
<tr>
<th>Codeword</th>
<th>Original IA, $k$</th>
<th>Good IA, $\Pi_{\text{good}}(k)$</th>
<th>Bad IA, $\Pi_{\text{bad}}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0.8393 - j \cdot 0.2939, -0.1677 + j \cdot 0.4256]^T$</td>
<td>1</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>$[-0.3427 + j \cdot 0.9161, 0.0498 + j \cdot 0.2019]^T$</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$[-0.2065 + j \cdot 0.3371, 0.9166 + j \cdot 0.0600]^T$</td>
<td>3</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$[0.3478 - j \cdot 0.3351, 0.2584 + j \cdot 0.8366]^T$</td>
<td>4</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>$[0.1049 + j \cdot 0.6820, 0.6537 + j \cdot 0.3106]^T$</td>
<td>5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$[0.0347 - j \cdot 0.2716, 0.0935 - j \cdot 0.9572]^T$</td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$[-0.7457 + j \cdot 0.1181, -0.4553 - j \cdot 0.4719]^T$</td>
<td>7</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>$[-0.7983 + j \cdot 0.3232, 0.5000 + j \cdot 0.0906]^T$</td>
<td>8</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>
The parameter $p$ in (5) reflects the quality of the feedback channel. We plot in Fig.3 the dependence of average SER on $p$. Although the good IA is designed for $p = 0.02$, it performs well in other values of $p$.

Similar SER experiments are carried out in the $4 \times 2$ system and the results are shown in Fig.4 and 5. 16QAM modulation is used. We can see that the good IA outperforms the bad IA, though it cannot reach the performance of error-free case.

In the Appendix, we prove that in MISO cases, redundant-free protection against the feedback errors. We derived a practical IA design criterion to improve the average SER, and showed that the proposed criterion follows the pseudo-Gray coding principle. Numerical results demonstrated that well-designed IA schemes improve the SER performance.

V. CONCLUSION

We studied MIMO beamforming systems with feedback errors, and used IA technique to provide the system with a redundancy-free protection against the feedback errors. We obtained a practical IA design criterion to improve the average SER, and showed that the proposed criterion follows the pseudo-Gray coding principle. Numerical results demonstrated that well-designed IA schemes improve the SER performance.

APPENDIX

In the Appendix, we prove that in MISO cases

\[
\mathbb{P}(\hat{h} \in S(c_k)) = \frac{1}{K} \quad \text{(35)}
\]

\[
\mathbb{E}(|\hat{h}^H c_l|^2 | \hat{h} \in S(c_k)) = \frac{1 - \rho}{N_t} + \rho|c_l^H c_l|^2 \quad \text{(36)}
\]

and in MIMO cases

\[
\mathbb{P}(u_1 \in S(c_k)) = \frac{1}{K} \quad \text{(37)}
\]

\[
\sum_{n=1}^{N_t} \lambda_n \mathbb{E}(|u_n^H c_l|^2 | u_1 \in S(c_k)) \leq \left(\frac{1 - \rho}{N_t} + \rho|c_l^H c_l|^2\right) \|H\|^2_F \quad \text{(38)}
\]

where $\|H\|^2_F = \sum_{n=1}^{N_t} \lambda_n$ is the squared Frobenius norm of the channel matrix $H$. (35) and (37) follow directly from [2, Lemma 2 and 4].

**Proof of (36)**: Since $\hat{h}$ is uniformly distributed on $\Omega_{N_t}$,\n
\[
\mathbb{E}(|\hat{h}^H c_l|^2 | \hat{h} \in S(c_k)) = \int_{S(c_k)} c_0 |\hat{h}^H c_l|^2 \, dh, \quad \text{(39)}
\]

where $c_0 = \frac{(N_t - 1)!}{2^{N_t} N_t!} (1 - \rho)^{(N_t - 1)}$, such that $\int_{S(c_k)} c_0 d\hat{u}_1 = 1$.

Let $\Theta = [c_k : \Theta_0]$ be a unitary matrix, where $\Theta_0$ is chosen arbitrarily with the constraint that $\Theta$ is unitary. Using the transformation $v = \Theta^H \hat{h}$, $S(c_k)$ is rotated to $S(e_1) = \{v \in \Omega_{N_t} : |v^H e_1|^2 > \rho\}$. The Jacobian for this transformation is 1, because $\Theta$ is unitary. Hence

\[
\int_{S(c_k)} c_0 |\hat{h}^H c_l|^2 dh = \int_{S(e_1)} c_0 |v^H \Theta^H c_l|^2 dv = c_l^H \Theta \int_{S(e_1)} vv^H dv \Theta^H c_l \quad \text{(40)}
\]

The surface integration $\int_{S(e_1)} vv^H dv$ is converted to $2N_t - 1$ dimensional multiple integration [16]. Let

\[
\mathcal{G} = \{y \in \mathbb{R}^{2N_t - 1} : -\pi < y_1 < \pi, y_2^2 + \cdots + y_{2N_t - 1}^2 < 1 - \rho\}
\]

Construct the transformation $g : \mathcal{G} \rightarrow S(e_1)$

\[
\mathcal{G} = \begin{cases}
\mathbb{R}(v) = g_1(y) = \sqrt{1 - y_2^2 - \cdots - y_{2N_t-1}^2} \cos y_1, \\
\mathbb{I}(v) = g_2(y) = \sqrt{1 - y_2^2 - \cdots - y_{2N_t-1}^2} \sin y_1, \\
\mathbb{R}(v_n) = g_{2n-1}(y) = y_{2n-1}, \quad n = 2, \ldots, N_t, \\
\mathbb{I}(v_n) = g_{2n}(y) = y_{2n-1}, \quad n = 2, \ldots, N_t
\end{cases}
\]

where $\mathbb{R}(v_n)$ and $\mathbb{I}(v_n)$ denote the real and imaginary parts of the $n$th entry of $v$ respectively. It is easily verified that $Jg = \sqrt{\det([Dg]^H(Dg))} = 1$, where $Dg$ is the differential matrix, with $[Dg]_{i,j} = \frac{\partial y_j}{\partial y_i}, i = 1, \ldots, 2N_t, j = 1, \ldots, 2N_t - 1$. 

![Fig. 4. Average SER vs SNR in the $4 \times 2$ system. ($p = 0.02$ except for the error-free case)](image1)

![Fig. 5. Average SER vs feedback quality in the $4 \times 2$ system. ($\gamma_0 = 12$ dB)](image2)
Then, under the transformation $v = g(y)$,

$$
\left[ \int_{S(e_1)} vv^H dv \right]_{n_1, t} = \int_{S(e_1)} \left\{ \Re(v_n)\Re(v_t) + \Im(v_n)\Im(v_t) \right\} dv + j \left\{ \Im(v_n)\Re(v_t) - \Re(v_n)\Im(v_t) \right\} dv
$$

$$
= \int_{g} \left\{ (g_{2n-1} g_{2t-1} + g_{2n} g_{2t}) + j (g_{2n} g_{2t-1} - g_{2n-1} g_{2t}) \right\} Jg dy
$$

$$
= \int_{g} g_{2n-1} g_{2t-1} dy + \int_{g} g_{2n} g_{2t} dy + j \left( \int_{g} g_{2n} g_{2t-1} dy - \int_{g} g_{2n-1} g_{2t} dy \right)
$$

The calculation of the integrations in above equation is straightforward, e.g. $\int_{g} g_{13} dy = \int_{g} g_{2} dy = \int_{g} dy = \int_{g} (1 - \rho) dy = \frac{v}{\pi}$. Collecting the results, one obtains

$$
c_0 \int_{S(e_1)} vv^H dv = \frac{1-\rho}{N_t} I_{N_t} + \rho e_1 e_1^H. \quad (41)
$$

Substituting (41) into (40) and (39), we get (36).

**Proof of (38):** Note that $u_1$ is uniformly distributed on $\Omega_{N_t}$ [3]. Therefore, according to (36)

$$
\mathbb{E} \left( |u_n^H c_i|^2 | u_1 \in S(c_k) \right) = \frac{1-\rho}{N_t} + \rho |c_k^H c_i|^2. \quad (42)
$$

Since $\sum_{n=1}^{N_t} |u_n^H c_i|^2 = c_i^H U U^H c_i = 1$ always holds, we have

$$
\sum_{n=1}^{N_t} \mathbb{E} \left( |u_n^H c_i|^2 | u_1 \in S(c_k) \right) = 1.
$$

Moreover, conditioned on a particular realization of $u_1$, $u_n, n = 2, \cdots, N_t$, is uniformly distributed on the subspace orthogonal to $u_1$ [3] [13], i.e., they have the same distribution, so the same expectation. Then

$$
\mathbb{E} \left( |u_n^H c_i|^2 | u_1 \in S(c_k) \right), \quad n = 2, \cdots, N_t
$$

$$
= \frac{1}{N_t-1} \left[ 1 - \mathbb{E} \left( |u_1^H c_i|^2 | u_1 \in S(c_k) \right) \right]
$$

$$
= \frac{1}{N_t-1} \left( 1 - \frac{1-\rho}{N_t} - \rho |c_k^H c_i|^2 \right)
$$

$$
\leq \frac{1}{N_t-1} + \rho |c_k^H c_i|^2. \quad (43)
$$

Substituting (42) and (43) into the left-hand side of (38), the desired result is obtained, i.e.,

$$
\sum_{n=1}^{N_t} \mathbb{E} \left( |u_n^H c_i|^2 | u_1 \in S(c_k) \right) \leq \frac{1}{N_t} + \rho |c_k^H c_i|^2 \sum_{n=1}^{N_t} \lambda_n
$$

$$
= \left( \frac{1}{N_t} + \rho |c_k^H c_i|^2 \right) \sum_{n=1}^{N_t} \lambda_n
$$

$$
= \left( \frac{1}{N_t} + \rho |c_k^H c_i|^2 \right) \|H\|^2_F.
$$

**REFERENCES**


