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POINCARÉ SERIES OF SUBSETS OF AFFINE WEYL GROUPS

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ABSTRACT. In this note, we identify a natural class of subsets of affine Weyl groups whose Poincaré series are rational functions. This class includes the sets of minimal coset representatives of reflection subgroups. As an application, we construct a generalization of the classical length-descent generating function and prove its rationality.

1. INTRODUCTION

The aim of this note is to prove the rationality of certain length generating functions in affine Weyl groups. Let $\widetilde{W}$ be an affine Weyl group with Coxeter generators $\widetilde{S} = \{s_i\}_{i=0}^n$ and length function $\ell(\cdot)$. For a subset X of $\widetilde{W}$, its Poincaré series (length generating function) is $X(q) := \sum_{w \in X} q^{\ell(w)}$. For many "natural" subsets X, $X(q)$ turns out to be a rational function; notable examples include the entire group $\widetilde{W}$, its parabolic subgroups $\widetilde{W}_I$ ($I \subset \widetilde{S}$), and the sets $\widetilde{W}^I$ of minimal length left coset representatives for $\widetilde{W}_I$ in $\widetilde{W}$ (in fact all this holds for arbitrary Coxeter groups $\widetilde{W}$).

One of our objectives is to study an interesting class of subgroups of $\widetilde{W}$-*reflection subgroups*. These are subgroups $W' \subset \widetilde{W}$ generated by reflections (conjugates of elements of $\widetilde{S}$); they exhibit many of the same properties as parabolic subgroups [3], [4], e.g. (i) W' is a Coxeter group in its own right, (ii) $\exists$ unique minimal length left coset representatives for W' in $\widetilde{W}$, etc. A complete classification of the reflection subgroups of affine Weyl groups (in terms of those of the underlying finite Weyl group) was given by Dyer in [4].

However, reflection subgroups are ill-behaved w.r.t. the length function. The length function of W' does not in general agree with the restriction of $\ell(\cdot)$ to W'. This makes it difficult to study the Poincaré series $W'(q)$; for instance, it does not seem to be known if $W'(q)$ is a rational function for all reflection subgroups W'. Our focus, however, will be on the set $X_{W'}$ of minimal length left coset representatives for W'; a complication here is that $X_{W'}(q) \neq \widetilde{W}(q)/W'(q)$ any more (this holds if W' is parabolic). This means that even in cases where $W'(q)$ is known to be a rational function (e.g. $\#W' < \infty$), we still cannot conclude that $X_{W'}(q)$ is rational. Our first goal is to show that $X_{W'}(q)$ is indeed a rational function for all reflection subgroups W' of $\widetilde{W}$. We thus get another natural class of subsets of $\widetilde{W}$ with rational

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Poincaré series. We remark that if $\widetilde{W}$ is allowed to be an arbitrary (non-finite, non-affine) Coxeter group, the rationality question for $X_{W'}(q)$ seems harder to decide; the article [11] is concerned with reflection subgroups of such Coxeter groups but only deals with the *growth type* of the set $X_{W'}$.

Next, we turn to a two-variable refinement of the Poincaré series of $\widetilde{W}$ - the classical length-descent generating function $\widetilde{W}(q,t) := \sum_{w\in\widetilde{W}} q^{\ell(w)}t^{\,\mathrm{des}(w)}$, where $\mathrm{des}(w) := \#\{s \in \widetilde{S} : \ell(ws) < \ell(w)\}$. This is well-known (see [7]) to be a polynomial in t with coefficients that are rational functions in q, i.e $\widetilde{W}(q,t) \in \mathbb{Q}(q)[t]$. To generalize this, let $\mathrm{refl}(\widetilde{W}) := \bigcup_{\sigma\in\widetilde{W}} \sigma\widetilde{S}\sigma^{-1}$ be the set of reflections in $\widetilde{W}$ and $A \subset \mathrm{refl}(\widetilde{W})$ be a finite subset. Define

$$\widetilde{W}(q,t,A) := \sum_{w\in\widetilde{W}} q^{\ell(w)}t^{\,\mathrm{des}_A(w)},$$

where $\mathrm{des}_A(w) := \#\{r \in A : \ell(wr) < \ell(w)\}$. Thus $\widetilde{W}(q,t,\widetilde{S}) = \widetilde{W}(q,t)$.

The second aim of this note is to show that $\widetilde{W}(q,t,A) \in \mathbb{Q}(q)[t]$ for all finite $A \subset \mathrm{refl}(\widetilde{W})$.

Both this and the earlier result on reflection subgroups will be shown to fit into a slightly more general framework. They will follow as simple consequences of our main theorem (Theorem 2.1), which also seems to be of independent interest.

2. The main theorem

2.1. **Preliminaries.** Let W be the finite Weyl group corresponding to an irreducible, crystallographic root system Φ. Let $\{s_i\}_{i=1}^n, \{\alpha_i\}_{i=1}^n, \{\Lambda_i\}_{i=1}^n$ be the simple reflections, the simple roots and the fundamental weights respectively. Let V be the $\mathbb{R}$-span of the simple roots and $(,)$ be a positive definite, W-invariant bilinear form on V. Similarly, one has the coroot system $\check{\Phi} \subset V$ with simple coroots $\check{\alpha}_i$ and fundamental coweights $\check{\Lambda}_i$. The root and coroot lattices will be denoted $Q = \mathbb{Z}\Phi$ and $\check{Q} = \mathbb{Z}\check{\Phi}$. For $\alpha \in V$, let t_α be the translation map $v \mapsto v + \alpha$. Let $T = T(\check{Q}) := \{t_\alpha : \alpha \in \check{Q}\}$ be the group of translations of V by elements of $\check{Q}$. The affine Weyl group $\widetilde{W}$ can be defined as the subgroup of the group of affine transformations of V generated by W and T; we have $\widetilde{W} = W \ltimes T$.

$\widetilde{W}$ is a Coxeter group with generators $\widetilde{S} := \{s_i\}_{i=1}^n \cup \{s_0\}$; here s_0 is the reflection about the affine hyperplane $\{v \in V : (v, \widetilde{\alpha}) = 1\}$ with $\widetilde{\alpha}$ = the highest root of Φ. Thus, $\widetilde{W}$ also has a *reflection representation* (or geometrical realization) $\widetilde{V}$. To construct this (see [10, §4] or [6, Chap 6]), set $\widetilde{V} := V \oplus \mathbb{R}\delta$ and extend $(,)$ to a positive semidefinite form on $\widetilde{V}$ by letting $(\delta,\delta) = 0, (\delta, v) = 0\,\forall v \in V$. The $\widetilde{W}$ action on $\widetilde{V}$ extends the W action on V via the following prescription: given $\sigma \in W, \alpha \in \check{Q}, v \in V$,

$$\sigma\delta = t_\alpha\delta = \delta,$$
$$t_\alpha(v) = v - (v,\alpha)\,\delta.$$

The root system of $\widetilde{W}$ is $\widetilde{\Phi} := \{\beta + k\delta : \beta \in \Phi, k \in \mathbb{Z}\} \subset \widetilde{V}$; the simple roots are $\{\alpha_i\}_{i=0}^n$, where $\alpha_0 = \delta - \widetilde{\alpha}$. The positive roots of $\widetilde{W}$ are

$$\widetilde{\Phi}^+ = \{\beta + k\delta : \beta \in \Phi^+, k \geq 0\} \cup \{\beta + k\delta : \beta \in \Phi^-, k \geq 1\}. \tag{2.1}$$

Given $\gamma \in \widetilde{\Phi}^+$, one has the map $s_\gamma \in GL(\widetilde{V})$ defined by $s_\gamma(\mu) := \mu - 2\frac{(\mu,\gamma)}{(\gamma,\gamma)}\,\gamma$. It is well-known that the set $\{s_\gamma : \gamma \in \widetilde{\Phi}^+\}$ is the image of $\mathrm{refl}(\widetilde{W})$ in $GL(\widetilde{V})$; thus $\widetilde{\Phi}^+$ is in bijection with $\mathrm{refl}(\widetilde{W})$.

2.2. **The main theorem and its corollaries.** Let $\ell(.)$ be the length function on $\widetilde{W}$ w.r.t. $\widetilde{S}$ (this extends the length function on W). Given a finite subset $A \subset \mathrm{refl}(\widetilde{W})$, let $\widetilde{W}^A := \{\sigma \in \widetilde{W} : \ell(\sigma r) > \ell(\sigma)\ \forall r \in A\}$. Observe that if A' is the corresponding set $\{\gamma \in \widetilde{\Phi}^+ : s_\gamma \in A\}$, we have the equivalent definition $\widetilde{W}^A = \{\sigma \in \widetilde{W} : \sigma(A') \subset \widetilde{\Phi}^+\}$. We also note that in the familiar case when $A = I \subset \widetilde{S}$, $\widetilde{W}^A$ is just the set of minimal length left coset representatives for the parabolic subgroup $\widetilde{W}_I$. Our main theorem is:

Theorem 2.1. *For any finite $A \subset \mathrm{refl}(\widetilde{W})$, the Poincaré series $\widetilde{W}^A(q) = \sum_{w \in \widetilde{W}^A} q^{\ell(w)}$ is a rational function.*

We postpone the proof to Section 3. We first use this theorem to give quick proofs of the two results mentioned in the introduction.

2.2.1. If $W' \subset \widetilde{W}$ is a reflection subgroup, it is a well-known theorem due (independently) to Deodhar [3] and Dyer [4] that W' is a Coxeter group w.r.t. a set $S' = \{s_{\gamma_i} : i = 1 \cdots k\}$ of reflection generators. Here, $\gamma_i \in \widetilde{\Phi}^+$ and satisfy $(\gamma_i, \gamma_j) \le 0\ \forall i \neq j$ [4, Theorem 4.4] (this theorem holds even when $\widetilde{W}$ is an arbitrary Coxeter group in which case S' need not be finite; for affine $\widetilde{W}$, however, it is easy to show that $\#S' < \infty$). In [4], Dyer also showed that there are unique minimal length elements in the left cosets of W'. Let $X_{W'}$ denote this set of minimal coset representatives; then $\sigma \in X_{W'} \Leftrightarrow \ell(\sigma s_{\gamma_i}) > \ell(\sigma)\ \forall i = 1 \cdots k$. Thus $X_{W'} = \widetilde{W}^{S'}$. As a consequence of Theorem 2.1 ,we have:

Corollary 1. *Let W' be any reflection subgroup of $\widetilde{W}$. Then $X_{W'}(q)$ is a rational function.*

2.2.2. We recall from the introduction that the generating function $\widetilde{W}(q,t) = \sum_{w \in \widetilde{W}} q^{\ell(w)} t^{\mathrm{des}(w)} \in \mathbb{Q}(q)[t]$. We refer to Reiner's article [7, Theorem 1] for the proof of this "folklore" result. The proof essentially consists in showing the following identity (in our notation):

$$\widetilde{W}(q,t) = \sum_{I \subset \widetilde{S}} t^{|I|}\,(1-t)^{|\widetilde{S}\setminus I|}\,\widetilde{W}^{\widetilde{S}\setminus I}(q). \tag{2.2}$$

Here $|\cdot|$ denotes set cardinality. To complete Reiner's argument, one observes that since $\widetilde{S}\setminus I \subset \widetilde{S}$, we have $\widetilde{W}^{\widetilde{S}\setminus I} = \dfrac{\widetilde{W}(q)}{\widetilde{W}_{\widetilde{S}\setminus I}(q)}$, which is a rational function.

It is now elementary to modify the above argument for the case where $\widetilde{S}$ is replaced by A. The analogue to equation (2.2) is now:

$$\widetilde{W}(q,t,A) = \sum_{B \subset A} t^{|B|}\,(1-t)^{|A\setminus B|}\,\widetilde{W}^{A\setminus B}(q), \tag{2.3}$$

where $\widetilde{W}(q,t,A) := \sum_{w \in \widetilde{W}} q^{\ell(w)}\,t^{\,\mathrm{des}_A(w)}$. Invoking Theorem 2.1, we conclude $\widetilde{W}^{A\setminus B}(q) \in \mathbb{Q}(q)$ and hence the following:

Corollary 2. $\widetilde{W}(q,t,A) \in \mathbb{Q}(q)[t]$ *for all finite subsets* $A \subset \mathrm{refl}(\widetilde{W})$.

3. Proof of the main theorem

3.1. Before embarking on the proof of our main theorem, we collect some well-known facts concerning $\widetilde{W}$ (good references are [5], [2]). We freely use the notation of Section 2. Let

$$C = \{v \in V : (v, \alpha_i) \geq 0\, \forall i = 1 \cdots n\}, \quad A_f = \{v \in C : (v, \widetilde{\alpha}) \leq 1\}$$

be the closures of the fundamental chamber and fundamental alcove, respectively. The finite Weyl group W is a parabolic subgroup of $\widetilde{W}$; let $\widetilde{W}^0$ be the set of minimal length right coset representatives for W in $\widetilde{W}$.

Fact 1. $\forall w \in \widetilde{W}$, $\exists! u \in W$ s.t. $uw \in \widetilde{W}^0$; this u is the unique element of W s.t. $uw(A_f) \subset C$.

Let $\rho \in V$ be the Weyl vector of Φ; it is determined by the conditions $(\rho, \check{\alpha}_i) = 1\, \forall i = 1 \cdots n$.

Fact 2. $\ell(t_\alpha) = \ell(t_{\sigma\alpha})\ \forall \sigma \in W, \alpha \in \check{Q}$; further $\ell(t_\alpha) = (\alpha, 2\rho)$ if α is a dominant element of $\check{Q}$.

For $u \in W$, define $T_u := \{t_\alpha \in T : ut_\alpha \in \widetilde{W}^0\}$. By Fact 1, this means $ut_\alpha(A_f) \subset C$. Let the vertices of the simplex A_f be $\{0, \theta_1, \theta_2, \cdots, \theta_n\}$. We note that $\theta_j \in C$, but in general they are not elements of the coweight lattice; we only have $(\theta_j, \alpha_i) \in \mathbb{Q}\ \forall i, j$. It is clear that $ut_\alpha(A_f) \subset C \Leftrightarrow u\alpha \in C$ and $u(\alpha + \theta_j) \in C\ \forall j = 1 \cdots n$. Let $m_i(u)$ be the smallest integer such that $m_i(u) \geq 0$ and $m_i(u) \geq -(u\theta_j, \alpha_i)\ \forall j = 1 \cdots n$. Since $u\alpha \in \check{Q}$, the above discussion implies:

Fact 3. The condition $t_\alpha \in T_u$ is equivalent to the system of inequalities:

$$(u\alpha, \alpha_i) \geq m_i(u) \ \ \forall i = 1 \cdots n. \tag{3.1}$$

3.2. **Proof of Theorem 2.1.** We refer to the statement of Theorem 2.1. We prefer to work with the set $A' = \{\gamma \in \widetilde{\Phi}^+ : s_\gamma \in A\}$ rather than with A itself. Thus $\widetilde{W}^A = \{\sigma \in \widetilde{W} : \sigma(A') \subset \widetilde{\Phi}^+\}$. First, we can assume w.l.o.g. that for each $\beta \in \Phi$, A' contains at most one element of the form $\beta + k\delta$. If not, suppose $\beta + k_1\delta,\ \beta + k_2\delta \in A'$ with $k_1 < k_2$. For $\sigma \in \widetilde{W}$, $\sigma(\beta + k_1\delta) = \sigma\beta + k_1\delta \in \widetilde{\Phi}^+ \ \Rightarrow \sigma\beta + k_2\delta \in \widetilde{\Phi}^+$, too. Thus we can delete $\beta + k_2\delta$ from A' without changing $\widetilde{W}^A$.

For each $\beta \in \Phi$, let k_β be the unique integer (if it exists) such that $\beta + k_\beta\,\delta \in A'$. If A' contains no element of the form $\beta + k\,\delta$, we set $k_\beta := \infty$. Let $F := \{\beta \in \Phi : k_\beta < \infty\}$.

Next, we'll analyze what it means for σ to be an element of $\widetilde{W}^A$. We write $\sigma = xt_\alpha$ with $x \in W$, $\alpha \in \check{Q}$. For each $\beta \in F$, we require

$$\sigma(\beta + k_\beta\,\delta) = xt_\alpha(\beta + k_\beta\,\delta) = x\beta + (k_\beta - (\alpha, \beta))\,\delta \in \widetilde{\Phi}^+.$$

By equation (2.1), this implies that α satisfies the following inequalities:

$$(\alpha, \beta) \leq \begin{cases} k_\beta & \text{if } x\beta \in \Phi^+, \\ k_\beta - 1 & \text{if } x\beta \in \Phi^-. \end{cases} \tag{3.2}$$

Note that by our convention of setting $k_\beta = \infty$ for $\beta \notin F$, we can state this as: $xt_\alpha \in \widetilde{W}^A \Leftrightarrow$ the inequalities (3.2) hold for *all* $\beta \in \Phi$ (not just for $\beta \in F$).

Now, for fixed $x, u \in W$, define $\widetilde{W}_{x,u} := \{xt_\alpha : t_\alpha \in T_u\}$ and $\widetilde{W}^A_{x,u} := \widetilde{W}^A \cap \widetilde{W}_{x,u}$. Then $\widetilde{W}^A = \bigsqcup_{x,u\in W} \widetilde{W}^A_{x,u}$. Given $\sigma \in \widetilde{W}_{x,u}$, we have $\sigma = xt_\alpha = (xu^{-1})(ut_\alpha)$; since $ut_\alpha \in \widetilde{W}^0$,

$$\ell(\sigma) = \ell(xu^{-1}) + \ell(ut_\alpha) = \ell(xu^{-1}) + \ell(t_\alpha) - \ell(u).$$

Thus

$$\sum_{\sigma\in\widetilde{W}^A_{x,u}} q^{\ell(\sigma)} = q^{\ell(xu^{-1})-\ell(u)} \sum_{xt_\alpha\in\widetilde{W}^A_{x,u}} q^{\ell(t_\alpha)}. \tag{3.3}$$

Let $f_{x,u}(q) := \sum_{xt_\alpha\in\widetilde{W}^A_{x,u}} q^{\ell(t_\alpha)}$; by Fact 2, we have $f_{x,u}(q) = \sum_{xt_\alpha\in\widetilde{W}^A_{x,u}} q^{\ell(t_{u\alpha})}$.

Claim. $f_{x,u}(q)$ is a rational function.

Proof. Observe that $xt_\alpha \in \widetilde{W}_{x,u}$ iff α satisfies the inequalities (3.1) and $xt_\alpha \in \widetilde{W}^A$ iff α satisfies the inequalities (3.2). Now, since $(\alpha,\beta) = (u\alpha, u\beta)\,\forall\beta\in\Phi$, inequalities (3.2) can be rewritten (with $\gamma = u\beta$) as:

$$\forall\gamma\in\Phi,\quad (u\alpha,\gamma) \le \begin{cases} k_{u^{-1}\gamma} & \text{if } xu^{-1}\gamma\in\Phi^+,\\ k_{u^{-1}\gamma}-1 & \text{if } xu^{-1}\gamma\in\Phi^-. \end{cases} \tag{3.4}$$

So, $xt_\alpha \in \widetilde{W}^A_{x,u}$ iff $u\alpha$ satisfies the systems of inequalities (3.1) and (3.4). Since $u\alpha \in C$, Fact 2 also gives $\ell(t_{u\alpha}) = (u\alpha, 2\rho)$. Thus

$$f_{x,u}(q) = \sum_{xt_\alpha\in\widetilde{W}^A_{x,u}} q^{(u\alpha,2\rho)} = \sum q^{(\pi,2\rho)},$$

where the last sum on the right runs over all $\pi \in C\cap\check{Q} \subset \check{Q}^+$ satisfying the systems of inequalities (3.1) and (3.4) (with π in place of $u\alpha$).

Now, since $\pi\in\check{Q}^+$ is a non-negative integer linear combination of simple coroots, the set of allowed π in the above summation can be thought of as the solution set in non-negative integers to a system of inequalities with integer coefficients. By the classical theory of such systems (see e.g. [9, §4.6], [8]), the generating series $f_{x,u}(q) = \sum q^{(\pi,2\rho)}$ is a rational function.

Finally, since $\widetilde{W}^A(q) = \sum_{x,u\in W} q^{\ell(xu^{-1})-\ell(u)} f_{x,u}(q)$, it is clear that $\widetilde{W}^A(q)$ is a rational function. This completes the proof of our main theorem. □

3.3. As a by-product of our method of proof above, we obtain the following fact concerning the rationality of the Poincaré series $T(q) = \sum_{t_\alpha\in T} q^{\ell(t_\alpha)}$. Since $T = \bigsqcup_{u\in W} T_u$, we have

$$\begin{aligned} T(q) &= \sum_{u\in W}\sum_{t_\alpha\in T_u} q^{\ell(t_\alpha)} = \sum_{u\in W}\sum_{t_\alpha\in T_u} q^{\ell(t_{u\alpha})} \\ &= \sum_{u\in W}\sum_{t_\alpha\in T_u} q^{(u\alpha,2\rho)}. \end{aligned}$$

The set $\{u\alpha : t_\alpha\in T_u\}$ is precisely the set of elements in $C\cap\check{Q}$ satisfying the inequalities (3.1); again by the general theory quoted above, we conclude that $\sum_{t_\alpha\in T_u} q^{(u\alpha,2\rho)}$ is a rational function. This proves

Corollary 3. *$T(q) = \sum_{t_\alpha \in T} q^{\ell(t_\alpha)}$ is a rational function.*

As our final remark, we compare the result of the above corollary with a related fact about T that can be derived from general considerations concerning finitely generated abelian groups. If K is any finite set of generators of the (free) abelian group T, we have the length function $\ell_K(t_\alpha)$, defined to be the length of the smallest word in $K \cup K^{-1}$ that represents t_α. It is well-known (see e.g. [1]) that the generating series $\sum_{t_\alpha \in T} q^{\ell_K(t_\alpha)}$ is a rational function. In our situation above, however, the length function ℓ on T is w.r.t. the Coxeter generators $\widetilde{S}$ of the ambient group $\widetilde{W}$ (note that none of these generators is in T).

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