Fault-tolerant cycle-embedding of crossed cubes ✩

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Abstract

The crossed cube CQ n introduced by Efe has many properties similar to those of the popular hypercube. However, the diameter of CQ n is about one half of that of the hypercube. Failures of links and nodes in an interconnection network are inevitable. Hence, in this paper, we consider the hybrid fault-tolerant capability of the crossed cube. Letting fe and fv be the numbers of faulty edges and vertices in CQ n, we show that a cycle of length l, for any 4 ≤ l ≤ |V(CQ n)| − fv, can be embedded into a wounded crossed cube as long as the total number of faults (fv + fe) is no more than n − 2, and we say that CQ n is (n − 2)-fault-tolerant pancyclic. This result is optimal in the sense that if there are n − 1 faults, there is no guarantee of having a cycle of a certain length in it.

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1. Introduction

Network topology is essential for parallel and distributed computation, and many topologies have been proposed, for example, hypercubes, butterfly graphs and star graphs. The hypercube is one of the most popular networks since it has a simple structure and is easy to implement. However, there are still some different points of view to construct new topologies; for example, a new topology having smaller diameter.

To lower the diameter, we may change some links of the hypercube. Some variations of the hypercube have been studied in the literature. In [2], Efe first studied the crossed cube CQ n, which has a structure similar to that of the hypercube, including recursive structure, the same number of vertices, and the same number of edges. However, the diameter of CQ n is only about one half of that of the hypercube, and the diameter is an important factor for parallel computing speed. Other studies have been done to explore more properties of the crossed cube CQ n, such as edge congestion of CQ n, as studied in [1]. Furthermore, embedding of binary trees, hamiltonian paths, and hamiltonian cycles into CQ n were discussed in [5,6].

The graph embedding problem asks if a guest graph is a subgraph of a host graph, and an important benefit of graph embedding is that we can apply...
existing algorithms for guest graphs to host graphs. This problem has attracted a burst of studies over the years.

The pancycle problem asks if a cycle of length \( l \) is a subgraph of a given graph with a given positive integer \( l \). Hwang [4] and Fan [3] et al. studied this problem on butterfly graphs and Möbius cubes, respectively. But they did not consider the possibilities of failures of nodes and/or links.

Failures are inevitable when a network is put in use. Therefore, the fault-tolerant capacity of an interconnection network is a crucial issue in parallel computing. Furthermore, both nodes and links may simultaneously be faulty in a network. Hence we study the hybrid fault tolerance of \( CQ_n \) in this paper. Letting \( f_v \) and \( f_e \) be the numbers of faulty vertices and edges in \( CQ_n \), respectively, we show that a cycle of length \( l \), for any \( 4 \leq l \leq |V(CQ_n)| - f_v \), is a subgraph of a wounded crossed cube with \( (f_v + f_e) \leq (n - 2) \). That is, \( CQ_n \) is \((n - 2)\)-fault-tolerant pancyclic. In addition, this result is optimal, and the reason is explained as follows: The \( n \)-dimensional crossed cube is \( n \)-regular. As a result, if there are \((n - 1)\) faulty edges incident to a single node, a hamiltonian cycle cannot be embedded into a wounded \( CQ_n \).

The rest of this paper is organized as follows: Section 2 includes the definition of the crossed cubes and some basic notation and terminologies. Then, the proof of the pancyclicity of \( CQ_n \) is given in Section 3. For the case \( n = 4 \), the proof is a little tedious, and we leave some parts of it to Appendix A.

2. Definitions and notation

Given a simple graph \( G \), we use \( V(G) \) and \( E(G) \) to denote the vertex and edge sets of \( G \), respectively. In order to define the crossed cube \( CQ_n \), as proposed by Efe [2], the pair related set \( R \) is introduced. Let \( R = \{(00,00),(10,10),(11,01),(01,11)\} \). Two binary strings \( a_1a_2 \) and \( b_1b_2 \) of length 2 are pair related, denoted by \( a_1a_2 \sim b_1b_2 \), if \( (a_1a_2, b_1b_2) \in R \). The following is the recursive definition of the \( n \)-dimensional crossed cube \( CQ_n \). \( CQ_n \) has \( 2^n \) vertices, each labeled by a binary string of length \( n \). \( CQ_1 \) is a complete graph with two vertices labeled 0 and 1, respectively. For \( n \geq 2 \), \( CQ_n \) is obtained by taking two copies of \( CQ_{n-1} \), denoted by \( CQ_{n-1}^0 \) and \( CQ_{n-1}^1 \), respectively, and adding \( 2^{n-1} \) edges as follows:

Let
\[
V(CQ_{n-1}^0) = \{0x_{n-2} \ldots x_1x_0: x_i = 0 \text{ or } 1\}
\]
and
\[
V(CQ_{n-1}^1) = \{1y_{n-2} \ldots y_1y_0: y_i = 0 \text{ or } 1\}.
\]

A vertex \( 0x_{n-2} \ldots x_1x_0 \in V(CQ_{n-1}^0) \) and a vertex \( 1y_{n-2} \ldots y_1y_0 \in V(CQ_{n-1}^1) \) are adjacent if
\[
\begin{align*}
(1) & \quad x_{n-2} = y_{n-2} \text{ if } n \text{ is even, and} \\
(2) & \quad x_{2i+1}x_{2i+2} \sim y_{2i}y_{2i+1}y_{2i+2} \text{ for } 0 \leq i < [(n - 1)/2].
\end{align*}
\]

We take \( CQ_3 \) and \( CQ_4 \) as examples and display them in Fig. 1 (a) and (b), respectively. In Fig. 1(c), we use a different way to draw \( CQ_3 \) in order to see its vertex-symmetry.

We now introduce some basic terminologies and notation needed for later discussion. A path is a sequence of vertices with any two consecutive vertices being adjacent in \( G \). We use \( \{u_1, u_2, \ldots, u_l\} \) to denote a path that begins with \( u_1 \) and ends with \( u_l \). In addition, \( \{u_1, u_2, \ldots, u_l\} \) is a cycle if \( u_1 = u_l \). A hamiltonian path is defined as a path which contains all the vertices of \( G \) exactly once. A graph \( G \) is hamiltonian connected if, for any two vertices of \( G \), there exists a hamiltonian path between them. We say that a graph \( G \) is pancyclic if \( G \) contains a cycle of length \( l \) as a subgraph, for every \( 4 \leq l \leq |V(G)| \). A cycle is a hamiltonian cycle if it traverses all the vertices of \( G \) exactly once. A graph \( G \) is hamiltonian if \( G \) contains a hamiltonian cycle.

To consider a wounded graph, we give the following terminologies and notation. Given a graph \( G \), let \( F_v \subseteq V(G) \) and \( F_e \subseteq E(G) \); and \( F = F_v \cup F_e \). Let \( G' \) be the graph obtained from \( G \) by deleting all the edges in \( F_e \). We use \( G - F \) to denote the subgraph of \( G' \) induced by \( V(G') - F_v \). We call a graph \( G \) k-fault-tolerant hamiltonian connected (abbreviated as \( k \)-hamiltonian connected) if \( G - F \) is hamiltonian connected for any \( F \) with \( |F| \leq k \). We call a graph \( G \) k-fault-tolerant hamiltonian (abbreviated as \( k \)-hamiltonian) if \( G - F \) is hamiltonian for any \( F \) with \( |F| \leq k \). A graph \( G \) is called \( k \)-fault-tolerant pancyclic (abbreviated as \( k \)-pancyclic) if \( G - F \) is pancyclic for any \( F \) with \( |F| \leq k \).
3. Main result

We use $CQ_{n-2}$ to denote an $(n-2)$-dimensional crossed cube which is a subgraph of $CQ_n$ induced by the vertices labeled $i\,x_{n-3}\ldots\,x_0$. We say that an edge is a critical edge of $CQ_n$ if it is an edge in $CQ_{n-1}$ with one endpoint in $CQ_{n-2}$ and the other in $CQ_{n-1}$ for $i \in \{0, 1\}$.

**Lemma 1.** Let $(u_1, u_2)$ be a critical edge of $CQ_n$ which is in $CQ_{n-1}$, and $v_1, v_2$ be the neighbors of $u_1$ and $u_2$ in $CQ_{n-1}$, respectively, for $n \geq 4$. Then $(v_1, v_2)$ is also a critical edge of $CQ_n$ in $CQ_{n-1}$.

**Proof.** We discuss two cases: (1) $n$ is even, and (2) $n$ is odd.

**Case 1.** $n$ is even. Without loss of generality, we assume that $u_1 = 00\,x_{n-3}\ldots\,x_1\,0$ and $u_2 = 01\,y_{n-3}\ldots\,y_1\,0$, where $x_2+i\,x_{2i} \sim y_2+i\,y_{2i}$ for $0 \leq i \leq \lfloor(n-3)/2\rfloor$. Then $v_1 = 10\,y_{n-3}\ldots\,y_1\,0$, and $v_2 = 11\,x_{n-3}\ldots\,x_1\,0$. By definition, $v_1$ and $v_2$ are adjacent, and $(v_1, v_2)$ is a critical edge in $CQ_{n-1}$.

**Case 2.** $n$ is odd. Without loss of generality, we assume that $u_1 = 00\,x_{n-3}\ldots\,x_1\,1\,0$. Suppose that $x_{n-3} = 0$. Then $u_1 = 00\,x_{n-4}\ldots\,x_1\,0$, $u_2 = 01\,y_{n-4}\ldots\,y_1\,0$, where $x_2+i\,x_{2i} \sim y_2+i\,y_{2i}$ for $0 \leq i \leq \lfloor(n-4)/2\rfloor$, $v_1 = 100\,y_{n-4}\ldots\,y_1\,0$, and $v_2 = 110\,x_{n-4}\ldots\,x_1\,0$. Thus, $v_1$ and $v_2$ are adjacent, and $(v_1, v_2)$ is a critical edge in $CQ_{n-1}$. It can be checked that the statement is also true for the case $x_{n-3} = 1$.

It is observed that vertices $u_1, u_2, v_1, v_2$ in the above lemma form a 4-cycle. We call this cycle a crossed 4-cycle in $CQ_n$. It is clear that, for each vertex $00\,x_{n-3}\ldots\,x_0$, there is exactly one crossed 4-cycle corresponding to the vertex. Thus, there are $2^{n-2}$ disjoint crossed 4-cycles in $CQ_n$. We note that a crossed 4-cycle contains two critical edges.

Huang et al. [5] showed the validity of the following theorem. Based on this theorem, we show the pancyclicity of the crossed cube by induction.

**Theorem 1** [5]. The crossed cube $CQ_n$ is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for $n \geq 3$.

The base case is $n = 3$, and the proof is given in the following.

**Theorem 2.** $CQ_3$ is 1-pancyclic.

**Proof.** Note that $CQ_3$ can be redrawn as Fig. 1(c), and it is vertex-transitive. We consider two cases: (1) one faulty vertex, and (2) one faulty edge as follows:

**Case 1.** One faulty vertex. Without loss of generality, we assume that vertex $x = 000$ is faulty. We list cycles of lengths from 4 to 7 as follows: (001, 111, 101, 011, 001), (001, 110, 101, 011, 011, 001, 110, 101, 011, 011, 001), (001, 110, 101, 011, 011, 001, 110, 101, 011, 011, 001).

**Case 2.** One faulty edge. Without loss of generality, we assume that the faulty edge $e$ is incident to 000. By case 1, there are cycles of lengths from 4 to 7 in the faulty $CQ_3$. For a cycle of length 8, suppose that $e = (000, 010)$. Then (000, 001, 111, 110, 010, 011, 101,
100, 000) is the desired one. Suppose that \( e = (000, 001) \). Then \((000, 010, 110, 111, 001, 011, 101, 100, 000)\) is a cycle of length 8. If \( e = (000, 000) \), the case is symmetric to the case \( e = (000, 001) \).

Let \( F \) be a set of faults in \( CQ_n \). We say that a vertex \( u \) in one subcube of \( CQ_n \) is a safe crossing-point in \( CQ_n - F \) if \( u \) still connects to the neighbor in the other subcube in \( CQ_n - F \), i.e., the corresponding neighbor \( u' \) in the other subcube of \( u \) is fault-free, and the edge \((u, u')\) is also fault-free. The main result is as follows.

**Theorem 3.** The crossed cube \( CQ_n \) is \( (n - 2) \)-pancyclic for \( n \geq 3 \).

**Proof.** We prove this by induction on \( n \). It follows from Theorem 2 that \( CQ_3 \) is 1-pancyclic. Now we proceed to the induction step. Suppose that \( CQ_{n-1} \) is \((n-3)\)-pancyclic for some \( n \geq 4 \). We will show that \( CQ_n \) is \((n-2)\)-pancyclic. Let \( F \subseteq V(CQ_n) \cup E(CQ_n) \) be the set of faults. We divide \( F \) into five disjoint parts:

\[
F_0^v = F \cap V(CQ_{n-1}^0), \quad F_0^e = F \cap E(CQ_{n-1}^0),
F_1^v = F \cap V(CQ_{n-1}^1), \quad F_1^e = F \cap E(CQ_{n-1}^1),
F^e = F \setminus \{(u, v) \mid (u, v) \text{ is an edge between } CQ_{n-1}^0 \text{ and } CQ_{n-1}^1\}.
\]

Let \( f = |F|, f_0^v = |F_0^v|, f_0^e = |F_0^e|, f_1^v = |F_1^v|, f_1^e = |F_1^e|, f^v = |F^v|, \) and \( f^e = |F^e| \). For convenience of discussion, we define the following subsets of \( F \): \( F_v = F \cap V(CQ_n) \), \( F_e = F \cap E(CQ_n) \), \( F_0^v = F_0^v \cup F_0^e \), and \( F_1^v = F_1^v \cup F_1^e \). And let \( f_v = |F_v|, f_e = |F_e|, f_0^v = |F_0^v|, \) and \( f_1^v = |F_1^v| \). Note that \( f^v + f^e = f - f_0^e \).

**Case 1.** There is a subcube containing all the \((n-2)\) faults. Without loss of generality, we assume that \( f_1^v = n - 2 \). Thus, there is no fault outside \( CQ_{n-1}^1 \), i.e., \( f^1 = f^e = 0 \). We discuss the existence of cycles of lengths from 4 to \( 2^n - f_1 \) according to the following cases.

**Case 1.1. Cycles of lengths from 4 to \( 2^{n-1} \).** Since \( CQ_{n-1} \) is \((n-3)\)-pancyclic, \( CQ_{n-1}^1 \) contains cycles of lengths from 4 to \( 2^{n-1} \) for \( n \geq 4 \). Clearly, \( CQ_n - F \) also contains cycles of these lengths.

**Case 1.2. A cycle of length \( 2^{n-1} + 1 \).** (See Fig. 2(a).) We want to construct a cycle containing \( 2^{n-1} - 1 \) vertices in \( CQ_{n-1}^1 \) and two vertices in \( CQ_{n-1}^0 \). To avoid faults in \( CQ_{n-1}^0 \), we introduce a term called the shadows of the faults. Let \( \{u_1, u_2, v_2, v_1, u_1\} \) be a crossed 4-cycle with \( u_1, u_2 \) in \( CQ_{n-1}^0 \) and \( v_1, v_2 \) in \( CQ_{n-1}^1 \), respectively. If there is a fault on this cycle but the fault is not in \( CQ_{n-1}^0 \), we call edge \((v_1, v_2)\) a shadow fault of \( F \) on \( CQ_{n-1}^1 \). (Similarly, we may define a shadow fault on \( CQ_{n-1}^0 \).) Let \( F^s = \{e \mid \text{edge } e \text{ is a shadow fault of } F \text{ on } CQ_{n-1}^i \} \). Since all crossed 4-cycles are vertex disjoint, \( |F^s| \leq n - 2 \). If \( |F^s| = n - 2 \), we arbitrarily pick an edge \( e_1 \) in \( F^s \), and let \( F^e = F^s - e_1 \), or else \( F^e = F^s \). Then \( |F^e| \leq n - 3 \) and \( CQ_{n-1}^i - F^e \) is still pancyclic. So there is a cycle \( C \) of length \( 2^{n-1} - 1 \) in \( CQ_{n-1}^1 - F^e \). Clearly, there are two critical edges on \( C \). Let \( (a, b) \neq e_1 \) be a critical edge on \( C \), so \( (a, b) \notin F^e \). Let \( a', b' \) be the neighbors of \( a \) and \( b \) in \( CQ_{n-1}^0 \), respectively. Then \( (a', a, b, a) \) is a fault-free crossed 4-cycle. Suppose that \( C = (a, Q, b, a) \). Then \( (a', a, Q, b, b', a') \) forms a cycle of length \( 2^n - f_1 \). (See Fig. 2(b).) By Theorem 1, \( CQ_{n-1}^0 \) is \((n-3)\)-hamiltonian and \( f^0 = n - 2 \), \( CQ_{n-1}^0 - F^0 \) still contains...
to Appendix A, and assume that so

2 in all lengths from 4 to 2 are still pancyclic. We discuss the existence of cycles between any two vertices in CQ. Since induction hypothesis, C_{Q_{n-1}} is (n-3)-hamiltonian. We thus have a hamiltonian cycle C = (u_0, u_1, \ldots, u_{2^n-1-f_v^0}, u_0) of length 2^n-1-f_v^0 in CQ_{n-1}-F^0. We claim that there exist two safe crossing-points u_i and u_j on C such that (u_i, u_j, u, v) \in CQ_{n-1}. Without loss of generality, we shall use some symmetric properties of CQ_3 to reduce the cases.

For convenience of discussion (see Fig. 3(a)), we call (000, 010), (001, 011), (111, 101), (110, 100) as inner edges of CQ_3 and (000, 001), (001, 111), (111, 110), (110, 010), (010, 011), (011, 101), (101, 100), (100, 000) as outer edges of CQ_3, respectively. Let x be a vertex of CQ_3. An inner edge e is said to be an N-edge of x if e connects to one of the endpoints of e. Hence CQ_3 has two N-edges of x. An inner edge e is said to be an H-edge of x if x is not incident to e, and e is not an N-edge of x. Therefore, CQ_3 has one H-edge of x.

To explore the pancyclicity of CQ_{n}-F, we need an observation, and it is stated in the following lemma.

Lemma 2. Let x be a faulty vertex in CQ_3. Then the two N-edges of x, e_1 and e_2, are on cycles of lengths from 4 to 7 in CQ_3-x. And the H-edge of x is on cycles of lengths 4, 5, and 7 in CQ_3-x.

Proof. Without loss of generality, we may assume that x = 000. Then the two N-edges of x are (001, 011) and (110, 100), and the H-edge of x is (111, 101). We
Then we can construct a cycle $C_l$ which is incident to $x$. Let $(u_1, u_2)$ be the two $N$-edges of $x$. By Lemma 2, $(u_1, u_2)$ and $(u_3, u_4)$ are on cycles of lengths from 4 to 7 in $CQ_3^1 - F^1$. Let $v_1, v_2, v_3, v_4$ be the neighbors of $u_1, u_2, u_3, u_4$ in $CQ_3^1$, respectively. It is not difficult to check that both $(v_1, v_2)$ and $(v_3, v_4)$ are inner edges of $CQ_3^1$. (See Fig. 3(b).) And $(v_1, v_2)$ cannot reach $(v_3, v_4)$ via exactly one edge of $CQ_3^1$. Suppose the other fault is a faulty vertex $y$ in $CQ_3^1$, or a faulty edge $e_2$ which is incident to $y$. Then $(v_1, v_2)$ or $(v_3, v_4)$, say $(v_1, v_2)$ is an $N$-edge or $H$-edge of $y$ in $CQ_3^1 - F^1$. By Lemma 2, $(v_1, v_2)$ is on cycles of lengths 4, 5, and 7 in $CQ_3^1 - F^1$. We use $C_l$ to denote a cycle of length $l$. Let $C_l$ and $C_j$ be cycles containing $(u_1, u_2)$ and $(v_1, v_2)$, respectively, $4 \leq l \leq 7$, $j = 4, 5, 7$. Then we can construct a cycle $C_l$ from $C_l$ and $C_j$ by adding $(u_1, v_1)$ and $(u_2, v_2)$, and deleting $(u_1, u_2)$ and $(v_1, v_2)$ for $8 \leq l \leq 14$. For example, Fig. 3(b) shows a cycle of length 12 in $CQ_4$ with one faulty vertex 0000 and one faulty edge (1100, 1110).

**Case 2. Cycles of lengths from 15 to 16.** If $f_e = 2$, we need only to find cycles of lengths from 4 to 14 which we did in the previous cases. If $f_e = 1$ and $f_v = 1$, we have to find a cycle of length 15. By Theorem 1, $CQ_4$ is 2-hamiltonian, so there is a cycle of length 15 in $CQ_4 - F$. If $f_e = 2$, since $CQ_4$ is 2-hamiltonian, there is also a cycle of length 16. Suppose that $F = \{(x_1, y_1), (x_2, y_2)\}$. Let $F' = \{(x_1, y_1), x_2\}$. Then there is a cycle of length 15 in $CQ_4 - F'$. This cycle is also fault-free in $CQ_4 - F$.

**References**


