A new prioritized multi-criteria outranking method: The prioritized PROMETHEE

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Abstract: The Multiple Criteria Decision Aiding (MCDA) has been a fast growing area of operations research and management science during the last decades, while the PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) is one of the well-known MCDA methods with ranking the alternatives from the best to the worst. In this paper, we propose a new outranking method called Prioritized PROMETHEE (P-PROMETHEE) method, which is used to deal with the ranking problem of Prioritized Multiple Criteria Decision Making (PMCDM). First, we use the idea of entropy to get the information of each criterion in the same prioritized level, which is used to get the weight of each criterion. Then, we use the weighted average operator to aggregate all the criteria in the same prioritized level. Moreover, we apply Yager’s prioritized measure-guided aggregation operator to the classical PROMETHEE method to get the P-PROMETHEE method. Finally, we give a practical example to illustrate the effectiveness of the P-PROMETHEE method.

Keywords: Entropy, prioritized measure, PROMETHEE method, weight vector, choquet integral

1. Introduction

As we know, the PROMETHEE method developed by Brans et al. [4, 6], is one of the effective outranking methods to solve the MCDM problems. As the PROMETHEE method has the properties of simplicity, clarityness and stability, it has been widely used to solve practical multi-criteria decision making problems, such as environment management, business and financial management, manufacturing and assembly, etc. [1, 2, 20].

The PROMETHEE method is based on the pairwise comparison of alternatives with respect to each criterion to get the ranking of all the alternatives. The PROMETHEE method includes: PROMETHEE I for partial ranking of the alternatives and PROMETHEE II for complete ranking of the alternatives [4]. Later, many useful extensions have been developed to enrich the PROMETHEE method. For instance, Brans and Mareschal developed PROMETHEE III for ranking based on intervals and PROMETHEE IV for solving a choice problem of an infinite set of alternatives. In 1988, Brans and Mareschal [15] proposed the visual interactive module GAIA (Geometrical Analysis for Interactive Aid) which provides a marvellous graphical representation supporting the PROMETHEE method. PROMETHEE V for MCDA including segmentation constraints and PROMETHEE VI for representation of the human brain was also developed by Brans and Mareschal [5, 16] in 1992 and 1995 respectively.

However, in order to implement the PROMETHEE method, we require three types of additional information: preference function shape, weights of criteria and thresholds. As Corrente et al. [10] pointed out, the preference information can be obtained through two ways: directly and indirectly. The Decision Maker (DM) can get all the preference information between alternatives when he provides directly all values of parameters in the model. The DM only provides some preference...
information between alternatives from which we get indirectly the compatible preference parameters. Obviously, we can find out that the indirect preference information needs less information from the DM than the direct preference information. Meanwhile, the indirect preference information is also relatively easy given by the DM. For this reason, the study of the PROMETHEE method based on indirect preference information is meaningful.

Up to now, many methods have been proposed to elicit the preference parameters. For example, Solymosi and Dombi [22] proposed the centralized weight of compatible preference parameters to represent the relative importance of criteria. Mareschal [14] gave a sensitivity analysis in order to obtain stability interval weights for criteria, in which we can get the same ranking of alternatives. Sun and Han [23] solved a linear programming problem to find the most discriminating set of weights compatible with the preference information provided by the DM. Eppe et al. [12] proposed a bi-objective optimization model which considers the number of inconsistencies and the robustness of parameter values to elicit the PROMETHEE II’s preference parameters.

The above mentioned papers mainly consider the MCDM problem with the criteria in the same level. Meanwhile, there are many methods to obtain the total satisfaction degree for each alternative to all criteria, such as the weighted average operator [27], the ordered weighted averaging (OWA) operator [22] and the ordered weighted geometric (OWG) operator [9, 25]. These aggregation operators allow to tradeoff between criteria. For example, if we use the weighted average operator, the total satisfaction degree can be obtained by $C(x) = \sum_{i=1}^{n} w_i C_i(x)$, where $C_i(x)$ represents the satisfaction degree of the alternative $x$ to the criterion $C_i$, and $w_i$ denotes the weight of the criterion $C_i$. In this case, we can compensate for a decrease of $A$ in satisfaction to the criterion $C_i$ by a gain $(w_j/w_i)A$ in satisfaction to the criterion $C_j$. However, there are lots of MCDM problems with the prioritized relation of criteria and we usually permit no compensation between criteria. Yager [28, 29] listed several practical MCDM problems to illustrate this kind of situations, such as selecting a bicycle, an organization decision-making problem, and a document retrieval problem. The typical example is the case of buying a car with respect to the criteria of safety and cost. In this case, the criterion safety has a higher priority than the criterion cost, it indicates that we are not willing to tradeoff the satisfaction of the criterion cost until the satisfaction of safety arrives some given level. For this situation, some methods have been developed to overcome this drawback [8, 19, 24, 28–30, 32, 33]. For example, Yager [28, 29] used the prioritized weights in which the weights associated with the lower prioritized criteria are related to the satisfaction of the higher prioritized criteria to express the prioritized relation between criteria. Based on Yager’s prioritized weights, Wei and Tang [24] proposed a generalized prioritized averaging operator and a generalized prioritized OWA operator. Yan [32] proposed a prioritized weighted aggregation operator based on the ordered weighted averaging (OWA) operator and the triangular norms (t-norms) to deal with multi-criteria decision making (MCDM) problems with multiple priorities. Yager [30] developed a prioritized aggregation operator which uses the monotonic measure set to convey the prioritization relationship. Based on Yager [30]’s idea, Chen [8] developed a general prioritized aggregation operator which allows the partial compensation between prioritized criteria.

However, the mentioned literature focuses on dealing with the prioritized MCDM problems, few papers pay attention to the outranking method with the prioritized relationship of criteria. As we know, Yu et al. [34] developed a new function which takes into account the influence of expectation levels to the prioritized weights in the PROMETHEE method. However, Yu et al. [34] only investigated the strict ordered prioritized PROMETHEE method. In this paper, we will mainly consider some much more general problems with the weak ordered prioritized criteria. As the above mentioned, the PROMETHEE method is a well-known outranking method. In order to meet the needs of weak ordered prioritized relationship, we will propose a new outranking method called Prioritized PROMETHEE method, which is an integrated approach between the prioritized measure and the PROMETHEE method.

The rest of the paper is organized as follows: Section 2 describes the classical PROMETHEE II. In Section 3, we recall Yager’s prioritized measure-guided outranking method. In Section 4, we propose a P-PROMETHEE method and employ a practical example to illustrate its effectiveness. Section 5 ends the paper with some conclusions.

2. The classical PROMETHEE methods

In what follows, we first introduce the basic principles and some basic concepts related to the PROMETHEE method (see Refs. [4, 6] for more
where $\pi$ represents the total degree of preference of the alternative $a_i$ to the alternative $a_j$ under the criterion $C_l$. We can write the function as follows:

$$P_l(a_i, a_j) = P_l(d(a_i, a_j)), \text{ for } a_i, a_j \in X$$  \hspace{1cm} (1)

where $d(a_i, a_j) = f(a_i) - f(a_j)$ and $P_l(\cdot)$ is a monotonically non-decreasing function varying from 0 to 1.

In order to facilitate the selection of the preference function, Brans and Vincke [7] proposed six basic types of preference functions: (1) usual criterion, (2) U-shape criterion, (3) V-shape criterion, (4) level criterion, (5) V-shape with indifference criterion, (6) Gaussian criterion. Certainly, the DM has to give the value of an indifference threshold $\psi_l$, and the value of a strict preference threshold $\rho_l$ for each criterion.

For any two alternatives $a_i, a_j \in X$, we have

$$\pi(a_i, a_j) = \sum_{l=1}^{n} w_l \cdot P_l(a_i, a_j)$$  \hspace{1cm} (2)

where $\pi(a_i, a_j)$ represents the total degree of preference of the alternative $a_i$ to the alternative $a_j$ taking into account all criteria. The weight $w_l$ represents the relative importance of the criterion $C_l$.

In order to get the ranking of all the alternatives, the PROMETHEE I method defines the positive outranking flow and negative outranking flow as follows:

$$\phi^+(a_i) = \frac{1}{m-1} \sum_{a_j \in X \setminus \{a_i\}} \pi(a_i, a_j)$$

$$\phi^-(a_i) = \frac{1}{m-1} \sum_{a_j \in X \setminus \{a_i\}} \pi(a_j, a_i)$$

where the positive outranking flow $\phi^+(a_i)$ represents how much the alternative $a_i$ prefers to all the other alternatives. The larger $\phi^+(a_i)$, the better the alternative $a_i$. Similarly, the negative outranking flow $\phi^-(a_i)$ represents how much all the other alternatives prefers to the alternative $a_i$. The smaller $\phi^-(a_i)$, the better the alternative $a_i$. The PROMETHEE I can get a partial ranking according to the value of the positive outranking flow and the value of negative outranking flow for all alternatives. The PROMETHEE II can get a complete ranking according to the value of net outranking flow for each alternative, which is obtained by $\phi(a_i) = \phi^+(a_i) - \phi^-(a_i)$.

3. Prioritized measure-guided aggregation operator

In order to capture the prioritized relationship of criteria, Yager [30] adopted the fuzzy measure and the fuzzy integral to develop a prioritized measure-guided aggregation operator, which has the characteristic that the higher prioritized criteria’s satisfaction can’t be compensated by the lower prioritized criteria’s satisfaction. Now we first give the definition of fuzzy measure:

Definition 1. [3, 17] Let $N = \{1, 2, \ldots, n\}$. A discrete fuzzy measure is a set function $\mu : 2^N \rightarrow [0, 1]$ which is monotonic (i.e., $\mu(A) \leq \mu(B)$ whenever $A \subseteq B \subseteq N$) and satisfies $\mu(\emptyset) = 0$ and $\mu(N) = 1$.

Fuzzy measure was first introduced by Sugeno [17], and has usually been used to deal with the MCDM problems associated with fuzzy integral. Meanwhile, fuzzy measures are not only used to represent the independent relationships of criteria, but also the dependent relationships of criteria.

Furthermore, Yager and Walker [31] first used a special kind of fuzzy measure to capture the strictly ordered prioritizations. They gave the following definitions:

$$L_j = \begin{cases} \{C_k \mid k = 1, 2, \ldots, j\}, & j = 1, \ldots, n \setminus \{0\} \\ \emptyset, & j = 0 \end{cases}$$  \hspace{1cm} (3)

where we assume $C_1 > C_2 > \cdots > C_n$ in the set of criteria. They are associated with each subset $L_j$ the value $j/n$. Then the prioritized measure $m : 2^C \rightarrow [0, 1]$ can be defined as:

$$m(A) = \max_j \left( \frac{j}{n} G_j(A) \right)$$  \hspace{1cm} (4)

where $A$ is a subset of $C$ and

$$G_j(A) = \begin{cases} 1, & L_j \subseteq A \\ 0, & L_j \not\subseteq A \end{cases}$$  \hspace{1cm} (5)

Besides, the prioritized measure can also be written as:

$$\mu(A) = \sum_{j=1}^{n} \frac{1}{n} G_j(A)$$  \hspace{1cm} (6)
The discrete Choquet integral is an adequate aggregation function that extends the OWA operator by taking the interactions of criteria into account.

Definition 2. [3, 13] The discrete Choquet integral with respect to a fuzzy measure \( \mu \) is given by

\[
\text{Choq}_\mu(a) = \sum_{j=1}^{n} a_j \left[ \mu(\{j\}, a_j \geq a_{j-1}) - \mu(\{j\}, a_j = a_{j-1}) \right]
\]

(7)

where \( a = (a_1, a_2, \ldots, a_n) \) is a non-increasing permutation of the input \( a \), and \( a_0 = \infty \) by convention.

As we know, Choquet integral is a typical type of fuzzy integrals, and it can be used to aggregate information from a set of resources with respect to a given fuzzy measure. In order to get the prioritized relationship of criteria, Yager and Walker [31] used Choquet integral to obtain an aggregation based on the prioritized measure. Thus, we can use the prioritized aggregation operator to calculate the overall satisfaction degree \( C(x) \) for each alternative \( x \) by

\[
C(x) = \text{Choq}_\mu(C_1(x), C_2(x), \ldots, C_n(x))
\]

(8)

where \( C_j = (C_{\text{ord}1}, C_{\text{ord}2}, \ldots, C_{\text{ord}j}) \), and \( \text{ind}(j) \) is the index of the \( j \)-th largest \( C_j(x) \). The measure \( \mu(x) \) is given by the formula (5).

The concept of entropy was first introduced by Shannon [21] to measure the uncertainty of a random variable. The entropy is very important for measuring the uncertain information in information theory. Recently, many scholars take the idea of entropy into the area of decision making, for example, O'Hagan [18] posed an entropy-based procedure to derive attribute weight vector of the OWA operator, Xu and Hu [26] suggested a maximum entropy method to determine the weight vector of the OWA operator, Xu and Hu [26] proposed an entropy-based procedure to derive attribute weights under intuitionistic fuzzy environment. The definition of entropy is given as follows:

Definition 3. [3, 21] For a given weighting vector \( w = (w_1, w_2, \ldots, w_n) \), where \( w_i \in [0, 1] \), and \( \sum_{i=1}^{n} w_i = 1 \), then the entropy of this weighting vector is defined by

\[
E(w) = -\sum_{i=1}^{n} w_i \ln w_i
\]

(9)

with the convention \( 0 \cdot \ln 0 = 0 \).

4. The prioritized PROMETHEE method

As Section 2 pointed out, the classical PROMETHEE method is a simple and effective outranking method. From the formula (1), we can find that the classical PROMETHEE method allow the tradeoff between criteria in order to get the total preference relationship of alternatives. However, this paper mainly studies a special ranking problem with the prioritized relationship between criteria. Since the higher prioritized criteria’s satisfaction can't be compensated by the lower prioritized criteria in the prioritized MCDM problems, the classical PROMETHEE method is not suitable to deal with the prioritized MCDM problems. Yager [30, 31] first introduced the prioritized measure and the prioritized measure-guided aggregation operator to deal with the prioritized MCDM problems. Fortunately, Yager’s prioritized measure-guided aggregation operator can capture the prioritized relationship between criteria, which means the lack of satisfaction to the higher prioritized criteria cannot be compensated by the increased satisfaction of those lower prioritized criteria. In order to deal with the prioritized outranking problems, we apply Yager’s prioritized measure-guided aggregation operator into the classical PROMETHEE method, and thus develop a prioritized-PROMETHEE method.

4.1. Problem formulation and the proposed approach

Before the introduction of the prioritized outranking problems, we first give the definition of the prioritized MCDM problems as follows:

Definition 4. [28, 29] In a MCDM problem, if the set of criteria, \( C = \{C_1, C_2, \ldots, C_n\} \), can be partitioned into \( q \) distinct prioritized hierarchies, \( H_1, H_2, \ldots, H_q \), such that \( H_k > H_l \) if \( k < l \), where \( H_k = \{C_{k1}, C_{k2}, \ldots, C_{kn}\} \subset C, C = \cup_{k=1}^{q} H_k \) (i.e., \( n = \sum_{k=1}^{q} n_k \)), and \( H_k \cap H_l = \emptyset \) for \( k, l \in \{1, 2, \ldots, q\} \) ("\( \emptyset \)" denotes the null set), then the problem is called a prioritized MCDM problem.

As Yager [28, 29] pointed out, the prioritized MCDM problems can be classified into two cases: (1) strictly ordered prioritizations, if each priority level has only one criterion, i.e., \( n_k = 1 \) for \( k = 1, 2, \ldots, q \); and (2) weakly ordered prioritizations, otherwise.

For the above prioritized MCDM problems, we mainly investigate the prioritized outranking problems, which aim to obtain the ranking of the alternatives based on the pairwise preference relationships of the
alternatives. For the prioritized outranking problem, we can obtain the preference matrix of alternatives for each criterion $C_l$:

\[
P_l(a_1, a_2) = \begin{bmatrix}
P_l(a_1, a_2) & \cdots & P_l(a_1, a_m) \\
\vdots & \ddots & \vdots \\
P_l(a_n, a_1) & \cdots & P_l(a_n, a_m)
\end{bmatrix}
\]

(10)

where the preference value $P_l(a_i, a_j)$ can be got by the formula (1). In order to visually compare the alternatives for each criterion $C_l$, Mareschal and Brans [15] introduced the normed flow of each alternative $a_i$ for each criterion $C_l$:

\[
\phi_l(a_i) = \frac{1}{m - 1} \sum_{a_j \in X \setminus \{a_i\}} (P_l(a_i, a_j) - P_l(a_j, a_i))
\]

(11)

where the preference value $P_l(a_i, a_j)$ can be got by the formula (1). For each criterion $C_l$, we should select a suitable preference function $P_l(\cdot, \cdot)$ to depict the preference value between two alternatives. In order to obtain the suitable preference function easily, Brans and Vincke [7] proposed six basic types of preference functions. Therefore, the preference function $P_l(\cdot, \cdot)$ is the key point to capture the criteria’s characteristics.

For the prioritized level $H_k = \{C_{k1}, C_{k2}, \ldots, C_{kn_k}\} \subset C$, we have

\[
\phi^k = \begin{bmatrix}
\phi_{k1}(a_1) & \phi_{k1}(a_2) & \cdots & \phi_{k1}(a_n) \\
\phi_{k2}(a_1) & \phi_{k2}(a_2) & \cdots & \phi_{k2}(a_n) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{kn_k}(a_1) & \phi_{kn_k}(a_2) & \cdots & \phi_{kn_k}(a_n)
\end{bmatrix}
\]

(13)

In order to get the total flow in the prioritized level $H_k$ for each alternative $a_i$, we have to aggregate the normed flow $\phi_{k}(a_i)$, $j \in \{1, 2, \cdots, n_k\}$ into a value $\phi^k(a_i)$. Usually, we can use the weight average operator to get the value $\phi^k(a_i)$ defined in the following way:

\[
\phi^k(a_i) = \sum_{j=1}^{n_k} w_j \phi_{k}(a_i)
\]

(14)

For the alternative $a_i$, the value $\phi^k(a_i)$ can be seen as the net flow of the pseudo criterion $C_k$, which is the aggregation of all the $k$th prioritized criteria. However, the DM can’t give the weight of each criterion in lots of actual problems especially when there are lots of criteria in the ranking problem. The DM can only estimate the criteria whether in the same prioritized level. Below we propose an entropy-based approach to eliciting the weights of criteria in the same prioritized level.

According to the formula, we can know that the normed flow $\phi_{k}(a_i)$ isn’t normal for all alternatives. So we introduce a transformation function to normalize the value $\phi_{k}(a_i)$ as follows:

\[
\phi'_{kj}(a_i) = \frac{\phi_{kj}(a_i) + 1}{m}, \quad j \in \{1, 2, \cdots, n_k\}, \quad i \in \{1, 2, \cdots, m\}
\]

(15)

Therefore, we have

\[
0 \leq \phi'_{kj}(a_i) < 1, \quad \sum_{a_i \in X} \phi'_{kj}(a_i) = 1
\]

(16)

We apply the information entropy formula to calculate the entropy for the vector $\phi'_{kj}(a_i)$, $j \in \{1, 2, \cdots, n_k\}$:

\[
E(\phi'_{kj}) = -\sum_{i=1}^{n_k} \phi'_{kj}(a_i) \ln \phi'_{kj}(a_i)
\]

(17)

Obviously, the greater the entropy $E(\phi'_{kj})$, the less deviation between all the alternatives under the criterion $C_{kj}$. If the entropy $E(\phi'_{kj})$ arrives the biggest value, we have $\phi'_{kj}(a_1) = \phi'_{kj}(a_2) = \cdots = \phi'_{kj}(a_{n_k})$ which means that all the alternatives have the same value under the criterion $C_{kj}$. It is reasonable to give the criterion $C_{kj}$ small weight for obtaining a complete rank. Using the formula, the weight of each criterion in the same prioritized level can be obtained by

\[
w_{kj} = \frac{1}{\sum_{j=1}^{n_k} E(\phi'_{kj})}
\]

(18)

Based on the formulas (13) and (17), we can have the net flow of each alternative $a_i$ in the prioritized level $H_k$. Meanwhile, all the results $\phi^k(a_i)$ $(k = 1, 2, \cdots, q; \quad i = 1, 2, \cdots, m)$ are listed in the following matrix:
\[ \phi = \left\{ \phi(a_1), \phi(a_2), \ldots, \phi(a_k) \right\} \]

Certainly, we have to aggregate the k net flows \( \phi(a_i), k = 1, 2, \ldots, q \) into a total net flow for each alternative \( a_i \). Due to the net flow in the different prioritized level, Yager’s prioritized measure guided aggregation operator is used to aggregate the total net flow. Utilizing the formula, we have the total net flow \( \phi(a_i) \) as follows:

\[
\phi(a_i) = \mathrm{Choq}_q(\phi^1(a_i), \phi^2(a_i), \ldots, \phi^q(a_i))
\]

\[ = \sum_{j=1}^{q} (\mu(P_j) - \mu(P_{j-1})) \phi^{\text{ind}_j}(a_i) \]  \hspace{1cm} (20)

where \( P_j = [C_{\text{ind}_0}^{\text{j}}, C_{\text{ind}_1}^{\text{j}}, \ldots, C_{\text{ind}_q}^{\text{j}}] \), and \( \text{ind}_j \) is the index of the \( j \)th largest \( \phi(a_i) \).

The prioritized measure can be obtained by the formulas as follows:

\[
\mu(A) = \sum_{j=1}^{q} G_j(A) \]  \hspace{1cm} (21)

\[
G_j(A) = \begin{cases} 1, & L_j \subseteq A \\ 0, & L_j \not\subseteq A \end{cases} \]  \hspace{1cm} (22)

\[
L_j = \begin{cases} \{C_j\} & j = 1, 2, \ldots, q, j \in \{1, 2, \ldots, q\} \\ \emptyset, & j = 0 \end{cases} \]  \hspace{1cm} (23)

Therefore, we can get a complete ranking of all the alternatives according to the values of the total net flows \( \phi(a_i), i = 1, 2, \ldots, m \).

Of course, if the DM provides partial information of his preference or experience, we employ Yager [30]’s generalized prioritized measure-guided aggregation operator to construct a model to get the total net flow. As Epe and Smet [11] pointed out, there are two types of partial preference information: Pairwise Action Comparisons (PAC) and Action Sub-rankings (ASR). The PAC gives the pairwise preference relation of alternatives, denoted by \( a_i \succ a_j \), which can be expressed by the constraint: \( (i, j), \forall h \in \{1, 2, \ldots, K^{\text{PAC}}\} \). The ASR considers a sub-ranking \( e = (i_1, i_2, \ldots, i_{K^{\text{ASR}}}) \) that expresses that \( a_{i_1} \succ a_{i_2} \succ \cdots \succ a_{i_{K^{\text{ASR}}}} \). Obviously, the information contained in the ASR information structure can be converted into PAC. For simplicity, we use the set \( P = \{(i, j) | l \in \{1, 2, \ldots, K^{\text{PAC}}\}\} \) to express all the preference information. The generalized prioritized measure-guided aggregation operator uses the following measure:

\[
\mu'(A) = \sum_{j=1}^{q} a_j \cdot G_j(A) \]  \hspace{1cm} (24)

where the quantity \( a_j \) satisfies \( a_j \in [0, 1] \) and \( \sum_{j=1}^{q} a_j = 1 \).

Combining the formulas (19) and (23), we have

\[
\phi(a_i) = \mathrm{Choq}_q(\phi^1(a_i), \phi^2(a_i), \ldots, \phi^q(a_i))
\]

\[ = \sum_{j=1}^{q} (\mu(P_j) - \mu(P_{j-1})) \phi^{\text{ind}_j}(a_i) \]  \hspace{1cm} (25)

In order to elicit the parameter \( a_i \), we establish the following model:

\[
\begin{align*}
\max & \quad \xi \\
\text{s.t.} & \quad \phi(a_i) - \xi \geq \phi(a_j), \\
& \quad (i, j) \in P, \\
& \quad a_1 \geq a_2 \geq \cdots \geq a_q \geq 0.
\end{align*} \]  \hspace{1cm} (26)

Through solving the linear programming model (25), we obtain the optimal value \( \xi^* \) and the parameter value \( a^* = (a_1^*, a_2^*, \ldots, a_q^*) \). If \( \xi^* \geq 0 \), then we can get the parameter \( a^* \) that is compatible with the DM’s preference information. If \( \xi^* < 0 \), then we can’t get the parameter \( a^* \) that is compatible with the DM’s preference information. In this case, we propose the following model to elicit the parameter:

\[
\begin{align*}
\max & \quad \sum_{i, j \in P} \xi_{ij} \\
\text{s.t.} & \quad \phi(a_i) - \xi_{ij} \geq \phi(a_j), \\
& \quad (i, j) \in P, \\
& \quad \sum_{i=1}^{q} a_{ij} = 1, \\
& \quad a_1 \geq a_2 \geq \cdots \geq a_q \geq 0.
\end{align*} \]  \hspace{1cm} (27)

With the weight parameter vector \( a^* = (a_1^*, a_2^*, \ldots, a_q^*) \) obtained by the above linear programming, we can compute the total net flow of the alternatives and rank the alternative from best to the worst one.
4.2. The proposed algorithm

Based on the above analysis, we develop a practical algorithm for the prioritized multi-criteria outranking problem in which the weights of criteria are completely unknown and the DM’s preference information is partially known or completely unknown, which involves the following steps:

Step 1. For a prioritized multi-criteria outranking problem, we use the formula \( d_l(a_i, a_j) = f(a_i) - f(a_j) \) to determine the difference between the evaluations of \( a_i \) and \( a_j \) with respect to each criterion.

Step 2. Apply the preference function (1) to calculate the preference of the alternative \( a_i \) with regard to the alternative \( a_j \) with respect to each criterion. The preference function \( F_l(\cdot) \) and the corresponding thresholds \( p \) and \( q \) can be given by the DM.

Step 3. Utilize the formula (10) to get the normed flow of the alternative \( a_i \) for the criterion \( C_l \). For each prioritized level, we utilize the formulas (17), (16) and (14) to obtain the weight vector \( w_k = (w_{k1}, w_{k2}, \cdots, w_{kn_k}) \).

Step 4. Utilize the formula (13) to determine \( \phi_k(a_i) \) which represents the net flow of the alternative \( a_i \) under the pseudo criterion \( C_k \).

Step 5. If the DM’s preference information of alternatives is completely unknown, then we use the formulas (19)-(22) to get the total net flow for each alternative. Based on the total net flow, we determine the rank of all alternatives; if the DM’s preference information of alternatives is partially known, then we use the model (MOD-1 or MOD-2) to obtain the vector of the weight parameter values \( \alpha^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_q^*) \), turn to next step.

Step 6. Utilize the formulas (19)-(23) to obtain the total net flow for each alternative and rank them from best to worst.

4.3. Illustrative example

In order to illustrate our approach, let us consider the strategic assessment of islands and reefs. As we know, China is a coastal country and has many islands and reefs. However, due to the limited comprehensive national strength, most of the islands and reefs are undeveloped. Therefore, it is necessary to rank all the islands and reefs according to the strategic status of islands and reefs. In this paper, the strategic status of islands and reefs are evaluated by the following three aspects: Sensitivity, Exposure and Defense. Meanwhile, the sensitivity contains criteria: history conflict \( (S_1) \), foreign relation \( (S_2) \), ethnic and religious relation \( (S_3) \) and strategic value \( (S_4) \). The exposure contains criteria: population \( (E_1) \), the value of island’s facility \( (E_2) \), the value of island’s resource \( (E_3) \) and strategic position \( (E_4) \). The defense contains criteria: weapon \( (D_1) \), the number of military bases and troops \( (D_2) \), detection capability \( (D_3) \), military support capability \( (D_4) \). The prioritized relationship of criteria is listed in Fig. 1.

While the population \( (E_1) \), the value of island’s facility \( (E_2) \), the value of island’s resource \( (E_1) \) and the number of military bases and troops \( (D_2) \) are all qualitative criteria. The assessments for history conflict \( (S_1) \), foreign relation \( (S_2) \), ethnic and religious relation \( (S_3) \), strategic value \( (S_4) \), strategic position \( (E_1) \), weapon \( (D_1) \), detection capability \( (D_3) \) and military support capability \( (D_4) \) are represented by linguistic terms. The linguistic terms used here and their corresponding INs (integer numbers) are shown in Table 1.

The weights of all criteria are completely unknown. For each criterion, we consider the linear preference function and define the thresholds as follows: the indifference threshold \( q \) is fixed at 0 and the preference
threshold \( p \) is equal to the highest difference between 
the islands' performances on the particular criterion 
(see Table 2).

In this example, there are six islands and reefs rep-
resented by \( a_1, a_2, a_3, a_4, a_5, a_6 \), which are needed 
to be ranked. The decision data used in the strategic 
asessment of islands and reefs are shown in Table 3.

Fortunately, the DM can give partial information of the 
assessment of islands and reefs, and the results are shown in Table 4.

In the following, we use the P-PROMETHEE method 
to rank the six islands. The solution process and the 
computation results are summarized as follows.

Firstly, we use the formula (10) to calculate the 
normed flow for each alternative on each criterion. 
The results are shown in Table 4.

Secondly, for each prioritized level, we utilize the 
formulas (17), (16) and (14) to obtain the weight vector 
\( w_k \) as follows:

\[
\begin{align*}
 w_1^1 &= (w_{11}, w_{12}, w_{13}, w_{14}) \\
 &= (0.2489, 0.2455, 0.2528, 0.2528) \\
 w_2^1 &= (w_{21}, w_{22}, w_{23}, w_{24}) \\
 &= (0.2499, 0.2493, 0.2466, 0.2542) \\
 w_3^1 &= (w_{31}, w_{32}, w_{33}, w_{34}) \\
 &= (0.2522, 0.2483, 0.2506, 0.2490)
\end{align*}
\]

Thirdly, we utilize the formula (13) to aggregate all 
the criteria of same prioritized level for each alternative. 
The obtained results are listed in Table 5.

Fourthly, based on the data presented in table, we use 
the model (MOD-1) to construct a linear programming 
model as follows:

\[
\begin{align*}
 \max \xi \\
 \text{s.t. } & \phi(a_i) - \xi \geq \phi(a_j), \\
 & \phi(a_i) - \xi \geq \phi(a_j), \\
 & \sum_{j=1}^{5} \alpha_{ij} = 1, \alpha_{i1} \geq \alpha_{i2} \geq \alpha_{i3} \geq 0.
\end{align*}
\]

By solving the above linear programming model, 
we can get the priority vector \( \alpha = (\alpha_{11}, \alpha_{21}, \alpha_{31}) \) = 
\((0.7227, 0.2773, 0)\).

At last, we use the formulas (19) (23) to obtain the 
total net flow of each alternative:

\[
\begin{align*}
 \phi(a_1) &= 0.2692 \cdot 0 + 0.1664 \cdot 0.7227 \\
 &= -0.0841 \cdot 0.2773 = 0.0991 \\
 \phi(a_2) &= 0.2247 \cdot 0 + 0.1266 \cdot 0 \\
 &= -0.0471 \cdot 1 = -0.0471
\end{align*}
\]
natives is a compatible with the DM's partial alternative is a
d. The alternatives is compatible with the DM's partial alternative is a

\[
\begin{array}{cccc}
\text{Criterion} & C_1 & C_2 & C_3 \\
D_1 & 0.4500 & 0.8000 & 0.0000 \\
D_2 & -0.1500 & -0.4000 & 0.6000 \\
D_3 & -0.7500 & -0.4000 & -0.6000 \\
D_4 & 0.4500 & -0.1000 & 0.0000 \\
D_5 & 0.1500 & 0.2000 & -0.6000 \\
D_6 & -0.3333 & -0.1333 & 0.7500 \\
D_7 & 0.4667 & 0.2667 & -0.4500 \\
D_8 & -0.7333 & -0.5333 & 0.1500 \\
D_9 & 0.0667 & 0.2667 & -0.4500 \\
D_{10} & 0.0667 & 0.6667 & -0.1500 \\
\end{array}
\]

Therefore, the final ranking order of all the alternatives is \( a_3 \succ a_1 \succ a_9 \succ a_8 \succ a_7 \), and the best alternative is \( a_3 \). Meanwhile, the ranking order of all the alternatives is compatible with the DM’s partial preference information of alternatives, which is \( a_3 \succ a_1 \succ a_9 \succ a_8 \succ a_7 \).

### 3.4. Comparative analysis

To illustrate the advantage of the P-PROMETHEE method, we make a comparative analysis with the similar methods. As mentioned previously, Yu et al. [34] proposed an approach based on idea of PROMETHEE to deal with the prioritized MCDM problems. However, Yu et al. [34] only investigated the strict ordered prioritized MCDM problems. Therefore, their method fails to deal with the ranking problem of weak ordered prioritized MCDM. Besides, Yan et al. [32] proposed a prioritized weighted aggregation operator based on the ordered weighted averaging (OWA) operator and the triangular norms (t-norms) to deal with the weak ordered prioritized MCDM problems, which are the same with our considered problems. The main procedures of Yan’s prioritized weighted aggregation operator [32] are summarized as follows:

**Step 1.** Normalize the decision matrix, in which each element indicating degree of satisfaction of a given alternative regarding criterion.

**Step 2.** Based on the DM’s attitudinal character \( \Omega_0 \) toward the prioritized level \( H_p \), we obtain the weights of criteria in the prioritized level \( H_p \) according to O’Hagan [18]’s OWA weight determination method as follows:

Therefore, the final ranking order of all the alternatives is \( a_3 \succ a_1 \succ a_9 \succ a_8 \succ a_7 \), and the best alternative is \( a_3 \). Meanwhile, the ranking order of all the alternatives is compatible with the DM’s partial preference information of alternatives, which is \( a_3 \succ a_1 \succ a_9 \succ a_8 \succ a_7 \).
Maximize \(-\sum_{i=1}^{n_q} w_i \ln w_i\)  

(MOD - 3) Subject to \(\sum_{i=1}^{n_q} \frac{n_q - i}{n_q} w_i = \Omega, \quad 0 \leq \Omega \leq 1\)

\(\sum_{i=1}^{n_q} w_i = 1, \quad w_i \in [0, 1], \quad i = 1, 2, \ldots, n_q\)

**Step 3.** Based on the obtained weights of criteria in the prioritized level \(H_q\), we get the degree of satisfaction of the prioritized level \(H_q\) by using the OWA operator.

\[\text{Sat}_q(a) = \text{OWA}_{D_q}[H_q] = \frac{1}{n_q} \sum_{k=1}^{n_q} \text{Sat}_q(a_k)\]  

where \(\text{Sat}_q(a_k)\) is the kth largest satisfaction degree in the prioritized level \(H_q\).

**Step 4.** Induce the priority weight for the prioritized level \(H_q\) using the following equation:

\[Z_q = T(Z_{q-1}(\cdot), \text{Sat}_{q-1}(\cdot)) = T_{\text{OWA}}^{-1}(\text{Sat}(\cdot))\]  

where \(T\) denotes the t-norms and \(Z_0(\cdot) = \text{Sat}(\cdot) = 1\).

**Step 5.** Get an aggregated value for each alternative under these prioritized criteria as:

\[V(a) = \sum_{q=1}^{Q} Z_q(a) \cdot \text{Sat}_q(a)\]  

**Step 6.** Based on the obtained aggregated values, we get the ranking of all the alternatives.

In what follows, we utilize Yan’s method [32] to solve the decision making problem mentioned in Section 4.3. Firstly, we calculate the normalized decision matrix, which is shown in Table 6. For simplicity, we let the DM’s attitudinal character \(D_q = 0.5, \quad q = 1, 2, 3\).

Therefore, we can easily obtain the weights of criteria in the same prioritized level by using O’Hagan’s OWA weight determination method (MOD-3). Then, we use the formula (27) to calculate the degree of satisfaction of each alternative in the prioritized level \(H_q\). The obtained results are displayed in Table 7. Furthermore, the priority weight of the prioritized level \(H_q\) can be obtained by using the formula (28), in which the product t-norm is used and the results are also listed in Table 7. Finally, we get the aggregated values of all the alternatives by the formula (29). The results are shown as follows:

\[V(a_1) = 0.2902, \quad V(a_2) = 0.1569, \quad V(a_3) = 0.2044\]
Consequently, it is easy to see that the P-PROMETHEE method, compared with Yan’s method [32], has the following two desirable advantages:

1. The P-PROMETHEE method can obtain the weights of criteria in the same prioritized level by the entropy-based method, which can capture the relation of criteria without additional information of weights. However, Yan’s method needs the DM’s attitudinal character $\Omega$ for obtaining the relative importance of each criterion.

2. The P-PROMETHEE method constructs a linear programming model to determine the prioritized weight of the prioritized level, which is compatible with the DM’s partial preference information of alternatives. However, Yan’s method obtains the prioritized weights based on the satisfaction degrees of the higher prioritized levels, which can’t use the partial preference information of alternatives. Therefore, the result may be not effective.

5. Concluding remarks

In this paper, we have extended the PROMETHEE method to the case of the prioritized MCDA, which had never been considered before. The set of criteria is not considered in the same level, but it has a weakly ordered prioritization relation. Then, we have developed a prioritized PROMETHEE method based on the prioritized measure and entropy to capture the ranking degree of all the alternatives. In most of the prioritized MCDM problems, it is difficult to give the weight of each criterion by the DM, especially when the number of criteria is large. For the criteria of same prioritized level, we have proposed an entropy-based method to derive the weight of each criterion.

Meanwhile, we have considered two cases where the preference information on alternatives is completely unknown and partial known. For the case where the preference information on alternatives given by the DM is completely unknown, we have employed Yager’s prioritized measure-guided aggregation operator [30] to calculate the total net flow of each alternative. For the case where the preference information on alternatives given by DM is partially known, we have proposed a linear programming method to derive the priority vectors. Based on the priority vectors, we can get the ranking of alternative compatible with the DM’s preference information. At last, the strategic assessment of islands and reefs has been used to illustrate and verify the P-PROMETHEE method. Surely, the P-PROMETHEE can also be used to deal with other prioritized multi-criteria ranking problem, such as document retrieval problems, risk assessment problems and selecting a car, etc. In the future, we will consider the general prioritized multi-criteria outranking method and extend the P-PROMETHEE method to uncertain situations.

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