A vertex operator algebra structure of degenerate minimal models, I

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Abstract

We study fusion rings for degenerate minimal models ($p = q$) for $N = 0$, $N = 1$ and $N = 2$ (super)conformal algebras. In the first part we consider a family of modules for the Virasoro vertex operator algebra $L(1,0)$, and show that a fusion ring of the family is isomorphic to a Grothendieck ring $\mathcal{R}ep(\mathfrak{sl}(2,\mathbf{C}))$.

In the second part, we used similar methods for the family of modules for $N = 1$ Neveu Schwarz vertex operator superalgebra $L(\frac{3}{2},0)$ and obtain a fusion ring isomorphic to $\mathcal{R}ep(\mathfrak{osp}(1|2))$. Finally, in the $N = 2$ case we shall study vertex operator superalgebra $L(3,0,0)$.

1 Introduction

The theory of vertex operator algebras is a powerful tool for constructing tensor categories predicted by the physicists [MS] . The most interesting class of vertex operator algebras are rational vertex operator algebras, i.e. algebras which have finitely many irreducible modules (up to equivalence), such that every module is semi–simple. A rational vertex operator algebra, which satisfy some extra conditions, gives us an intertwining operator algebra (which is roughly genus–zero weakly holomorphic conformal field theory [H2]), and the vertex tensor category [HL3]. Some examples of such constructions are given in [H1], [HL2] and [HM].

Vertex operator algebras which are neither rational or quasi–rational (every module is semisimple) are not well–understood at this point. Certain vertex operator algebras have familiar fusion rings (or subrings), but constructing tensor category from the present theory is a difficult problem. In [KL1–4] the construction of some tensor categories which raise from non-rational models are given, but their approach is stemming from algebraic geometry, and also does not depend on the rigid structure of vertex operator algebra and the notions like rationality, intertwining operators, etc.

In this paper we concentrate on the particular example of vertex operator (super)algebras which do not share these nice properties. Let us explain the outline. In [H1] author constructed tensor categories associated to the vertex operator algebras $L(c_{p.q},0)$, where $c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$ and $(p,q) = 1$. This result rely bith on the tensor product theory of vertex operator algebras and results obtained in [W] about the rationality of vertex operator algebra $L(c_{p,q},0)$.

We consider *degenerate* case "$p=q$", with central charge $c=1$. This case is substantially different for many reasons. The vertex operator algebra $L(1,0)$ is not rational (or quasi–rational) but it has distinguished family of irreducible modules $\mathcal{F}_1$, which consists of modules isomorphic to $L(1,\frac{m^2}{4})$ for some $m\in\mathbb{N}$. These modules have a quite simple embedding structure ([KR], [FF2]). Then by using (a version) of the Frenkel-Zhu's fromula we calculated the fusion ring for the family $\mathcal{F}_1$. Since formula for the fusion rules from [FZ] is not true in general (see counterexample [L1]-[L2]) it is worth mentioning that we have used a version of Frenkel-Zhu's formula in order to get a result. Slight mismatch with the original fusion coefficients is related to the embedding structure of the Verma modules. This explain why some fusion rules are 2 if we use Frenkel-Zhu's formula, even though the fusion coefficients are known to be ≤ 1.

Finally, we show that the fusion ring for the family $\mathcal{F}_1$ is isomorphic to a Grothendieck ring $\mathcal{R}ep(\mathfrak{sl}(2,\mathbb{C}))$, i.e. that we "formally" have

$$L\left(1,\frac{n^2}{4}\right)\times L\left(1,\frac{m^2}{4}\right)=$$

$$L\left(1,\frac{(n+m)^2}{4}\right)+L\left(1,\frac{(n+m-2)^2}{4}\right)\ldots+L\left(1,\frac{(n-m)^2}{4}\right),$$

where $m,n\in\mathbb{N}$ and $n\geq m$. This result was predicted in [FKRW] as a part of more general conjecture concerning fusion rings for $W(\mathfrak{gl}_N)$ algebras). In the superalgebra case the conjecture holds for $\mathfrak{osp}(1|2)$ (see bellow) and we hope that some version is true for $\mathfrak{osp}(2|2)$ as well. Physicists seem to be familiar with fhese fusion rules for a while (it is hard to trace this result, but certanly it emanates from [BPZ]). We stress also, that in [FM] the similar result is quoted, but for the vertex operator algebra $L(c,0)$ at the generic level.

We have to stress that these fusion coefficients are simply derived from the space of intertwining operators between irreducible modules. In other words it is *not* true that the only modules which "fuse" with $L\left(1,\frac{n^2}{4}\right)$ and $L\left(1,\frac{m^2}{4}\right)$ are completely reducible. This fact makes impossible to implement $P(z)$–tensor product construction from [HL1]. The resolution might be constructing (a new) tensor product which takes only irreducible modules into account, but this approach is very cumbersome. Better approach would be working in the larger family $\bar{\mathcal{F}}_1$, which consists of all quotients of Verma modules $M(1,\frac{m^2}{4})$. The possible constructions will be discussed elsewhere.

Later we showed how to construct all intertwining operators from the lattice vertex operator algebra V_L and its irreducible module $V_{L+1/2}$.

The second part is a natural supplement to a paper [HM] where authors constructed vertex tensor categories and intertwining operator algebras associated to the minimal models for the Neveu-Schwarz superalgebra. Here we discuss the fusion ring of the family of modules $\mathcal{F}_{3/2}$ for the Neveu-Schwarz $N=1$ vertex operator superalgebra $L(\frac{3}{2},0)$ which consists of modules isomorphic to $L(\frac{3}{2},\frac{q^2}{2})$, where $q \in \mathbb{N}$. Some calculations have been done in this direction by physicists but these result are not satisfactory (from our point of view). We proved (see [M]) that a fusion ring is isomorphic to the Grothendieck ring $\mathcal{R}ep(\mathfrak{osp}(1|2))$, i.e. that we formally have

$$L\left(\frac{3}{2}, \frac{r^2}{2}\right) \times L\left(\frac{3}{2}, \frac{q^2}{2}\right) =$$

$$L\left(\frac{3}{2}, \frac{(r+q)^2}{2}\right) + L\left(\frac{3}{2}, \frac{(r+q-1)^2}{2}\right) \ldots + L\left(\frac{3}{2}, \frac{(r-q)^2}{2}\right),$$

for $r, q \in \mathbb{N}$, $r \geq q$.

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2 Representation of the Virasoro algebra at level $c = 1$

Representation theory of the Virasoro algebra has been studied intensively in the last two decades ([KR], [FF1]–[FF2]). Kac's determinant formula is the most important tool in the highest weight theory. From the Kac's determinant formula it follows that the highest weight Verma module with a central charge $c(t) = 13 - 6t - 6t^{-1}$ and the weight

$$h_{p,q}(t) = \frac{1-p^2}{4}t^{-1} - \frac{1-pq}{2} + \frac{1-q^2}{4}t,$$

has a singular vector of weight $h_{p.q}(t) + pq$, $t \in \mathbb{C}$. We are interested in the case $t = 1$, i.e $c = 1$. It follows that $M(1, h)$ is irreducible if and only if $h \neq \frac{m^2}{4}$ for some $m \in \mathbb{N}$. In the case $h = \frac{m^2}{4}$ we have the following description

Proposition 2.1 *Verma module $M(1, \frac{m^2}{4})$ has the unique singular vector of the weight $\frac{m^2}{4} + (m+1)$. This vector generate the maximal submodule. In other words we have the following exact sequence*

$$0 \to M\left(1, \frac{(m+2)^2}{4}\right) \to M\left(1, \frac{m^2}{4}\right) \to L\left(1, \frac{m^2}{4}\right) \to 0. \quad (1)$$

Even though do not exist in general, in the case $h_{1,q}(t)$, i.e. $p = 1$ there are explicit formulas at each level $c(t)$ (which covers our special case $t = 1$). When $c = 1$ Benoit and S. Aubin's formula [BSA1] gives us

$$v_{1,q} = \sum_{\substack{I=\{i_1,\ldots,i_n\} \\ |I|=q}} c_q(i_1, ..., i_n) L_{i_1} \ldots L_{i_n}, \quad (2)$$

where

$$c_r(i_1,\ldots,i_n) = \prod_{\substack{1\leq k<r \\ k\neq \sum_{j=1}^{s} i_j}} k(r-k).$$

Remark 2.1 Note that every singular vector (2) has form $L(-1)^{m+1} + \ldots$, where dots represent lower degree terms (with respect to the universal enveloping algebra grading).

3 Vertex operator algebra $L(1,0)$ and a family $\mathcal{F}_1$

We recall the definition of vertex operator algebra and modules from [FHL] or [FLM]. Also, we recall well know results about the representation theory of Virasoro vertex operator algebras (see [FZ], [W], [L1], [H1]). We define $L(1,0) = M(1,0)/ < L(-1)\mathbf{1} >$, which is a simple vertex operator algebra. To every vertex operator superalgebra we associate Zhu's associative algebra $A(V)$ [Z], [KW]. In the special case of vacuum module $V = L(c,0)$ we know (see [KW]) that $A(V) \cong \mathbb{C}[y]$. where $y = [L(-2) - L(-1)]$. Here, we have chosen a multiplication in $A(V)$ as in [W] (which is slightly different then in [FZ]),

$$a * b = \mathrm{Res}_x Y(a,x)\frac{(1-x)^{deg(a)}}{x} b,$$

where $a, b \in A(V)$

By using standard techniques (see [FZ], [W]) it follows.

Proposition 3.1 *Every irreducible module for the vertex operator algebra $L(1,0)$ is isomorphic to $L(1,h)$, for some $h \in \mathbb{C}$.*

Proof: Since $A(L(1,0)) \cong \mathbb{C}[x]$, and every finite dimensional irreducible $\mathbb{C}[x]$–module is one dimensional , we have the proof. ∎

Now, we define a family $\mathcal{F}_1$ of irreducible modules for $L(1,0)$, such that every irreducible module from $\mathcal{F}_1$ is isomorphic to $L(1,\frac{m^2}{4})$, for some $m \in \mathbb{N}$.

Since the notation of intertwining operators is more subtle we include here the original definition as stated in [FHL].

Definition 3.1 Let W_1, W_2 and W_3 be a triple of modules for vertex operator algebra V. The mapping

$$\mathcal{Y} \mapsto W_1 \otimes W_2 \to W_3\{x\},$$

is called intertwining operator of the type $\binom{W_3}{W_1\ W_2}$, if it satisfies the following properties

1. The *truncation* property: For any $w_i \in W_i$, $i = 1, 2$,

$$(w_1)_n w_2 = 0,$$

   for n large enough.

2. The $L(-1)$-*derivative property*: For any $v \in V$,

$$\mathcal{Y}(L(-1)w_1, x,) = \frac{\partial}{\partial x}\mathcal{Y}(w_1, x),$$

3. The *Jacobi identity*: In $\mathrm{Hom}(W_1 \otimes W_2, W_3)\{x_0, x_1, x_2\}$, we have

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)\mathcal{Y}(w_1, x_2)w_2 \\ -x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_1, x_2)Y(u, x_1)w_2 \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y(u, x_0)w_1, x_2)w_2 \end{aligned} \qquad (3)$$

   for $u \in V$ and $w_1 \in W_1$, $w_2 \in W_2$.

We denote the space of all intertwining operators of the type $\binom{W_3}{W_1\ W_2}$ by $I\binom{W_3}{W_1 W_2}$. The dimension of the space of intertwining operators (also known as "*fusion rule*")of the type $\binom{W_3}{W_1 W_2}$ we denote by $\mathcal{N}_{W_1,W_2}^{W_3}$.

Our goal is to find fusion rules of the type

$$\binom{L(1, \frac{r^2}{4})}{L(1, \frac{p^2}{4})L(1, \frac{q^2}{4})}.$$

Since our modules are irreducible we want to introduce Frenkel-Zhu's formula which (roughly) gives us prescription of calculating fusion rules when W_1, W_2 and W_3 are irreducible V–modules. It is not hard to see, by using the Jacobi identity, that the space $I\binom{L(1, \frac{r^2}{4})}{L(1, \frac{p^2}{4})L(1, \frac{q^2}{4})}$ is at most one dimensional.

Now for every module M, we associate $A(V)$–bimodule $A(M) := M/O(M)$, where $O(M)$ is spanned by the elements of the form

$$\mathrm{Res}_x Y(u, x)\frac{(1 - x)^{deg(a)}}{x^2}v,$$

$u \in V$, $v \in M$. In the case $M = M(c, h)$,

$$O(M(c, h)) = \{(L(-n - 3) - 2L(-n - 2) + L(-1))v, n \in \mathbb{N}, v \in M(c, h)\}. \quad (4)$$

It follows that

$$A(M(c, h)) \cong \mathbb{C}[x, y],$$

where

$$y = [L(-2) - L(-1)], \quad x = [L(-2) - 2L(-1) + L(0)],$$

and

$$y * p(x,y) = xp(x,y), \quad p(x,y) * y = yp(x,y),$$

where $p(x,y) \in A(M(c,h))$.

Now, Frenkel-Zhu's formula states that it is possible to calculate the dimension of the space $\binom{M_3}{M_1\,M_2}$ by knowing $A(V)$, $A(M_1)$, $M_2(0)$ and $M_3(0)$. But this formula does not apply in general. Instead giving original formula from [FZ], we state the following refinement given in [L1]-[L2].

Theorem 3.1 *Let M_1, M_2 and M_3 be irreducible V–modules such that M_2 and M_3' are generalized Verma V–modules. Then we have*

$$\mathcal{N}_{M_1 M_2}^{M_3} = \dim\ \mathrm{Hom}_{A(V)}(A(M_1) \otimes_{A(V)} M_2(0), M_3(0)),$$

where $M_i(0)$, $i = 1,2,3$, are the "top" levels of M_i, respectively, equipped with the $A(V)$-module structure.

Our modules $L(1, \frac{m^2}{4})$ *are not* generalized Verma V–modules, so this formula does not apply. Thus we have to analyze Frenkel-Zhu's formula in more details.

First "half" of the Frenkel-Zhu's formula is easy to prove and it states:

Lemma 3.1 *Let M_3 be an irreducible V–module then*

$$\mathcal{N}_{M_1 M_2}^{M_3} \leq \dim\ \mathrm{Hom}_{A(V)}(A(M_1) \otimes_{A(V)} M_2(0), M_3(0)).$$

Remark 3.1 E Even though Frenkel-Zhu's formula does not apply we want to get description of

$$\mathrm{Hom}_{A(V)}(A(M_1) \otimes_{A(V)} M_2(0), M_3(0)), \tag{5}$$

for various reasons. First, we show that this calculations (and in super case) correspond to a Lie algebra homology (see [W]) , and secondly, we want to show that Frenkel-Zhu's formula is not "so far" from the actual fusion rules.

Define an infinite dimensional Lie algebra $\mathcal{L}$ spanned by

$$L(-n-2) - 2L(-n-1) + L(-n),$$

for $n \in \mathbb{N}$. In the case of minimal models the calculations from [FF2], about the homology of $\mathcal{L}$ with the coefficients in $L(c,h)$, have been used in order to determine the fusion rules. For the Verma modules we have (cf. [W], [L1])

Proposition 3.2 *0-th homology, $H_0(\mathcal{L}, M(1,h))$ with the coefficients in the Verma modules is isomorphic to $\mathbb{C}[x,y]$ as a $A(L(1,0))$–bimodule.*

The following result is application of more general theory [FF1] in our special case.

Theorem 3.2 *We have*

(a)

$$dim\ \mathrm{H}_0\left(\mathcal{L}, L\left(1, \frac{m^2}{4}\right)\right) = \infty.$$

(b) $\mathrm{H}_0(\mathcal{L}, L(1, \frac{m^2}{4}))$ *is finitely generated as a (left)* $A(L(1,0))$*–module.*

(c) Let $m \geq n$. *Then*

$$\mathrm{Ext}^1\left(L(1, \frac{m^2}{4}), L(1, \frac{n^2}{4})\right) = \begin{cases} \mathbb{C} & \text{if } m = n+2 \\ 0 & \text{otherwise} \end{cases}$$

Proof: a) Since the maximal submodule of $M(1, \frac{m^2}{4})$ is generated by one vector, in the projection (or homology) $A(L(1, \frac{m^2}{4}))$ is isomorphic to $\frac{\mathbb{C}[x,y]}{I}$, where I is a cyclic submodule (with respect to the left and right action) generated by some polynomial $p(x, y)$ which is a projection of $v_{1,m}$ in $\mathbb{C}[x, y]$. It is clear that this space is infinite dimensional.
b) Note first that $[L(-1)v] = (y - x + deg(v))[v]$. By using Remark 2.1 it follows that

$$[v_{sing}] = p(x, y) = \prod_{i=1}^{m+1} (x - y + i) + q(x, y).$$

where $\deg(q) < (m + 1)$. Thus, the pure monomials in $p(x, y)$ with the highest powers are x^{m+1} and y^{m+1}. Since, I is spanned by $p(x, y)\mathbb{C}[x]$, here we consider only the left action, it follows that $\frac{\mathbb{C}[x,y]}{I}$ is finitely generated. The similar argument holds for the right action.
c) Apply Ext functor for the exact sequence (1) and use the embedding structure of Verma modules. The corresponding non-split exact sequence is given by

$$0 \to L(1, \frac{(m+2)^2}{4}) \to M(1, \frac{m^2}{4})/M(1, \frac{(m+4)^2}{4}) \to L(1, \frac{m^2}{4}) \to 0.$$

∎

For every $m, n \in \mathbb{N}$ (we exclude the case $mn = 0$), fix the multiset $J_{m,n} = \{m+n, m+n-2, \ldots, m-n\}$. Let $\mathcal{F}_{\lambda,\mu}$ be a "density" module for the Virasoro algebra. $\mathcal{F}_{\lambda,\mu}$ is spanned by w_r, $r \in \mathbb{Z}$ and the action is given by

$$L_n.w_r = (\mu + r + \lambda(m+1))w_{r-n}.$$

In [FF1] the projection formula for the singular vectors (considered as a element of the enveloping algebra) on $\mathcal{F}_{\lambda,\mu}$ (more precisely w_0) was found. We want to relate the projection of the singular vectors on $\mathcal{F}_{\lambda,\mu}$ with the projection inside $A(M(1, \frac{m^2}{4})) \otimes_{C[y]} L(1, \frac{n^2}{4}))$. It is easy to see that

$$[L(-j_1) \ldots L(-j_k)v_{m^2/4}] =$$

$$\prod_{r=1}^{k}(j_r\frac{n^2}{4}-y+\beta(r,k)).[v_{m^2/4}]=$$
$$\prod_{r=1}^{k}(j_r\frac{n^2}{4}-x+\beta(r,k))v_{m^2/4} \tag{6}$$

where $v_{m^2/4}$ is the higest weight vector and

$$\beta(r,m)=j_{r+1}+\ldots+j_k+\frac{m^2}{4}.$$

But the last factor in (6) is the same as the $P(j_1,..,j_k)$ where

$$L(-j_1)\ldots L(-j_k).w_0=P(j_1,...,j_k)w_{j_1+\ldots+j_k},$$

and the projection is in $\mathcal{F}_{\lambda,\mu}$ for $\lambda=-\frac{n^2}{4}$ and $\mu=\frac{n^2}{4}+\frac{m^2}{4}-x$.

In the remarkable paper [FF2], projection formulas for all singular vectors on the density modules were found. In the slightly different notation, for the singular vectors we consider, these formulas appeared in [K]. The result is

$$v_{1,m+1}.w_0=\prod_{i\in J_{m,n}}(x-\frac{i^2}{4})w_{m+1}, \tag{7}$$

up to a multiplicative constant.

Now, by using (7) fact and the discussion above (cf. [W]) we obtain

Lemma 3.2 *As a $A(L(1,0)$–module $A(L(1,\frac{m^2}{4}))\otimes_{A(L(1,0))}L(1,\frac{n^2}{4})(0)$ is isomorphic to $\frac{\mathbb{C}[x]}{<\prod_{i\in J_{m,n}}(x-i^2/4)>}$.*

Notice that as $A(L(1,0))$– module

$$A(L(1,\frac{m^2}{4}))\otimes_{A(L(1,0))}L(1,\frac{n^2}{4})(0)\cong\bigoplus_{i\in J_{m,n}}\mathbb{C}v_i,$$

where v_i is an irreducible $A(L(1,0))$–module such that $y.v_i=i^2/4v_i$. Hence, if $m\leq n$, then some modules appear twice in the above decomposition. Therefore, the Frenkel–Zhu's formula does not apply (fusion rules are at most 1). The same failure was already noticed in [L1]). On the other hand this does not happen when $n\leq m$, i.e. in the decomposition every module appear once. Anyhow, using the formula Lemma 3.2 and Lemma 3.1 we obtain

Proposition 3.3 *Let $L(1,\frac{m^2}{4})$, $L(1,\frac{n^2}{4})$ and M irreducible $L(1,0)$–modules. Then we have the following upper bounds*

$$\dim I\binom{M}{L(1,\frac{m^2}{4})L(1,\frac{n^2}{4})}\leq\begin{cases}1 \text{ if } M\cong L\left(1,\frac{r^2}{4}\right) \text{ for } r\in J^*_{m,n},\\ 0 \text{ otherwise}\end{cases} \tag{8}$$

where $J^*_{m,n}=\{m+n,\ldots,|m-n|\}$.

■

Now, we shall show that actually there is an equality in the equation (8). There are several ways of doing this. We will provide two different proofs. One which uses the properties of Verma modules and the other one which uses free field realization of the modules $L(1, \frac{m^2}{4})$.

In [H1] it was showed that every $A(V)$ homomorphism from $A(W_1) \otimes_{A(V)} W_2(0)$ to $W_3(0)$ does not necessary lead to an intertwining operator of the form $\binom{W_3}{W_1\,W_2}$ but rather to $\binom{F(W_3(0)^*)'}{W_1\,F(W_2(0))}$, where for a $A(V)$–module U we denote by $F(U)$ a generalized Verma V–module (for the definition see [L1] or [L2]) . In the case when V is rational $F(W_2(0)) \cong W_2$ and $F(W_3(0)^*)' \cong W_3$ ([L2]). But the main difficulty is that if vertex operator algebra is not rational then the generalized Verma module $F(W_2(0))$ may not be isomorphic to W_2. Suppose $m \leq n$, then $F(L(1, \frac{m^2}{4})(0)) \cong M(1, \frac{m^2}{4})$, thus py picking an arbitrary homomrphism we obtain an intertwining operator of the form

$$\binom{M(1, \frac{r^2}{4})}{L(1, \frac{m^2}{4})M(1, \frac{n^2}{4})},$$

for every $r \in \{m+n, \ldots, m-n\}$. This operator can be pushed down to a (non–trivial) intertwining operator of the type

$$\binom{L(1, \frac{r^2}{4})}{L(1, \frac{m^2}{4})M(1, \frac{n^2}{4})}.$$

Let $\mathcal{Y}$ be any such operator. The aim is to construct intertwining operator of the form $\binom{L(1, \frac{r^2}{4})}{L(1, \frac{m^2}{4})L(1, \frac{n^2}{4})}$. The following isomorphism is part of the general theory (see [FHL]),

$$I\binom{L(1, \frac{r^2}{4})}{L(1, \frac{m^2}{4})M(1, \frac{n^2}{4})} \cong I\binom{L(1, \frac{r^2}{4})}{M(1, \frac{n^2}{4})L(1, \frac{m^2}{4})}.$$

Under this isomorphism $\mathcal{Y} \mapsto \mathcal{Y}^{opp}$. Note, that every irreducible module which appears in the decomposition of

$$A\left(L\left(1, \frac{m^2}{4}\right)\right) \otimes_{A(L(1,0))} L\left(1, \frac{n^2}{4}\right)(0)$$

already appears in the decomposition of

$$A\left(L\left(1, \frac{n^2}{4}\right)\right) \otimes_{A(L(1,0))} L\left(1, \frac{m^2}{4}\right)(0),$$

(but now with the multiplicity is at most 1).

Since

$$\mathrm{Hom}_{A(L(1,0))}(A(M(1, \frac{n^2}{4})) \otimes_{A(L(1,0))} L(1, \frac{m^2}{4})(0), L(1,h)(0)),$$

is one dimensional and isomorphic to $L(1,h)(0)$ (and the corresponding homomorphism is the evaluation map), this homomorphism corresponds to $\mathcal{Y}^{opp}$. We want to project the intertwining operator $\mathcal{Y}^{opp}$ to $L(1,\frac{n^2}{4})$. But we know that

$$A(L(1,\frac{n^2}{4}))\otimes_{A(L(1,0))} L(1,\frac{m^2}{4})(0)\cong \frac{\mathbb{C}[x]}{p(x)\mathbb{C}[x]}.$$

where $p(x)$ is a certain polynomial, with roots of $p(x)$ within the set $J_{m,n}$, so we see that in the case when homomorphism is non-trivial

$$A(M(1,\frac{(n+2)^2}{4}))\otimes_{A(L(1,0))} L(1,\frac{m^2}{4})(0)\mapsto 0.$$

Thus since $M(1,\frac{(n+2)^2}{4})$ is a maximal submodule of $M(1,\frac{n^2}{4})$ the intertwining operator can be projected to $L(1,\frac{n^2}{4})$. Thus we obtain (in the non-canonical way) a non–trivial intertwining operator $\bar{\mathcal{Y}}^{opp}$ of the type $\binom{L(1,\frac{r^2}{4})}{L(1,\frac{m^2}{4})L(1,\frac{n^2}{4})}$. So we have proved

Theorem 3.3 *In the Proposition 3.3 the equality holds.*

Remark 3.2 In general if M is any $L(1,0)$–module and

$$\mathcal{Y}\in I\binom{M}{L(1,\frac{m^2}{4})L(1,\frac{n^2}{4})},$$

then M is not necessary completely reducible. Also, note that we exluded the case $mn=0$. If m or n are equal to zero then we deal with intertwining operators among two irreducible modules and vertex operator algebras, which are well known.

This yields.

Theorem 3.4 *Let $\mathcal{A}$ be a free Abelian group on the set $\{a(m):m\in\mathbb{N}\}$ and*

$$\times:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$$

a binary operation defined by the formula

$$a(m)\times a(n)=\sum_{r\in\mathbb{N}\cup 0}\mathcal{N}_{L(1,\frac{m^2}{4})L(1,\frac{n^2}{4})}^{L(1,\frac{r^2}{4})}a(r).$$

Then $\mathcal{A}$ is a commutative associative ring with the multiplication

$$a(m)\times a(n)=a(m+n)+a(m+n-2)+\ldots+a(|m-n|),$$

i.e. $\mathcal{A}$ is isomorphic to a Grothendieck ring $\mathcal{R}ep(\mathfrak{sl}(2,\mathbb{C}))$.

∎

Note that in our proof we actually analyzed more carefully the failure of Frenkel-Zhu's formula. One should not expect to apply our procedure in the more general setting, because Virasoro vertex operator algebra has a quite simple structure. Certainly it would be interesting to study a class of vertex operator algebra for which

$$A(W_1) \otimes_{A(V)} W_2(0) \cong A(W_2) \otimes_{A(V)} W_1(0), \tag{9}$$

for any choice of irreducible modules W_1 and W_2. Then we hope that for this class of vertex algebras some version of Frenkel-Zhu's formula indeed apply. Assumption (9) turns out to be very natural since

$$I\binom{W_3}{W_1 W_2} \cong I\binom{W_3}{W_2 W_1}. \tag{10}$$

4 Construction of all intertwining operators for the family $\mathcal{F}_1$

4.1 V_L vertex operator algebra and its irreducible modules

Let L be a rank one even lattice with a generator β normalized such that $\langle \beta, \beta \rangle = 1$ and let $\alpha = \sqrt{2}\beta$. Thus $\langle \alpha, \alpha \rangle = 2$. As in [FLM], [DL] we define V_L as a vector space

$$V_L = M(1) \otimes \mathbb{C}[L],$$

where $M(1)$ is the level one irreducible module for Heisenberg algebra $\hat{h}_Z$ associated to one–dimensional abelian algebra $h = L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{C}[L]$ is the group algebra of L with a generator e^{α}. Put $\omega = \frac{1}{2}\beta(-1)^2$. Then V_L is a vertex operator algebra (see [FLM]) with the Virasoro element ω. We have a decomposition

$$V_L = \bigoplus_{m \in \mathbb{Z}} M(1) \otimes e^{m\alpha}.$$

Let L^o be a dual lattice, $L^o/L \cong \mathbb{Z}/2\mathbb{Z}$. Then (as in [DL]), for a nontrivial coset representative, we obtain irreducible V_L–module $V_{L+1/2}$, which can be decomposed as

$$V_{L+1/2} = \bigoplus_{m \in \mathbb{Z}} M(1) \oplus e^{m\alpha + 1/2\alpha}.$$

Moreover, $V_{L+1/2}$, V_L is (up to equivalence) complete list of irreducible V_L–modules. Furthermore, one can equip the space $W = V_L \oplus V_{L+1/2}$ (as in [DL]) with the structure of the generalized vertex operator algebra. We will neglect this fact in our considerations.

For every module W for the Virasoro algebra on which $L(0)$ acts semisimple we define a formal character (or a q-graded dimension) by

$$ch_q(W) = \sum_{n \in \mathrm{Spec} L(0)} \dim(W_n) q^n.$$

From the Proposition 2.1 it follows that

$$ch_q(L(1, \frac{m^2}{4})) = \frac{q^{\frac{m^2}{4}} - q^{\frac{(m+2)^2}{4}}}{q^{-1/24}\eta(q)}.$$

Then it is not hard to obtain

$$ch_q(V_L) = \sum_{n\geq 0}(2n+1)ch_q(L(1,n^2))$$

$$ch_q(V_{L+1/2}) = \sum_{n\geq 0}(2n+2)ch_q\left(L\left(1, \frac{(2n+1)^2}{4}\right)\right). \qquad (11)$$

Consider the vectors

$$x = e^{\alpha},\ y = e^{-\alpha},\ h = \alpha(-1)\iota(0),$$

which span $(V_L)_1$. These vectors span a Lie algebra isomorphic to $\mathfrak{s}l(2,\mathbb{C})$. x_0, y_0 and h_0 as act derivatives on W. The following result was obtained in [DG].

Proposition 4.1 *As* $(L(1,0),\mathfrak{sl}_2)$*–module*

$$V_L \cong \bigoplus_{m\geq 0} L(1,m^2)\otimes V(2m),$$

where $V(2m)$ *is an irreducible* $2m+1$ *dimensional* $\mathfrak{sl}_2$*–module.*

∎

The proof uses the result from [DLM], [DM1] about the decomposition of the vertex operator algebra V with respect to a "dual" pair (V^G, G) where $G = Aut(G)$ is a compact (or finite) group and V^G is a G–stable subvertex operator algebra. This can be modified when instead of group G we work with the Lie algebra.

Since $V_{L+1/2}$ is a module for the pair $(V_L^{\mathfrak{sl}_2}, \mathfrak{sl}_2)$ then by using (11) we derive

$$V_{L+1/2} \cong \bigoplus_{m\geq 0} L(1, \frac{(2m+1)^2}{4})\otimes V(2m+1), \qquad (12)$$

where $V(2m+1)$ is a $2m+2$–dimensional $\mathfrak{sl}_2$–module. Since $e^{m\alpha+1/2\alpha}$ is a highest weight vector of the weight $2m+2$ for sl_2 and it generates the representation of Virasoro algebra isomorphic to $L(1, \frac{(2m+1)^2}{4})$. Then it follows that $V(2m+1)$ is irreducible $\mathfrak{sl}_2$–module.

Remark 4.1 Note that V^{sl_2} (sl_2–stable vertex operator algebra) is exactly V^G where $G \cong SO(3)$ is a (full) group of automorphisms of V_L. It is well known that every irreducible representation can be obtain as a representation of $SL(2,\mathbb{C})$, since $PSL(2,\mathbb{C}) \cong SO(3)$. In particular every such representation is odd–dimensional. Thus, one has to consider the double cover, i.e. $SL(2,\mathbb{C})$ to obtain all representations. Therefore one has to consider space which is "twice bigger", i.e. $W = V_L \oplus V_{L+1/2}$.

Since, $V_{L+1/2}$ is an irreducible V_L–module we have the Jacobi identity

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)w\\
&-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)w\\
&=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)w,
\end{aligned}\tag{13}$$

for every $u \in V_L$, $v \in V_{L+1/2}$ and $w \in W$. Also, for

$$\mathcal{Y} \in I\binom{V_L}{V_{L+1/2}V_{L+1/2}},$$

we have

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)\mathcal{Y}(v,x_2)w\\
&-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)w\\
&=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(u,x_0)v,x_2)w.
\end{aligned}\tag{14}$$

Remark 4.2 Note that W can not be equipped with a vertex operator superalgebra structure. If $u, v \in V_{L+1/2}$ then we do not get Jacobi identity in the form (13) or (14), but rather generalized identity where the delta function is suitably multiplied with the terms of the type $\left(\frac{x_1-x_0}{x_2}\right)^{1/2}$. Studying this (generalized) Jacobi identity is useful for studying convergence and the extension properties for the intertwining operators (cf. [H1]).

4.2 Intertwining operators for the family $\mathcal{F}_1$.

Let $V(i)$, $i \in \mathbb{N}$ be an irreducible sl_2–module considered as a subspace of W which corresponds to the decompositions (4.1) and (12). Fix a positive integer j. We introduce a basis $u_j(m)$, $m \in \{j, j-2, \ldots, -j\}$ for $V(j)$, such that the following relations are satisfied,

$$\begin{aligned}
h.u_j(m) &= m u_j(m)\\
x.u_j(m) &= \frac{\sqrt{(j+m+2)(j-m)}}{2}u_j(m+2)\\
y.u_j(m) &= \frac{\sqrt{(j+m)(j-m+2)}}{2}u_j(m-2),
\end{aligned}\tag{15}$$

where $u_j(k) = 0$ for $k \notin \{j, \ldots, -j\}$. Also, we choose a dual basis $u_j^*(m)$ for $V(j)^*$ such that $< u_j^*(m), u_j(n) >= \delta_{m,n}$. Define $< g.u^*, v >= - <$

$u^*, g.v >$. Then $V(j)^*$ became a sl_2–module and an isomorphism from $V(j)$ to $V(j)^*$ is given by $\mu(u_j(m)) = (-1)^{j-m} u_j^*(-m)$. By using this identification, for $j_1, j_2, j_3 \in \mathbb{N}$ and $-j_i \leq m_i \leq j_i$, $i = 1, 2, 3$, we introduce real numbers (essentially Clebsch–Gordan coefficients) $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$, such that

$$u_{j_1}(m_1) \otimes u_{j_2}(m_2) = \sum_{j_3=|j_1-j_2|}^{j_3=j_1+j_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1+m_2 \end{pmatrix} u_{j_3}(m_1+m_2). \quad (16)$$

First we need an auxiliary result which is slightly modified result from [DM1] and [DG].

Proposition 4.2 *Suppose that V is a vertex operator algebra and W_1, W_2 and W_3 three irreducible V–modules. Let $v_i \in W_1, w_i \in W_2$, $i = 1, ..., k$ be homogeneous elements such that $v_i \neq 0$ and w_i are linearly independent. Then*

$$\sum_{i=1}^{k} \mathcal{Y}(v_i, x) w_i \neq 0.$$

∎

Now. let us go back to our vertex operator algebra V_L. Let $\mathcal{Y}$ be any intertwining operator of the type

$$\binom{V_L}{V_{L+1/2} V_{L+1/2}}, \binom{V_{L+1/2}}{V_L V_{L+1/2}} \text{ or } \binom{V_L}{V_L V_L}. \quad (17)$$

By using the Proposition 4.2 the map

$$\mathcal{Y}(\ , x)\ : V(j_1) \otimes V(j_2) \rightarrow W\{x\}$$

is injective. and for every m_1, m_2 and j_1, j_2 there is a $p \in \mathbb{C}$ such that

$$u_{j_1}(m_1)_p u_{j_2}(m_2) = \sum_{j_3=|j_1-j_2|}^{j_3=j_1+j_2} k(j_1, j_2, j_3, m_1, m_2, m_1+m_2) u_j(m_1+m_2),$$

where $k(j_1, j_2, j_3, m_1, m_2, m_1+m_2)$ is a (non–zero) multiple of

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1+m_2 \end{pmatrix}.$$

(in the special case $\mathcal{Y} = Y$ this fact was noticed in [DG]).

Now it is clear that if $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1+m_2 \end{pmatrix} \neq 0$, then the $L(1,0)$–module generated by $\mathcal{Y}(u_{j_1}(m_1), x) u_{j_2}(m_2)$ contains a copy of $L(1, \frac{j_3^2}{4})$. Since $L(1,0)$ is contained in V_L and $L(1, \frac{m^2}{4})$ is a $L(1,0)$–module then we obtain the following Jacobi identity

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)\mathcal{Y}(v,x_2)w - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)w$$
$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(u,x_0)v,x_2)w, \tag{18}$$

for $u \in L(1,0)$, $v \in L(1, \frac{j_1^2}{4})$ and $w \in L(1, \frac{j_2^2}{4})$ (here v and w lie in Vir–submodules generated by $u_{j_1}(m_1)$ and $u_{j_2}(m_2)$, respectively).

Now we can push down $\mathcal{Y}$ to $L(1, \frac{j_3^2}{4})$, which is generated by the vector $u_{j_3}(m_1+m_2)$. Since for every j_1, j_2 and $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$ we can choose a pair m_1, m_2 and a $\mathcal{Y}$ of the appropriate type (17) such that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1+m_2 \end{pmatrix} \neq 0.$$

Hence, we obtain an intertwining operator of the type

$$\binom{L(1, \frac{j_3^2}{4})}{L(1, \frac{j_1^2}{4})L(1, \frac{j_2^2}{4})}.$$

So this is the end of the construction.

Remark 4.3 In the last part of the construction we "pushed down" intertwining operators for V_L–modules to an intertwining operator for a subalgebra $L(1,0)$. Since we use the the definition of the subalgebra from [FHL] (compare with [FZ]), i.e. $\omega_W = \omega_V$, Virasoro relation and $L(-1)$–property hold automatically.

5 Concluding remarks

At the end let us stress a few possible applications of these result which is left for a further studies. Also we raised an open questions.

1. As we mention in the beginning, one needs to develop a tensor product theory (as in [HL1]) which apply in this setting. Even in the generic level case there are some subtielties one can indeed do this because the maximal submodule is irreducible, thus for certain family of modules, Huang-Lepowsky tensor product theory applies. Some of related things are discussed in [FM].

2. We know that is possible to obtain *intertwining operator algebra* (see [H2]) from the rational vertex operator algebra (which satisfies some natural convergence and extension conditions). Since the notation of intertwining operator algebra can be extended more general (such that undelying

structure is infinite–dimensional associative, commutative algebra) one hopes that it is possible to construct such a structure for the families $\mathcal{F}_1$ and $\mathcal{F}_{3/2}$. In the language of the conformal field theory this involves explicit calculations of correlation function for both products and iterates of intertwining operators (cf. Remark 4.2).

3. For rational vertex operator algebras, construct a *natural* isomorphism

$$\mathrm{Hom}_{A(V)}(A(M_1) \otimes_{A(V)} M_2(0)) \cong \mathrm{Hom}_{A(V)}(A(M_2) \otimes_{A(V)} M_1(0)).$$

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