Generating OWA Weights from Individual Assessments

José Luis García-Lapresta, Bonifacio Llamazares and Teresa Peña

Abstract In this contribution we propose a method for generating OWA weighting vectors from the individual assessments on a set of alternatives in such a way that these weights minimize the disagreement among individual assessments and the outcome provided by the OWA operator. For measuring that disagreement we have aggregated distances between individual and collective assessments by using a metric and an aggregation function. We have paid attention to Manhattan and Chebyshev metrics and arithmetic mean and maximum as aggregation functions. In this setting, we have proven that medians and the mid-range are the solutions for some cases. When a general solution is not available, we have provided some mathematical programs for solving the problem.

1 Introduction

In 1988 Yager [13] introduced OWA operators as a tool for aggregating numerical values in multi-criteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with concrete variables and, therefore, they are anonymous. Moreover, they satisfy other interesting properties, such as monotonicity, unanimity, continuity and compensativeness, i.e., the value of an OWA operator is always located between the minimum and the maximum values of the variables. Because of these properties, OWA operators have been widely used in the literature (see, for instance, Yager and Kacprzyk [18]).

Initially, the weights of an OWA operator may be fixed taking into account the importance we want to give to the assessments. So, the outcome of an OWA operator

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may be the maximum, the minimum, the average or a median of the individual assessments, among a large number of possibilities.

It is important to note that the determination of the weights of OWA operators is a relevant issue since the origins of the theory of OWA operators. In this way, Yager [13] proposes to use linguistic quantifiers for generating the OWA weights; O'Hagan [10] generates the OWA weights by maximizing their entropy whenever a degree of orness has been fixed; Filev and Yager [5] consider an exponential smoothing approach for generating the OWA weights. After these seminal papers, a large variety of techniques have been proposed in the literature (see, for instance, Yager [14, 15, 16], Wang and Parkan [11], Xu [12], Liu [8], Llamazares and García-Lapresta [9] and Ahn [1]).

In our proposal, we do not fix the OWA weighting vector, but we generate an OWA operator for each profile of individual assessments, just one that minimizes the disagreement (or equivalently, maximizes the consensus) in the group with respect to the outcome provided by the OWA operator. More concretely, once the agents opinions are known, we first calculate the distances among individual assessments on the alternatives and the collective assessments generated by an arbitrary OWA operator. Secondly, we use an aggregation function for obtaining a representative measure of disagreement from the individual assessments to the collective one. By solving a mathematical program, we obtain the weighting vector(s) that maximize(s) the consensus among individual and collective opinions.

Within the general framework we have chosen the arithmetic mean and the maximum as aggregation functions. Thus the general mathematical programs falls into the *minisum* and the *minimax* procedures, respectively (see, for instance, Brams, Kilgour and Sanver [3]).

The paper is organized as follows. Section 2 is devoted to introduce notation and some basic notions. Section 3 contains our proposal for generating an OWA operator for each profile of individual assessments and some results for the minisum and minimax procedures with two specific metrics (Manhattan and Chebyshev). Section 4 includes an illustrative example. Finally, Section 5 contains some concluding remarks.

2 Preliminaries

An *aggregation function* is a continuous mapping $A : [0,1]^m \longrightarrow [0,1]$ that satisfies the following conditions:

1. *Monotonicity*: $A(x_1,\ldots,x_m) \leq A(y_1,\ldots,y_m)$ for all $(x_1,\ldots,x_m),(y_1,\ldots,y_m) \in [0,1]^m$ such that $x_i \leq y_i$ for every $i \in \{i,\ldots,m\}$.
2. *Unanimity* (or *idempotency*): $A(x,\ldots,x) = x$ for every $x \in [0,1]$.

It is easy to see that every aggregation function is *compensative*, i.e.,

$$\min\{x_1,\ldots,x_m\} \leq A(x_1,\ldots,x_m) \leq \max\{x_1,\ldots,x_m\},$$

for every $(x_1,\dots,x_m) \in [0,1]^m$.

On aggregation functions, see Fodor and Roubens [6], Grabisch, Orlovski and Yager [7], Calvo, Kolesárova, Komorníková and Mesiar [4] and Beliakov, Pradera and Calvo [2], among others.

2.1 OWA operators

OWA operators are anonymous aggregation functions that are defined by weighting vectors in the following way.

Given a weighting vector $\boldsymbol{w} = (w_1,\dots,w_m) \in [0,1]^m$ such that $\sum_{i=1}^m w_i = 1$, the *OWA operator associated with* $\boldsymbol{w}$ is the mapping $F_{\boldsymbol{w}} : [0,1]^m \longrightarrow [0,1]$ defined by

$$F_{\boldsymbol{w}}(x_1,\dots,x_m) = \sum_{i=1}^m w_i \cdot y_i$$

where y_i is the i-th greatest number of $\{x_1,\dots,x_m\}$.

The set of weighting vectors will be denoted by

$$\mathscr{W} = \left\{ \boldsymbol{w} \in [0,1]^m \;\Big|\; \sum_{i=1}^m w_i = 1 \right\}.$$

In some cases it is interesting to consider OWA operators associated with weighting vectors satisfying specific requirements. Some of the most used in the literature are the following:

1. Weighting vectors with a fixed *orness* (or *attitudinal character*) $\alpha \in (0,1)$ (see Yager [13] and O'Hagan [10]):

$$\mathscr{W}^1_\alpha = \left\{ \boldsymbol{w} \in \mathscr{W} \;\Big|\; \frac{1}{m-1} \sum_{i=1}^m (m-i) w_i = \alpha \right\}.$$

2. Symmetric weights:

$$\mathscr{W}^2 = \left\{ \boldsymbol{w} \in \mathscr{W} \mid w_i = w_{m+1-i} \quad \forall i \in \left\{1,\dots,\left[\tfrac{m}{2}\right]\right\} \right\}.$$

3. Centered weights (after Yager [17]):

$$\mathscr{W}^3 = \left\{ \boldsymbol{w} \in \mathscr{W}^2 \mid w_1 \leq w_2 \leq \cdots \leq w_{\left[\frac{m+1}{2}\right]} \right\}.$$

4. Trimmed weights:

$$\mathscr{W}^4_1 = \{ \boldsymbol{w} \in \mathscr{W} \mid w_1 = w_m = 0 \}.$$
$$\mathscr{W}^4_2 = \{ \boldsymbol{w} \in \mathscr{W} \mid w_1 = w_2 = w_{m-1} = w_m = 0 \}.$$

. . . .

$$\mathscr{W}^4_{\left[\frac{m-1}{2}\right]} = \left\{ \boldsymbol{w} \in \mathscr{W} \mid w_1 = \cdots = w_{\left[\frac{m-1}{2}\right]} = w_{\left[\frac{m}{2}\right]+2} = \cdots = w_m = 0 \right\}.$$

It is worth noting that $\mathscr{W}^3 \subseteq \mathscr{W}^2 \subseteq \mathscr{W}^1_{0.5} \subseteq \mathscr{W}$ and $\mathscr{W}^4_{\left[\frac{m-1}{2}\right]} \subseteq \cdots \subseteq \mathscr{W}^4_2 \subseteq \mathscr{W}^4_1$.

Some well-known aggregation functions are specific cases of OWA operators. For instance:

1. The *maximum*, given by the weighting vector $(1,0,\ldots,0)$.
2. The *minimum*, given by the weighting vector $(0,\ldots,0,1)$.
3. The *arithmetic mean*, given by the weighting vector $\left(\frac{1}{m},\ldots,\frac{1}{m}\right)$.
4. The *mid-range*, given by the weighting vector $\widehat{\boldsymbol{w}} = (0.5,0,\ldots,0,0.5)$.
5. *Medians*, given by the following weighting vectors $\widetilde{\boldsymbol{w}}$:

   a. If m is odd

$$\widetilde{w}_i = \begin{cases} 1, \text{ if } i = \frac{m+1}{2}, \\ 0, \text{ otherwise.} \end{cases}$$

   b. If m is even

$$\widetilde{w}_i = \begin{cases} \theta, & \text{if } i = \frac{m}{2}, \\ 1-\theta, & \text{if } i = \frac{m}{2}+1, \\ 0, & \text{otherwise,} \end{cases}$$

   for some $\theta \in [0,1]$.

   Notice that $\widetilde{\boldsymbol{w}} \in \mathscr{W}^4_{\left[\frac{m-1}{2}\right]}$.

2.2 Collective assessments

Consider a set of agents (experts or voters) $V = \{1,\ldots,m\}$ $(m \geq 2)$ who show their opinions on a set of alternatives $A = \{a_1,\ldots,a_n\}$ $(n \geq 2)$ through numbers in the interval $[0,1]$.

A *profile* is a $m \times n$ matrix

$$P = \begin{pmatrix} a_1^1 & \cdots & a_j^1 & \cdots & a_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^i & \cdots & a_j^i & \cdots & a_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^m & \cdots & a_j^m & \cdots & a_n^m \end{pmatrix}$$

where $a_j^i \in [0,1]$ is the assessment that agent i assigns to alternative a_j. The set of profiles is denoted by $\mathscr{P}$.

In order to aggregate individual assessments, we consider an OWA operator $F_{\boldsymbol{w}}$ associated with a weighting vector $\boldsymbol{w} \in \mathscr{W}^*$, where $\mathscr{W}^*$ may be $\mathscr{W}$ or any of the subsets of weighting vectors mentioned in the previous subsection. Taking into account the j-th column of P, $(a_j^1, \ldots, a_j^m)$, that includes individual opinions on the alternative a_j, we generate the collective assessment on that alternative through $F_{\boldsymbol{w}}$

$$v_j(\boldsymbol{w}) = F_{\boldsymbol{w}} \left(a_j^1, \ldots, a_j^m\right).$$

With $\boldsymbol{v}(\boldsymbol{w}) = (v_1(\boldsymbol{w}), \ldots, v_n(\boldsymbol{w}))$ we denote the vector that contains the collective assessments on the alternatives of A generated by the OWA operator $F_{\boldsymbol{w}}$. On the other hand, the i-th row of P includes the assessments of individual $i \in \{1, \ldots, m\}$ on the alternatives of A and will be denoted by $\boldsymbol{a}^i = \left(a_1^i, \ldots, a_n^i\right)$.

3 A model for generating OWA weights

In our proposal, we do not fix the OWA weighting vector, but we generate an OWA operator for each profile of individual assessments, just one that maximizes the consensus (or, equivalently, minimizes the disagreement) in the group with respect to the outcome provided by the OWA operator. For this, it is necessary to fix two ingredients:

- A metric $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$.
- An aggregation function $A : [0,1]^m \longrightarrow [0,1]$.

By means of the metric d, we calculate the distances among individual assessments on the alternatives and the collective assessments generated by an OWA operator associated with a weighting vector belonging to $\mathscr{W}^*$. On the other hand, we use an aggregation function for obtaining a representative measure of disagreement from the individual assessments to the collective one. Thus, given a profile $P \in \mathscr{P}$, we propose to find weighting vector(s) $\boldsymbol{w} \in \mathscr{W}^*$ being solution(s) of the following mathematical program:

$$\begin{aligned} \min \quad & A\Big(d(\boldsymbol{a}^1, \boldsymbol{v}(\boldsymbol{w})), \ldots, d(\boldsymbol{a}^m, \boldsymbol{v}(\boldsymbol{w}))\Big) \\ \text{s. t.}: \quad & \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{1}$$

Notice that from continuity of A and compactness of $\mathscr{W}^*$, the existence of solution(s) in (1) is always guaranteed.

Among the large variety of aggregation functions and metrics that we may use in (1), we present with more detail those cases where the aggregation functions are the arithmetic mean and the maximum in combination with Manhattan and Chebyshev metrics. These metrics are widely used in the literature and allow us to obtain the OWA weighting vector solving simple linear optimization problems.

The *Manhattan metric* is defined by

$$d_1\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \sum_{i=1}^{n} |x_i - y_i|.$$

The *Chebyshev metric* is defined by

$$d_\infty\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) = \max\{|x_1-y_1|,\ldots,|x_n-y_n|\}.$$

3.1 Minisum outcomes

If we consider the arithmetic mean as aggregation function, then (1) is equivalent to find the weighting vectors that minimize the sum of distances between the individual assessments and the collective assessments generated by the OWA operator associated with those weighting vectors. In other words, (1) becomes

$$\begin{aligned} &\min \quad \sum_{i=1}^{m} d\Big((a_1^i,\ldots,a_n^i),(v_1(\boldsymbol{w}),\ldots,v_n(\boldsymbol{w}))\Big) \\ &\text{s. t. :} \quad \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{2}$$

1. If we use the Manhattan metric, then (2) is transformed in the following mathematical program:

$$\begin{aligned} &\min \quad \sum_{i=1}^{m} \Big(|a_1^i - v_1(\boldsymbol{w})| + \cdots + |a_n^i - v_n(\boldsymbol{w})|\Big) \\ &\text{s. t. :} \quad \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{3}$$

2. If we use the Chebyshev metric, then (2) is now transformed in the following mathematical program:

$$\begin{aligned} &\min \quad \sum_{i=1}^{m} \max\Big\{ |a_1^i - v_1(\boldsymbol{w})|,\ldots,|a_n^i - v_n(\boldsymbol{w})| \Big\} \\ &\text{s. t. :} \quad \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{4}$$

In the first case, i.e., in Problem (3), it is possible to give the analytical solution of the problem when $\mathscr{W}^* = \mathscr{W}$.

Proposition 1. *For $\mathscr{W}^* = \mathscr{W}$, the solutions of Problem (3) are the medians.*

Proof. Taking into account $\mathscr{W}^* = \mathscr{W}$ in Problem (3), we obtain the following mathematical program:

$$\min_{\boldsymbol{w}\in\mathscr{W}} \sum_{i=1}^{m}\sum_{j=1}^{n} |a_j^i - v_j(\boldsymbol{w})| = \min_{\boldsymbol{w}\in\mathscr{W}} \sum_{j=1}^{n}\sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})|. \tag{5}$$

Moreover, the following inequality is satisfied:

$$\min_{\boldsymbol{w}\in\mathscr{W}} \sum_{j=1}^{n}\sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})| \geq \sum_{j=1}^{n} \min_{\boldsymbol{w}\in\mathscr{W}} \sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})|.$$

On the other hand, it is known that given $x_1,\ldots,x_m \in \mathbb{R}$, medians are the solutions of the following problem:

$$\min_{x\in\mathbb{R}} \sum_{i=1}^{m} |x_i - x|.$$

Therefore, they are also the solution of

$$\min_{\boldsymbol{w}\in\mathscr{W}} \sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})|$$

for every $j \in \{1,\ldots,n\}$. Consequently,

$$\sum_{j=1}^{n} \min_{\boldsymbol{w}\in\mathscr{W}} \sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})| = \min_{\boldsymbol{w}\in\mathscr{W}} \sum_{j=1}^{n} \sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})|$$

and medians are the solutions of Problem (5). □

Since medians belong to $\mathscr{W}_i^4$, $i \in \left\{1,\ldots,\left[\frac{m-1}{2}\right]\right\}$, they are also the solutions of Problem (3) when $\mathscr{W}^* = \mathscr{W}_i^4$. Moreover, the median for $\theta = 0.5$ also belongs to $\mathscr{W}^2$ and $\mathscr{W}^3$. Therefore, the solution of Problem (3) when $\mathscr{W}^* = \mathscr{W}^2$ or $\mathscr{W}^* = \mathscr{W}^3$ is the median with $\theta = 0.5$.

In relation to $\mathscr{W}_\alpha^1$, we are going to calculate the orness of medians. We distinguish two cases:

1. If m is odd:

$$orness(\widetilde{\boldsymbol{w}}) = \frac{1}{m-1}\sum_{i=1}^{m}(m-i)\widetilde{w}_i = \frac{1}{m-1}\,\frac{m-1}{2} = 0.5.$$

2. If m is even:

$$\begin{aligned} orness(\widetilde{\boldsymbol{w}}) &= \frac{1}{m-1}\sum_{i=1}^{m}(m-i)\widetilde{w}_i = \frac{1}{m-1}\left(\frac{m}{2}\theta + \left(\frac{m}{2}-1\right)(1-\theta)\right) \\ &= \frac{1}{m-1}\left(\frac{m}{2} - 1 + \theta\right). \end{aligned}$$

In the first case, if $\alpha = 0.5$, then the median belongs to $\mathscr{W}_{0.5}^1$ and is the solution of Problem (3) when $\mathscr{W}^* = \mathscr{W}_{0.5}^1$.

In the second case, the minimum and the maximum orness are reached when $\theta = 0$ and $\theta = 1$, respectively. Therefore, if m is even, then

$$\frac{m-2}{2(m-1)} \le orness(\widetilde{\boldsymbol{w}}) \le \frac{m}{2(m-1)}.$$

Consequently, for these values of α, the median, with $\theta = (m-1)\alpha + 1 - \frac{m}{2}$, belongs to $\mathscr{W}_\alpha^1$ and is the solution of Problem (3) when $\mathscr{W}^* = \mathscr{W}_\alpha^1$.

When m is odd and $\alpha \neq 0.5$ or when m is even and $\alpha \notin \left[\frac{m-2}{2(m-1)}, \frac{m}{2(m-1)}\right]$, it is possible to find a numerical solution of Problem (3). For this, we can replace this problem by the following equivalent smooth linear problem that uses some auxiliary variables:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} (\lambda_1^i + \cdots + \lambda_n^i) \\ \text{s.t.}: \quad & -\lambda_1^i \leq v_1(\boldsymbol{w}) - a_1^i \leq \lambda_1^i, \qquad i = 1,\dots,m \\ & \dots\dots\dots\dots\dots\dots\dots\dots \\ & -\lambda_n^i \leq v_n(\boldsymbol{w}) - a_n^i \leq \lambda_n^i, \qquad i = 1,\dots,m \\ & \lambda_j^i \geq 0, \qquad i = 1,\dots,m,\ j = 1,\dots,n \\ & \boldsymbol{w} \in \mathscr{W}_\alpha^1 \end{aligned} \tag{6}$$

In a similar way, when Problem (4) is considered, i.e., when the Chebyshev metric is used, the solution of the problem can be obtained by solving the following equivalent smooth linear problem that uses some auxiliary variables:

$$\begin{aligned} \min \quad & \lambda_1 + \cdots + \lambda_m \\ \text{s.t.}: \quad & -\lambda_1 \leq v_j(\boldsymbol{w}) - a_j^1 \leq \lambda_1, \qquad j = 1,\dots,n \\ & \dots\dots\dots\dots\dots\dots\dots\dots \\ & -\lambda_m \leq v_j(\boldsymbol{w}) - a_j^m \leq \lambda_m, \qquad j = 1,\dots,n \\ & \lambda_i \geq 0, \qquad i = 1,\dots,m \\ & \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{7}$$

3.2 Minimax outcomes

We now consider the maximum as aggregation function. In this case, (1) is equivalent to find the weighting vectors that minimize the maximum distance between the individual assessments and the collective assessments generated by the OWA operator associated with those weighting vectors. In other words, (1) becomes

$$\begin{aligned} \min \quad & \max_{i=1,\dots,m} \left\{ d\Big((a_1^i, \dots, a_n^i), (v_1(\boldsymbol{w}), \dots, v_n(\boldsymbol{w})) \Big) \right\} \\ \text{s. t.}: \quad & \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{8}$$

1. If we use the Manhattan metric, then (8) is transformed in the following mathematical program:

$$\begin{aligned} \min \quad & \max_{i=1,\ldots,m} \Big\{ |a_1^i - v_1(\boldsymbol{w})| + \cdots + |a_n^i - v_n(\boldsymbol{w})| \Big\} \\ \text{s. t.}: \quad & \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{9}$$

2. If we use the Chebyshev metric, then (8) is transformed in the following mathematical program:

$$\begin{aligned} \min \quad & \max_{i=1,\ldots,m} \max \Big\{ |a_1^i - v_1(\boldsymbol{w})|, \ldots, |a_n^i - v_n(\boldsymbol{w})| \Big\} \\ \text{s. t.}: \quad & \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{10}$$

Problem (9) is equivalent to solve the following smooth linear problem that uses some auxiliary variables:

$$\begin{aligned} \min \quad & \delta \\ \text{s.t.}: \quad & \lambda_1^i + \cdots + \lambda_n^i \le \delta, && i = 1, \ldots, m \\ & -\lambda_1^i \le v_1(\boldsymbol{w}) - a_1^i \le \lambda_1^i, && i = 1, \ldots, m \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & -\lambda_n^i \le v_n(\boldsymbol{w}) - a_n^i \le \lambda_n^i, && i = 1, \ldots, m \\ & \lambda_j^i \ge 0, && i = 1, \ldots, m,\ j = 1, \ldots, n \\ & \boldsymbol{w} \in \mathscr{W}^* \end{aligned} \tag{11}$$

In the following proposition, we show that the solution of Problem (10) is the average between the maximum and the minimum, i.e., the mid-range, whenever $\mathscr{W}^* = \mathscr{W}$.

Proposition 2. *For $\mathscr{W}^* = \mathscr{W}$, the solution of Problem (10) is the mid-range.*

Proof. When $\mathscr{W}^* = \mathscr{W}$ in Problem (10), we obtain the following mathematical program:

$$\min_{\boldsymbol{w} \in \mathscr{W}} \max_{i=1,\ldots,m} \max_{j=1,\ldots,n} \Big\{ |a_j^i - v_j(\boldsymbol{w})| \Big\} = \min_{\boldsymbol{w} \in \mathscr{W}} \max_{j=1,\ldots,n} \max_{i=1,\ldots,m} \Big\{ |a_j^i - v_j(\boldsymbol{w})| \Big\}. \tag{12}$$

Moreover, the following inequality is satisfied:

$$\min_{\boldsymbol{w} \in \mathscr{W}} \max_{j=1,\ldots,n} \max_{i=1,\ldots,m} \Big\{ |a_j^i - v_j(\boldsymbol{w})| \Big\} \ge \max_{j=1,\ldots,n} \min_{\boldsymbol{w} \in \mathscr{W}} \max_{i=1,\ldots,m} \Big\{ |a_j^i - v_j(\boldsymbol{w})| \Big\}.$$

On the other hand, it is clear that for every $j \in \{1, \ldots, n\}$, the weighting vector $\widehat{\boldsymbol{w}}$ is the solution of the following problem:

$$\min_{\boldsymbol{w} \in \mathscr{W}} \max_{i=1,\ldots,m} \Big\{ |a_j^i - v_j(\boldsymbol{w})| \Big\}.$$

Therefore,

$$\max_{j=1,\ldots,n} \min_{\boldsymbol{w}\in\mathscr{W}} \max_{i=1,\ldots,m} \left\{ |a^i_j - v_j(\boldsymbol{w})| \right\} = \min_{\boldsymbol{w}\in\mathscr{W}} \max_{j=1,\ldots,n} \max_{i=1,\ldots,m} \left\{ |a^i_j - v_j(\boldsymbol{w})| \right\}$$

and $\widehat{\boldsymbol{w}}$ is also the solution of Problem (12). □

Since $\widehat{\boldsymbol{w}}$ belongs to $\mathscr{W}^1_{0.5}$ and $\mathscr{W}^2$, it is also the solution of Problem (10) for $\mathscr{W}^* = \mathscr{W}^1_{0.5}$ and $\mathscr{W}^* = \mathscr{W}^2$. On the other hand, when $\mathscr{W}^*$ is any of the remaining subsets, the solution of Problem (10) can be obtained by solving the following equivalent linear problem:

$$\begin{aligned}
&\min \quad \delta \\
&\text{s.t.}: \quad -\delta \le v_1(\boldsymbol{w}) - a^i_1 \le \delta, \qquad i = 1,\ldots,m, \\
&\qquad\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad\quad -\delta \le v_n(\boldsymbol{w}) - a^i_n \le \delta, \qquad i = 1,\ldots,m, \\
&\qquad\quad \boldsymbol{w} \in \mathscr{W}^*
\end{aligned} \tag{13}$$

4 An illustrative example

In this section we provide an example to show the solutions obtained when the previous models are used. Suppose that the opinions of four experts on three alternatives are given by the following matrix:

$$P = \begin{pmatrix} 0.7 & 0.6 & 0.1 \\ 0 & 0.5 & 0.8 \\ 0.6 & 0.1 & 1 \\ 0.6 & 0.7 & 0 \end{pmatrix}$$

From the individuals' preferences shown in this matrix, the collective assessment on each alternative generated by an OWA operator is given by

$$\begin{aligned}
v_1(\boldsymbol{w}) &= 0.7w_1 + 0.6w_2 + 0.6w_3, \\
v_2(\boldsymbol{w}) &= 0.7w_1 + 0.6w_2 + 0.5w_3 + 0.1w_4, \\
v_3(\boldsymbol{w}) &= w_1 + 0.8w_2 + 0.1w_3.
\end{aligned}$$

Next, using the models proposed in the previous section, we seek the weighting vector (belonging to some of the weighting vector sets given in Subsection 2.1) that minimizes the disagreement among individual assessments on the alternatives and the collective assessments. It is worth noting that when the set of weighting vector is $\mathscr{W}^1_\alpha$, we show the solutions for $\alpha = 0.75$, $\alpha = 0.5$ and $\alpha = 0.25$. On the other hand, when the analytical solution is unknown, we have used LINGO software to solve the corresponding linear programming problems.

4.1 Minisum outcomes

When the arithmetic mean and the Manhattan metric are used, the solutions obtained are shown in Table 1. It is worth noting that we have solved Problem (6) to obtain the solution when the set of weighting vectors is $\mathscr{W}^1_{0.75}$ or $\mathscr{W}^1_{0.25}$. For the remaining sets of weighting vectors, the solution is already known (see Subsection 3.1).

Table 1 Solutions for the arithmetic mean and the Manhattan metric

	w_1	w_2	w_3	w_4
$\mathscr{W}$	0	θ	$1-\theta$	0
$\mathscr{W}^1_{0.75}$	0.5	0.3334	0.0833	0.0833
$\mathscr{W}^1_{0.5}$	0	0.5	0.5	0
$\mathscr{W}^1_{0.25}$	0	0	0.75	0.25
$\mathscr{W}^2$	0	0.5	0.5	0
$\mathscr{W}^3$	0	0.5	0.5	0
$\mathscr{W}^4_1$	0	θ	$1-\theta$	0

When the arithmetic mean and the Chebyshev metric are used, the solutions can be obtained solving Problem (7) for the different sets of weighting vectors considered in this contribution. Table 2 summarizes these solutions.

Table 2 Solutions for the arithmetic mean and the Chebyshev metric

	w_1	w_2	w_3	w_4
$\mathscr{W}$	0.4706	0	0	0.5294
$\mathscr{W}^1_{0.75}$	0.75	0	0	0.25
$\mathscr{W}^1_{0.5}$	0.5	0	0	0.5
$\mathscr{W}^1_{0.25}$	0	0.375	0	0.625
$\mathscr{W}^2$	0.5	0	0	0.5
$\mathscr{W}^3$	0.25	0.25	0.25	0.25
$\mathscr{W}^4_1$	0	0.1429	0.8571	0

4.2 Minimax outcomes

When the arithmetic mean and the Manhattan metric are used, the solutions can be obtained by solving Problem (11) for the different sets of weighting vectors considered in this contribution. These solutions are given in Table 3.

Table 3 Solutions for the maximum and the Manhattan metric

	w_1	w_2	w_3	w_4
$\mathscr{W}$	0.6667	0	0.2222	0.1111
$\mathscr{W}^1_{0.75}$	0.673	0	0.2307	0.0963
$\mathscr{W}^1_{0.5}$	0	0.7222	0.0556	0.2222
$\mathscr{W}^1_{0.25}$	0	0.375	0	0.625
$\mathscr{W}^2$	0	0.5	0.5	0
$\mathscr{W}^3$	0	0.5	0.5	0
$\mathscr{W}^4_1$	0	0.8334	0.1666	0

When the Manhattan metric is replaced by the Chebyshev metric, the solution is known if the set of weighting vector is $\mathscr{W}$, $\mathscr{W}^1_{0.5}$ or $\mathscr{W}^2$. In other cases, the solutions can be obtained by solving Problem (13). Table 4 shows these solutions.

Table 4 Solutions for the maximum and the Chebyshev metric

	w_1	w_2	w_3	w_4
$\mathscr{W}$	0.5	0	0	0.5
$\mathscr{W}^1_{0.75}$	0.625	0	0.375	0
$\mathscr{W}^1_{0.5}$	0.5	0	0	0.5
$\mathscr{W}^1_{0.25}$	0	0.375	0	0.625
$\mathscr{W}^2$	0.5	0	0	0.5
$\mathscr{W}^3$	0.25	0.25	0.25	0.25
$\mathscr{W}^4_1$	0	0.7143	0.2857	0

5 Concluding remarks

In this contribution we have proposed a general method for generating weighting vectors of OWA operators from the individual assessments in such a way that the obtained OWA operator maximizes the consensus among the agents. This endogenous procedure is based on a metric and an aggregation function that provide a measure of the disagreement among the agents with respect to the outcome provided by the OWA operator in each case. We have paid special attention to Manhattan and Chebyshev metrics and the arithmetic mean and maximum as aggregation functions. In these cases, the outcomes may be obtained by solving some mathematical linear programs. Moreover, in some specific situations we have obtained that medians and mid-range are the solutions of the problems.

It is worth noting that the solutions obtained in the above mentioned models might not be unique. Even more, depending the weighting vector we choose, the outcomes provided by the corresponding OWA operator could be different. In this way, it would be necessary to provide an appropriate procedure for choosing a single weighting vector among the set of multiple solutions.

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