The Normalized Banzhaf Value and the Banzhaf Share Function

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THE NORMALIZED BANZHAF VALUE AND
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Abstract

A cooperative game with transferable utilities—or simply a TU-game—describes a situation in which players can obtain certain payoffs by cooperation. A value function for these games is a function which assigns to every such a game a distribution of payoffs over the players in the game. A famous solution concept for TU-games is the Banzhaf value. This Banzhaf value is not efficient, i.e., in general it does not distribute the payoff that can be obtained by the ‘grand coalition’ consisting of all players cooperating together.

In this paper we consider the normalized Banzhaf value which distributes the payoff that can be obtained by the ‘grand coalition’ proportional to the Banzhaf values of the players. This value does not satisfy certain axioms underlying the Banzhaf value. In this paper we discuss some characterizations of the normalized Banzhaf value and compare these with other solution concepts such as, for example, the (non-normalized) Banzhaf value and the Shapley value.

Another approach to analyze efficient value functions is to consider share functions being functions which assign to every player in a TU-game its share in the worth of the ‘grand coalition’. We discuss the characterization of a class of such share functions containing the Banzhaf and Shapley share functions.

Finally, we generalize the concept of the potential function of a game as introduced by Hart and Mas-Colell to a class of potential functions and characterize any element of the class of share functions by the normalized marginal function of the corresponding potential function.

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1. Introduction

A situation in which a finite set of \( n \) agents can obtain certain payoffs by cooperation can be described by a cooperative game with transferable utilities—or simply a TU-game—being a pair \((N, v)\), where the finite set of players \( N \) is defined by the set \( N = \{1, \ldots, n\} \) representing the agents and where \( v: 2^N \to \mathbb{R} \) such that \( v(\emptyset) = 0 \) is the characteristic function yielding for any subset \( E \subseteq N \) the payoff \( v(E) \) that can be achieved by coalition \( E \). Throughout the paper we use the convention that for \( E \subseteq N \) a coalition of players, \(|E|\) denotes the number of players in \( E \).

In this paper we only consider monotone TU-games. A TU-game \((N, v)\) is monotone if \( v(E) \leq v(F) \) for all \( E \subseteq F \subseteq N \). The collection of all monotone TU-games is denoted by \( G \). Note that the null games \((N, v^0)\) with \( v^0(E) = 0 \) for all \( E \subseteq N \) are part of \( G \). In many applications we may restrict ourselves to the subclass \( G^0 = \{(N, v) \in G \mid v \neq v^0\} \) of all monotone games that are not null games. The unanimity game of a coalition \( T \subseteq N \) is the game \((N, u_T)\) with characteristic function \( u_T \) defined by \( u_T(E) = 1 \) if \( T \subseteq E \) and \( u_T(E) = 0 \) for any other coalition \( E \subseteq N \). Note that all unanimity games \( u_T, T \neq \emptyset \), are in \( G^0 \).

A value function for monotone TU-games is a function \( f \) that assigns to every game \((N, v)\) in \( G \) an \( n \)-dimensional real vector \( f(N, v) \in \mathbb{R}^n \). This vector can be seen as a distribution of payoffs over the individual players in the game. A famous value function is the Banzhaf value (Banzhaf (1965)). Axiomatizations of the Banzhaf value can be found in, e.g., Lehrer (1988) and Haller (1994).

A value function \( f \) is efficient if for every TU-game \((N, v)\) it exactly distributes the payoff that can be obtained by the ‘grand coalition’ \( N \) consisting of all players, i.e., if for every TU-game \((N, v)\) it holds that the sum of the components of \( f(N, v) \) is equal to \( v(N) \). The Banzhaf value is not efficient. Therefore, it is not adequate in allocating the worth \( v(N) \) of the ‘grand coalition’ over the players in \( N \).

In this paper we consider the normalized Banzhaf value which distributes the value \( v(N) \) of the grand coalition proportional to the Banzhaf values of the players. Thus, the normalized Banzhaf value is efficient. As is noted in, e.g., Dubey and Shapley (1979) the normalized Banzhaf value does not satisfy some important properties that are satisfied by the Banzhaf value. An axiomatic characterization of the normalized Banzhaf value has been given in van den Brink and van der Laan (1995) by stating some new types of axioms. We remark here that the ‘multiplicative’ normalized Banzhaf value considered here is different from the ‘additive’ normalized Banzhaf value as considered in Ruiz, Valenciano and Zarzuelo (1996). In the additive normalization the same value is added to or subtracted from every players’ Banzhaf values in order to distribute \( v(N) \). Another famous solution concept for TU-games is the Shapley value (Shapley (1953)). Axiomatizations of this value can be found in, e.g., Shapley (1953), Young (1985) and van den Brink (1995). Also in van den Brink and van der Laan (1995) an axiomatization of the Shapley value is given by using axioms that are similar to the axioms used in their characterization of the normalized Banzhaf value.

A new type of solution concepts has been introduced in van der Laan and van den Brink (1995), namely the concept of share functions. A share function assigns to every TU-game \((N, v)\) an \( n \)-dimensional real vector which \( i \)th component is player \( i \)'s share in the value to be distributed. In their paper van der Laan and van den Brink provide an axiomatic characterization of a class of share functions containing the Banzhaf share function and
the Shapley share function as special cases. The Banzhaf (respectively Shapley) share function is the share function that assigns to every player its Banzhaf value (respectively Shapley value) divided by the sum of the Banzhaf values (respectively Shapley values) of all players in $N$. Consequently, the Banzhaf (respectively Shapley) share function multiplied with the worth of the grand coalition yields the normalized Banzhaf value (respectively Shapley value). Results on the class of share functions mentioned above can be translated to results on the corresponding efficient value functions.

A different line of approach to characterize the Banzhaf value has been given by Dragan (1996a, 1996b) by applying the concept of the potential function of a game as introduced by Hart and Mas-Colell (1988, 1989). In particular, Dragan shows that the Banzhaf value of a game is equal to the marginals of the Banzhaf potential function. Using these marginals again to define the so-called induced potential game associated with the original game it also follows that the Banzhaf value of a game is equal to the Shapley value of its induced potential game. In van den Brink and van der Laan (1998) the potential function is generalized according to the lines of van der Laan and van den Brink (1995). With this generalized potential function we can characterize the class of share functions mentioned in the previous paragraph and their corresponding value functions.

In this paper we want to bring together the ideas mentioned above. In Section 2 we discuss the concept of Banzhaf value function and state an axiomatization of the normalized Banzhaf value as introduced by van den Brink and van der Laan (1995). In this section we also recall the similar axiomatization of the Shapley value. In Section 3 we introduce the concept of share functions and recall the main results given in van der Laan and van den Brink (1995) on a specific class of share functions. In Section 4 we show that a potential function can be defined for any share function in this class as done in van den Brink and van der Laan (1998). This also gives a full class of potential functions containing the potential functions considered by Hart and Mas-Colell (1988, 1989) and Dragan (1996a, 1996b).

We conclude this introduction by remarking that for notational convenience we state all results in this paper on the class $G$ of all monotone TU-games. We will remark where the results can be stated more general.

2. The normalized Banzhaf value

In this section we recall the axiomatization of the normalized Banzhaf value as given in van den Brink and van der Laan (1995). First, given a game $(N, v) \in G$, for all $E \subseteq N$ and all $i \in T$, let $m^E_i(N, v)$ defined by

$$m^E_i(N, v) = v(E) - v(E \setminus \{i\})$$

be the marginal contribution of player $i$ to coalition $E$ in game $(N, v)$. Then the Banzhaf value on the class $G$ of monotone games is given by the following definition.

**Definition 1** The Banzhaf value on $G$ is the value function $\bar{v}^B$ given by

$$\bar{v}^B_i(N, v) = \sum_{E \in \mathcal{N}} \frac{1}{2^{n-1}} m^E_i(N, v) \text{ for all } i \in N.$$
In the literature various axiomatizations of the Banzhaf value have been given. The following four axioms on a value function \( f \) have been used by Haller (1994). First, observe that for every pair \((N,v)\) and \((N,w)\) in \( \mathcal{G} \) and nonnegative real numbers \( a \) and \( b \) the weighted sum game \((N,av + bw)\), \( a, b \geq 0 \), is the game \((N,z)\) with the characteristic function \( z = av + bw \) defined by \( z(E) = av(E) + bw(E) \) for all \( E \subseteq N \). The linearity axiom states that the value of a weighted sum game is equal to the weighted sum of the values of the separate games.

**Axiom 2 (Linearity)** For every \((N,v),(N,w) \in \mathcal{G} \) and every real number \( c > 0 \) it holds that (i) \( f(N, v + w) = f(N, v) + f(N, w) \) and (ii) \( f(N, cv) = cf(N, v) \).

A value function satisfies *additivity* if (i) holds. A player \( i \in N \) is a *dummy player* in \((N,v) \in \mathcal{G} \) if \( v(E \cup \{i\}) = v(E) + v(\{i\}) \) for all \( E \subseteq N \setminus \{i\} \). The dummy player axiom states that a dummy player \( i \) receives its own worth \( v(\{i\}) \).

**Axiom 3 (Dummy player property)** For every \((N,v) \in \mathcal{G} \) and every dummy player \( i \in N \) in the game \((N,v) \) it holds that \( f_i(N,v) = v(\{i\}) \).

For a game \((N,v) \in \mathcal{G} \) and a permutation \( \pi: N \rightarrow N \), the associated permuted game \((N,\pi v) \) is given by \( (\pi v)(E) = v(\cup_{i \in E} \pi(i)) \) for all \( E \subseteq N \). The anonymity axiom states that if players change roles according to some permutation \( \pi \) then their payoffs change accordingly.

**Axiom 4 (Anonymity)** For every \((N,v) \in \mathcal{G} \) and every permutation \( \pi: N \rightarrow N \) it holds that \( f_i(N, v) = f_{\pi(i)}(N, \pi v) \).

Finally, let \((N,v)\) be the game in which player \( i \in N \) acts as a *proxy* for player \( j \in N \) in the game \((N,v) \in \mathcal{G}, \) i.e., the game \((N,v^{ij}) \) is given by

\[
v^{ij}(E) = \begin{cases} 
v(E \cup \{j\}) & \text{if } i \in E \\
v(E \setminus \{j\}) & \text{else.}
\end{cases}
\]

(1)

The proxy agreement property states that the sum of the payoffs of two players does not change if one of them goes to act as a proxy for the other.

**Axiom 5 (Proxy agreement property)** For every \((N,v) \in \mathcal{G} \) and every pair \( i,j \in N \) it holds that \( f_i(N,v) + f_j(N,v) = f_i(N,v^{ij}) + f_j(N,v^{ji}) \).

The following theorem has been proved by Haller (1994)\(^2\).

**Theorem 6 (Haller (1994))** The Banzhaf value is the unique value function on the class \( \mathcal{G} \) of monotone TU-games that satisfies linearity, the dummy player property, anonymity, and the proxy agreement property.

\(^2\)Haller does not restrict himself to monotone TU-games. Moreover, his characterization is stated more general for classes \( \mathcal{G}^N \) of games with arbitrary fixed player set \( N \).
Observe that the anonymity property implies that the value is also symmetric. Two players \(i\) and \(j\) are said to be symmetric in the game \((N, v)\) if for all \(E \subset N\) such that \(i, j \in E\) it holds that \(v(E \setminus \{i\}) = v(E \setminus \{j\})\). A value function is symmetric when symmetric players receive the same payoff.

As said in the introduction, the Banzhaf value is not adequate in allocating the worth \(v(N)\) of the ‘grand coalition’ because it is not efficient, i.e., it does not necessarily distribute the worth \(v(N)\) of the grand coalition over the players in \(N\). More precisely, by summation over all components of the expression given in Definition 1 it follows that

\[
\sum_{i \in N} \varphi^B_i(N, v) = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{E \subset N \setminus \{i\}} m^i_E(N, v) = \frac{1}{2^{n-1}} \sum_{E \subset N} (2\|E\| - n)v(E).
\]

Note that \(\sum_{E \subset N}(2\|E\| - n)v(E) > 0\) and hence \(\sum_{i \in N} \varphi^B_i(N, v) > 0\) when \((N, v) \in \mathcal{G}^0\). For null games it holds that \(\sum_{i \in N} \varphi^B_i(N, v) = 0\), and thus that \(\varphi^B(N, v)\) is efficient if \(v = v^0\). To divide the worth of the grand coalition according to the Banzhaf value, for non-null games we have to replace the Banzhaf value by the normalized Banzhaf value.

**Definition 7** The normalized Banzhaf value on \(\mathcal{G}\) is the value function \(\varphi^B\) given by

\[
\varphi^B_i(N, v) = \begin{cases} \frac{\sum_{E \subset N \setminus \{i\}} m^i_E(N, v)}{\sum_{j \in N} \varphi^B_j(N, v)} & \text{for all } i \in N, \text{ if } (N, v) \in \mathcal{G}^0 \\ 0 & \text{for all } i \in N, \text{ if } v = v^0. \end{cases}
\]

Note that \(\varphi^B(N, v) = \frac{\varphi^B(N, v)}{\sum_{j \in N} \varphi^B_j(N, v)}\) for \((N, v) \in \mathcal{G}^0\). Thus, the normalized Banzhaf value \(\varphi^B(N, v)\) is an efficient value function that distributes the worth \(v(N)\) of the grand coalition proportional to the Banzhaf values of the players. The normalized Banzhaf value satisfies anonymity. However, it does not satisfy linearity, the dummy player property, nor the proxy agreement property. So, as argued already by Dubey and Shapley (1979) this normalization is not as innocent as it seems. Next we present five axioms on a value function \(f\) on \(\mathcal{G}\) that uniquely determine the normalized Banzhaf value. The first axiom is the familiar efficiency axiom.

**Axiom 8 (Efficiency)** For every \((N, v) \in \mathcal{G}\) it holds that \(\sum_{i \in N} f_i(N, v) = v(N)\).

Player \(i \in N\) is a null player in \((N, v) \in \mathcal{G}\) if he is a dummy player in \((N, v)\) with \(v(\{i\}) = 0\), i.e., if \(v(E) = v(E \setminus \{i\})\) for all \(E \subset N\). The second axiom states that deleting a null player in a game does not change the payoffs of the other players. This property is analyzed extensively in Derks and Haller (1994). The restriction of a game \((N, v) \in \mathcal{G}\) to a coalition \(T \subset N\) is the \(|T|\)-player game \((T, v_T)\) with the characteristic function \(v_T\) on \(T\) defined by \(v_T(E) = v(E)\) for all \(E \subset T\).

**Axiom 9 (Null player out property)** Let \(i \in N\) be a null player in \((N, v) \in \mathcal{G}\). Then \(f_j(N \setminus \{i\}, v_N \setminus \{i\}) = f_j(N, v)\) for all \(j \in N \setminus \{i\}\).

A monotone game \((N, v) \in \mathcal{G}\) is additive if for every player there is a (non-negative) weight such that the value of every coalition is equal to the sum of the weights of the players in that coalition, i.e., there exists a non-negative weight vector \(\lambda \in \mathbb{R}_+^n\) such that \(v(E) = \sum_{i \in E} \lambda_i\) for all \(E \subset N\). The third axiom states that in a monotone additive game every player earns its own weight.
Axiom 10 (Additive game property) Let $(N, v) \in \mathcal{G}$ be an additive game with weight vector $\lambda \in \mathbb{R}^n_+$. Then $f(N, v) = \lambda$.

Observe that a value function $f$ satisfies the additivity game property if $f$ satisfies the dummy player property, so that the additive game property is weaker than the dummy player property. Further, for a game $(N, v) \in \mathcal{G}$, let be given two other monotone characteristic functions $w$ and $z$ such that $v = w + z$, i.e., $v$ can be split into $w$ and $z$. Suppose that for a permutation $\pi$ on $N$ we change in game $z$ the roles of the players according to $\pi$. The next axiom states that the payoff of players who do not change roles does not change.

Axiom 11 (Independence of irrelevant permutations) Let $(N, v), (N, w), (N, z) \in \mathcal{G}$ be such that $v = w + z$, and let $\pi$ be a permutation on $N$. Then $f_i(N, w + \pi z) = f_i(N, v)$ for all $i \in N$ with $\pi(i) = i$.

Again suppose that the characteristic function $v$ can be split in two other monotone characteristic functions $w$ and $z$ and let $z^{ij}$ be the game in which player $i \in N$ acts as a proxy for player $j \in N$ in the game $(N, z) \in \mathcal{G}$. Further, let $h$ be a non-null player in $(N, v)$ who is a null player in $(N, z)$. The fifth axiom now states that under this proxy agreement the ratio between the sum of the payoffs of players $i$ and $j$ and the payoff of player $h$ does not change.

Axiom 12 (Proportional proxy agreement property) Let $(N, v), (N, w), (N, z) \in \mathcal{G}$ be such that $v = w + z$, let $i, j \in N$ be non-null players in $(N, z)$, and let $h \in N \setminus \{i, j\}$ be a non-null player in $(N, v)$ and a null player in $(N, z)$. Then

$$\frac{f_i(N, w + z^{ij}) + f_j(N, w + z^{ij})}{f_h(N, w + z^{ij})} = \frac{f_i(N, v) + f_j(N, v)}{f_h(N, v)}.$$ 

The five axioms stated above uniquely determine the normalized Banzhaf value for monotone TU-games, as is shown in van den Brink and van der Laan (1995).

Theorem 13 (van den Brink and van der Laan (1995)) The normalized Banzhaf value is the unique value function $f$ on the class $\mathcal{G}$ of monotone TU-games that satisfies efficiency, the null player out property, the additive game property, independence of irrelevant permutations and the proportional proxy agreement property.

We remark that the axiom of independency of irrelevant permutations and the proportional proxy agreement property can be replaced by two weaker axioms. To state these axioms, recall from the literature that the characteristic function $v$ of every game $(N, v)$ can be expressed a linear combination of the characteristic functions of the unanimity games $(N, u^T)$, $T \subset N$, by $v = \sum_{T \subset N} \Delta_u(T) u^T$ with $\Delta_u(T)$ the dividend of coalition $T \subset N$ given by $\Delta_u(T) = \sum_{E \in T} (-1)^{|T|-|E|} u(E)$ (see Harsanyi (1959)). Now, suppose that in game $(N, v)$ we replace the unanimity game of a coalition $T \subset N$ by the unanimity game of a coalition $H \subset N$ with $|H| = |T|$. The weaker version of independence of irrelevant permutations states that this replacement does not change the payoff of every player that belongs to both $T$ and $H$ or does not belong to $T$ or $H$. 

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Axiom 14 (Independence of irrelevant unanimity replacements) For \((N, v) \in \mathcal{G}\), let \(T, H \subseteq N\) be such that \(|T| = |H|\). Further, let the game \((N, w) \in \mathcal{G}\) be given by \(w = v - \Delta_v(T)u^T + \Delta_v(T)u^H\). Then
\[
f_i(N, w) = f_i(N, v) \text{ for all } i \in (T \cap H) \cup (N \setminus (T \cup H)).
\]

Again, consider a game \((N, v) \in \mathcal{G}\) with characteristic function \(v = \sum_{T \subseteq N} \Delta_v(T)u^T\) expressed as a linear combination of the characteristic functions of unanimity games. For a coalition \(T \subseteq N\), \(2 \leq |T| \leq n - 1\), let \(h\) be a non-null player in the game \((N, v) \in \mathcal{G}\) who does not belong to \(T\). Further, for a player \(j \in T\), let some player \(i \in T \setminus \{j\}\) act as a proxy for player \(j\) in the unanimity game of coalition \(T\), i.e., in the expression of \(v\) as a linear combination of the unanimity games, the characteristic function \(u^T\) of coalition \(T\) is replaced by the characteristic function \(u^{T \setminus \{j\}}\) of the unanimity game of coalition \(T \setminus \{j\}\).\(^3\)

The next axiom is a weaker version of the proportional proxy agreement property and states that the ratio between the sum of the payoffs of players \(i\) and \(j\) and the payoff of player \(h\) does not change when \(i\) is going to act as a proxy of \(j\) in the unanimity game \((N, u^T)\).

Axiom 15 (Unanimity proxy property) Let \((N, v) \in \mathcal{G}\), \(T \subseteq N\) be such that \(2 \leq |T| \leq n - 1\), and let \(j \in T\). Further, let \((N, w) \in \mathcal{G}\) be the game with characteristic function \(w = v - \Delta_v(T)u^T + \Delta_v(T)u^{T \setminus \{j\}}\). For every non-null player \(h \in N \setminus T\) it holds that
\[
\frac{f_i(N, w) + f_j(N, w)}{f_h(N, w)} = \frac{f_i(N, v) + f_j(N, v)}{f_h(N, v)} \text{ for all } i \in T \setminus \{j\}.
\]

The proof of Theorem 13 as given in van den Brink and van der Laan (1995) immediately yields the next corollary.

Corollary 16 (van den Brink and van der Laan (1995)) The normalized Banzhaf value is the unique value function \(f\) on the class \(\mathcal{G}\) of monotone \(TU\)-games that satisfies efficiency, the null player out property, the additive game property, independence of irrelevant unanimity replacements and the unanimity proxy property.

We conclude this section with some remarks on the Shapley value and its relationship to the Banzhaf value and the normalized Banzhaf value. First of all, it is well-known that when we replace in Definition 1 the factor \(2^{-(n-1)}\) by \(\frac{(|E|-1)!(n-|E|)!}{n!}\) we obtain the Shapley value instead of the Banzhaf value, i.e., the Shapley value \(c.p^S(N, v)\) on \(\mathcal{G}\) is the value function \(\varphi^S\) defined by
\[
\varphi^S_i(N, v) = \sum_{E \subseteq N, E \neq N} \frac{(|E|-1)!(n-|E|)!}{n!} m^i_E(N, v) \text{ for all } i \in N.
\]

This shows the well-known similarity between the Banzhaf value \(\varphi^B(N, v)\) and the Shapley value \(\varphi^S(N, v)\). They only differ in the weights putted on the marginal contributions of the players. A similar analogon can be observed when we express the value functions in terms of the dividends of the characteristic function. Van den Brink and van der Laan (1995) proved that the Banzhaf value can be expressed by these dividends by showing that the value function defined in this way satisfies the four axioms mentioned in Theorem 6.

\(^3\)Note that, in fact, in the unanimity game of coalition \(T \setminus \{j\}\) every player in \(T \setminus \{j\}\) acts as a proxy for player \(j\) in the unanimity game of coalition \(T\).
Theorem 17 (van den Brink and van der Laan (1995)) For every \((N,v) \in \mathcal{G}\) and every \(i \in N\) it holds that
\[
\varphi_i^B(N,v) = \sum_{E \subseteq N} \frac{\Delta_i(E)}{2^{|E|-1}}.
\]

It is again well-known that we obtain the Shapley value if we replace the denominator \(2^{|E|-1}\) by \(|E|\) in the expression of Theorem 17. As the Banzhaf value, also the Shapley value satisfies linearity, the dummy player property, and anonymity, but it does not satisfy the proxy agreement property. Instead it is efficient, so that efficiency, linearity, the dummy player property and anonymity characterize the Shapley value\(^4\). Compared to the normalized Banzhaf value the Shapley value satisfies efficiency, the null player out property, the additive game property, and the axiom of independency of irrelevant permutations. However, it does not satisfy the proportional proxy agreement property nor the weaker unanimity proxy property. Instead, it satisfies another kind of proxy agreement property.

To state this property, again write the characteristic function \(v\) of a game \((N,v) \in \mathcal{G}\) as a linear combination the characteristic function of unanimity games. Let \(h\) be a non-null player in \((N,v)\) and let \(T\) be a coalition of at least two players not containing player \(h\). Suppose that some player \(i \in T\) acts as a proxy for player \(j \in T\) in the unanimity game of coalition \(T\), and thus replace in the weighted sum of unanimity games the unanimity game of coalition \(T\) by the unanimity game of coalition \(T \setminus \{j\}\). As noted before, in this case all players in \(T \setminus \{j\}\) act as proxy players for \(j\). The coalitional unanimity proxy property states that the ratio between the sum of the payoffs of all players in \(T\) and the payoff of player \(h\) does not change if we let the players in \(T \setminus \{j\}\) act as proxy players for player \(j\). Moreover, the payoffs of the players in coalition \(T \setminus \{j\}\) change by the same amount.

Axiom 18 (Coalitional unanimity proxy property) Let \((N,v) \in \mathcal{G}, T \subseteq N\) be such that \(2 \leq |T| \leq n - 1, j \in T,\) and let \(h \in N \setminus T\) be a non-null player in \((N,v)\). Further, let \(w = v - \Delta_v(T)u^T + \Delta_v(T)u^{T \setminus \{j\}}.\) Then
\[
\frac{\sum_{i \in T} f_i(N,w)}{f_h(N,w)} = \frac{\sum_{i \in T} f_i(N,v)}{f_h(N,v)}.
\]

Moreover, there exists a \(c^* \in \mathbb{R}\) such that
\[
f_i(N,w) - f_i(N,v) = c^* \text{ for all } i \in T \setminus \{j\}.
\]

We can characterize the Shapley value by replacing the proportional proxy agreement property in Theorem 13 by this coalitional unanimity proxy property.

Theorem 19 (van den Brink and van der Laan (1995)) The Shapley value is the unique value function \(f\) on the class \(\mathcal{G}\) of monotone TU-games that satisfies efficiency, the null player out property, the additive game property, independence of irrelevant permutations, and the coalitional unanimity proxy property.

\(^4\)In characterizing the Shapley value the additivity property of the linearity axiom is sufficient.
Similarly as in the characterization of the normalized Banzhaf value the axiom of independence of irrelevant permutations can be replaced by the weaker independence of irrelevant unanimity replacements.

3. Share functions

As shown in the previous section we have that \( \sum_{E \subseteq N} \varphi^{B}(N,v) = \frac{1}{2^{n-1}} \sum_{E \subseteq N} (|E| - n)v(E) \), i.e., the sum of the components of the Banzhaf value is equal to a weighted sum of the worths of the coalitions in the characteristic function with weight \( \frac{1}{2^{n-1}}(2|E| - n) \) for every coalition \( E \subseteq N \). The sum of the components of the Shapley value equals \( v(N) \). Both the Banzhaf and Shapley value can be expressed as weighted sums of marginal contributions. More general, for given \( n \in \mathbb{N} \), let \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x_i > 0, i = 1, \ldots, n \} \) be an arbitrarily chosen vector of positive real numbers. Then we can generalize the Banzhaf value function and the Shapley value function to a class of value functions defined by

\[
\varphi^{\omega}(N,v) = \sum_{E \subseteq N} \omega_{|E|} m^{\omega}_E(N,v) \text{ for all } i \in N,
\]

obtaining symmetric probabilistic values as studied by Weber (1988). Clearly, we obtain the Banzhaf value function by taking \( \omega_t = 2^{-(n-1)} \) and we have the Shapley value function by taking \( \omega_t = \frac{(t-1)!(n-t)!}{n!} \) for all \( t = 1, \ldots, n \). Although the Shapley value exactly distributes the worth \( v(N) \) of the 'grand' coalition, in general the components of \( \varphi^{\omega} \) do not sum up to \( v(N) \).

An alternative approach to divide the worth of the grand coalition amongst its players is given by the concept of share function introduced by van der Laan and van den Brink (1995). The only difference with the concept of value functions is that a share function assigns to each player his share in the worth of the grand coalition, i.e., a share function on the class \( G \) of monotone games is a function \( \rho \) giving player \( i \in N \) the share \( \rho_i(N,v) \) in the worth \( v(N) \) of the grand coalition. So, for given game \( (N,v) \) a share function \( \rho \) gives a payoff \( \rho_i(v) v(N) \) to player \( i \in N \). Of course the total payoff equals \( v(N) \) if and only if \( \sum_{i \in N} \rho_i(N,v) = 1 \). Therefore, for a share function \( \rho \) on \( G \), the axiom of efficiency is redefined as follows.

**Axiom 20 (Efficient shares)** For every \( (N,v) \in G \) it holds that \( \sum_{i \in N} \rho_i(N,v) = 1 \).

Note that share functions are not interesting for null games, since irrespective of the shares all players get a zero payoff when multiplying the shares with \( v(N) \). Therefore, in this section we consider the class \( G^0 \) of monotone games not being null games. In van der Laan and van den Brink (1995) a class of share functions on the set \( G^0 \) has been introduced by defining a share function for any function \( \mu : G^0 \to \mathbb{R} \) satisfying certain conditions. Such a function \( \mu \) assigns a real value \( \mu(N,v) \) to any game \( (N,v) \) in the set \( G^0 \) of non-null monotone games. First of all we define certain properties of such functions.

**Definition 21 (Properties of \( \mu \)-functions)**

1. The function \( \mu : G^0 \to \mathbb{R} \) is positive on \( G^0 \) if \( \mu(N,v) > 0 \) for any \( (N,v) \in G^0 \).
2. The function \( \mu : G^0 \to \mathbb{R} \) is additive on \( G^0 \) if for any pair \( (N,v), (N,w) \in G^0 \), it holds that \( \mu(N,v + w) = \mu(N,v) + \mu(N,w) \).
(3) The function \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) is linear on \( \mathcal{G}^0 \) if it is additive and for every \((N,v) \in \mathcal{G}^0\) and \(c > 0\) it holds that \( \mu(N, cv) = c \mu(N, v) \).

(4) The function \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) is symmetric on \( \mathcal{G}^0 \) if for every \((N,v) \in \mathcal{G}^0\) and every pair of symmetric players \(i, j\) in \((N,v)\) it holds that \( \mu(E \setminus \{i\}, v_{E \setminus \{j\}}) = \mu(E \setminus \{j\}, v_{E \setminus \{i\}}) \) for all \( E \subseteq N, E \supseteq \{i,j\} \).

Observe that the assumption of symmetry states that for any two symmetric players \(i\) and \(j\) in the game \((N,v)\) it holds that the change in the value of \( \mu \) in case that \(i\) is deleted from any subgame containing both \(i\) and \(j\) is equal to the change in the value of \( \mu \) when \(j\) is deleted from this subgame.

To characterize the class of share functions we replace the axiom of linearity by the more general concept of \( \mu \)-linearity of a share function \( \rho \) on \( \mathcal{G}^0 \).

**Axiom 22 (\( \mu \)-Linearity)** Let \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) be given. Then for every \((N,v), (N,w) \in \mathcal{G}^0\) and every real number \(c > 0\) it holds that (i) \( \mu(v + w) \rho(N,v + w) = \mu(N,v) \rho(N,v) + \mu(N,w) \rho(N,w) \) and (ii) \( \mu(N, cv) \rho(N,cv) = c \mu(N,v) \rho(N,v) \).

A share function satisfies the weaker axiom of \( \mu \)-additivity if only part (i) of the \( \mu \)-linearity axiom holds. Finally, we replace the dummy player property and the anonymity property by the null player property and the symmetry property respectively, of a share function on \( \mathcal{G}^0 \).

**Axiom 23 (Null player property)** For every \((N,v) \in \mathcal{G}^0\) and every null player \(i \in N\) in the game \((N,v)\) it holds that \( \rho_i(N,v) = 0 \).

**Axiom 24 (Symmetry)** For every \((N,v) \in \mathcal{G}^0\) and any pair of symmetric players \(i, j \in N\) it holds that \( \rho_i(N,v) = \rho_j(N,v) \).

The following result has been proven in van der Laan and van den Brink (1995)\(^5\).

**Theorem 25 (van der Laan and van den Brink (1995))** Let \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) be positive and symmetric on \( \mathcal{G}^0 \). Then there exists a unique share function \( \rho^\mu \) on \( \mathcal{G}^0 \) satisfying the axioms of efficient shares, null player property, symmetry and \( \mu \)-additivity (respectively \( \mu \)-linearity) if and only if \( \mu \) is additive (respectively linear) on \( \mathcal{G}^0 \).

The proof of the theorem is based on the fact that any characteristic function \( v \) can be expressed as a linear combination of characteristic functions of unanimity games. Although in van der Laan and van den Brink (1995) it is not stated explicitly in the conditions of the theorem that \( \mu \) must be symmetric, this fact is used implicitly within the proof. Clearly, the symmetry axiom can not be true for any \((N,v)\) when \( \mu \) is not symmetric\(^6\). However, without symmetry of \( \mu \) the theorem is still true when symmetry is

\(^5\)The results in this section are stated on the class \( \mathcal{G}^0 \) of monotone games not being null games. In van der Laan and van den Brink (1995) they are stated on classes \( \mathcal{C} \subseteq \mathcal{G} \) with arbitrary fixed player set \( N \), which contain all unanimity games on \( N \).

\(^6\)Since in van der Laan and van den Brink (1995) Theorem 25 is stated on classes of games with arbitrary fixed player set \( N \), symmetry cannot be used in that paper. Therefore, in that paper symmetry should be replaced by the stronger anonymity of \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) meaning that for every \((N,v) \in \mathcal{G}^0\) and permutation \( \pi: N \rightarrow N \) it holds that \( \mu(N,v) = \mu(N, \pi v) \).
only assumed to be held on the subclass of all unanimity games on coalitions of at least two players. So, in fact we have the even stronger result that for any positive and additive (respectively linear) function \( \mu \) there is a unique share function \( \rho^\mu \) satisfying the axioms of efficient shares, null player property and \( \mu \)-additivity (respectively \( \mu \)-linearity) on \( \mathcal{G}^0 \) and the symmetry axiom on the subset of unanimity games on coalitions of at least two players.

Next we consider the class of share functions that are determined by the value functions defined in (3) with the Banzhaf and Shapley share functions as special cases.

**Definition 26**

The **Shapley share function** \( \rho^S \) on \( \mathcal{G}^0 \) is given by \( \rho^S_i(N, v) = \varphi^S_i(N, v)/v(N) \), for all \( i \in N, (N, v) \in \mathcal{G}^0 \).

The **Banzhaf share function** \( \rho^B \) on \( \mathcal{G}^0 \) is given by \( \rho^B_i(N, v) = \varphi^B_i(N, v)/v(N) = \sum_{E \subseteq N, |E| = |N| - i} \varphi^B_i(N, v), \) for all \( i \in N, (N, v) \in \mathcal{G}^0 \).

For given positive vectors \( \omega \in \mathbb{R}_+^n, n \in \mathbb{N} \), let the function \( \sigma^\omega: \mathcal{G}^0 \rightarrow \mathbb{R} \) be defined as the sum of the components of the corresponding value function, i.e.,

\[
\sigma^\omega(N, v) = \sum_{i \in N} \varphi^\omega_i(N, v) = \sum_{i \in N} \sum_{E \subseteq N} \omega_E m^i_E(N, v).
\]

In case \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) is given by \( \mu(N, v) = \sigma^\omega(N, v) \) for a given vector of positive weights \( \omega \) putted to the marginal contributions of the players, the following result has been shown in van der Laan and van den Brink (1995).

**Theorem 27 (Weighted marginal contributions share function)** Let \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) be given by \( \mu(N, v) = \sigma^\omega(N, v) \) for a given positive vector of weights \( \omega \in \mathbb{R}_+^n \). Then the share function \( \rho^\omega \) given by

\[
\rho^\omega_i(N, v) = \frac{\sum_{E \subseteq N \setminus \{i\}} \omega_E m^i_E(N, v)}{\sigma^\omega(N, v)}, \quad i \in N,
\]

is the unique share function satisfying the axioms of efficient shares, null player property, symmetry and \( \sigma^\omega \)-linearity on \( \mathcal{G}^0 \).

The theorem shows that any choice of positive weights on the marginal contributions defines a share function satisfying efficiency, the null player property, symmetry and \( \sigma^\omega \)-linearity. Now, for \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n, n \in \mathbb{N}, \) a vector of real numbers, consider the function \( \mu: \mathcal{G}^0 \rightarrow \mathbb{R} \) given by \( \mu(N, v) = \eta^\beta(N, v) \), where the function \( \eta^\beta: \mathcal{G}^0 \rightarrow \mathbb{R} \) is defined by

\[
\eta^\beta(N, v) = \sum_{E \subseteq N} \beta_E v(E),
\]

i.e., \( \beta \) is a vector of weights putted to the worths \( v(E), E \subseteq N, \) of the characteristic function. In van der Laan and van den Brink (1995) it is shown that \( \sigma^\omega(N, v) \) is equal to \( \eta^\beta(N, v) \) for all \( (N, v) \in \mathcal{G} \) if and only if

\[
\begin{align*}
\beta_n &= n \omega_n \\
\beta_t &= t \omega_t - (n - t) \omega_{t+1}, \quad t = n - 1, \ldots, 1.
\end{align*}
\]
Observe that for arbitrarily chosen positive numbers \( \omega_t, t = 1, \ldots, n \), some of the corresponding values of \( \beta_t \), \( t = 1, \ldots, n \), may be negative. However, the functions \( \sigma^\omega \) (respectively \( \eta^\beta \)) satisfy the properties given in Definition 21 for any positive vector \( \omega \), (respectively the corresponding vector \( \beta \)) of weights. In particular, we have that for any positive vector \( \omega \) the function \( \sigma^\omega \) (being equal to the function \( \eta^\beta \) for corresponding weight vector \( \beta \)) is positive on the class \( G^0 \). Solving the recursive system (4) we find that \( \beta_t = (2t - n)2^{-(n-1)} \) when \( \omega_t = 2^{-(n-1)} \), \( t = 1, \ldots, n \), which corresponds to the expression for the sum of the Banzhaf values given in equation (2), whereas \( \beta_n = 1 \) and \( \beta_t = 0 \) for all \( t < n \) when we take the Shapley weights in the vector \( \omega \), which corresponds to the fact that the Shapley value is efficient.

Alternatively, these choices of the weight vector \( \beta \) yield the Shapley share function and the Banzhaf share function on the class \( G^0 \), and thus the next corollary follows immediately from Theorem 27.

**Corollary 28 (van der Laan and van den Brink (1995))**

Let the function \( \eta^S: G^0 \rightarrow \mathbb{R} \) be defined by \( \eta^S(N, v) = v(N) \). Then the Shapley share function \( \rho^S \) is the unique share function on \( G^0 \) satisfying the axioms of efficient shares, null player property, symmetry and \( \eta^S \)-linearity.

Let the function \( \eta^B: G^0 \rightarrow \mathbb{R} \) be defined by \( \eta^B(N, v) = 2^{-(n-1)} \sum_{E \subseteq N} (|E| - n) v(E) \). Then the Banzhaf share function \( \rho^B \) is the unique share function on \( G^0 \) satisfying the axioms of efficient shares, null player property, symmetry and \( \eta^B \)-linearity.

Besides the Shapley and Banzhaf share functions, many other share functions can be obtained by particular choices of weight vectors \( \omega \) (or corresponding \( \beta \)). The Deegan-Packel share function \( \rho^{DP} \) given by \( \rho^{DP}(N, v) = \sum_{i \in N} \frac{\varphi^{DP}(N, v)}{\varphi^S_i v(E)}, i \in N \), where \( \varphi^{DP} \) is the Deegan-Packel value (Deegan and Packel (1979)) given by

\[
\varphi^{DP}_i(N, v) = \sum_{\ell \in C_N} \frac{v(T)}{|T|} \quad \text{for all } i \in N.
\]

This share function satisfies the axioms of efficient shares, symmetry and \( \eta^T \)-linearity with \( \eta^T(N, v) = \sum_{E \subseteq N} v(E) \), i.e., \( \eta^T \) measures the sum of the worths of all coalitions of \( N \). But it does not satisfy the null player property and thus does not belong to the class of share functions discussed above. However, according to Theorem 25 there exists a unique share function satisfying efficient shares, symmetry, \( \eta^T \)-linearity and the null player property. Taking \( \beta_t = 1, t = 1, \ldots, n \), and solving the recursive system (4) for the corresponding vector of weights \( \omega \), we find that \( \omega_n = \frac{1}{n} \) and \( \omega_t = \frac{1+(n-1)\delta_{t1}}{t} \), \( t = n-1, \ldots, 1 \). The share function \( \rho^T \) is found by using these weights in Theorem 27.

Another well-known efficient value function is the \( \tau \)-value (Tijs (1981)) being the value function given by

\[
\varphi^\tau_i(N, v) = \alpha(N, v) R_i(N, v) + (1 - \alpha(N, v))(r_i(N, v)),
\]

with the 'utopia' vector \( R(N, v) \in \mathbb{R}^n_+ \) given by

\[
R_i(N, v) = m^i_N(N, v), i = 1, \ldots, n,
\]
weights $W_t$ of Theorem 27

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>weights $\omega_t$ of Theorem 27</th>
<th>weights $\beta_t$ of Corollary 28</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^S$</td>
<td>$\omega_t = \frac{(t-1)!((n-1)!}{n!}$</td>
<td>$\beta_t = \begin{cases} 1 &amp; \text{if } t = n \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>$\rho^B$</td>
<td>$\omega_t = 2^{-(n-1)}$</td>
<td>$\beta_t = (2t - n)2^{-(n-1)}$</td>
</tr>
<tr>
<td>$\rho^T$</td>
<td>$\omega_t = \begin{cases} \frac{1}{1+(n-1)t} \omega_{t-1} &amp; \text{if } t = n \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\beta_t = 1$</td>
</tr>
<tr>
<td>$\rho^M$</td>
<td>$\omega_t = \begin{cases} 1 &amp; \text{if } t = n \ 0 &amp; \text{otherwise} \end{cases}$</td>
<td>$\beta_t = \begin{cases} n &amp; \text{if } t = n \ -1 &amp; \text{if } t = n - 1 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
</tbody>
</table>

Table 1: Some share functions and their weights

the 'minimal right' vector $r(N, v) \in \mathbb{R}^n$ given by

$$r_i(N, v) = \max_{E \subseteq N \setminus \{i\}} \left( v(E) - \sum_{j \in N \setminus \{i\}} R_j(N, v) \right), \quad i = 1, \ldots, n,$$

and where $\alpha(N, v)$ is determined by the efficiency condition $\sum_{i \in N} \varphi_i^*(N, v) = v(N)$. Again the corresponding $\tau$-share function $\rho^\tau$ given by $\rho_i^\tau(N, v) = \frac{\varphi_i^*(N, v)}{\sum_{j \in N} \varphi_j^*(N, v)}$, $i \in N$, does not belong to the class of share functions considered above. The $\tau$-share function satisfies the axioms of efficient shares, null player property and symmetry, but there is no linear $\mu$-function such that $\rho^\tau$ is $\mu$-linear. However, let $G^0_+ \subseteq G^0_+$ be the subclass of non-null monotone games satisfying $\sum_{i \in N} R_i(N, v) = nv(N) - \sum_{i \in N} v(N \setminus \{i\}) > 0$, i.e., $G^0_+$ is the subclass on which the sum of the elements in the 'utopia' vector $R(N, v)$ is positive. Observe that this class contains all unanimity games $(N, u^T)$, $T \neq 0$. Now, let $\eta^M : G^0_+ \rightarrow \mathbb{R}_+$ be defined by $\eta^M(N, v) = \sum_{i \in N} R_i(N, v)$. Then the function $\eta^M$ is positive, symmetric and linear on $G^0_+$ and applying Theorem 25 on the subclass $G^0_+$ containing all unanimity games yields that there exists a unique share function $\rho^M$ on $G^0_+$ satisfying efficient share, the null player property, symmetry and $\mu^M$-linearity. Taking $\beta_n = 1$, $\beta_{n-1} = -1$, and $\beta_t = 0$, $t = 1, \ldots, n - 2$ and solving the recursive system (4) for the corresponding vector of weights $\omega$, we find that $\omega_n = 1$ and $\omega_t = 0$, $t = n - 1, \ldots, 1$. Although not all these weights are positive, nevertheless the corresponding $\sigma^\omega(N, v) = \eta^M(N, v) > 0$ on the subclass $G^0_+$, which is sufficient to apply Theorem 27, to find the share function $\rho^M$ on $G^0_+$.

Table 1 summarizes the share functions mentioned above. The corresponding properties are summarized in Table 2.

4. Potential functions

In the previous sections we described two approaches to characterize the normalized Banzhaf value for efficiently allocating the value $v(N)$. In Section 2 we gave a 'direct' characterization, while in Section 3 we approached it using share functions. In terms of share functions the Banzhaf and normalized Banzhaf value are the same. In this section we

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Table 2: Properties of some share functions

<table>
<thead>
<tr>
<th>share</th>
<th>efficient shares</th>
<th>null player</th>
<th>symmetry</th>
<th>μ-additivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^S$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\rho^B$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\rho^{DP}$</td>
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<td>+</td>
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<td>$\rho^T$</td>
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<td>+</td>
</tr>
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<td>$\rho^*$</td>
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</tr>
<tr>
<td>$\rho^M$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

take another approach to share functions by generalizing the concept of potential function as introduced in Hart and Mas-Colell (1988, 1989). Hart and Mas-Colell define a potential function on $\mathcal{G}$ to be a function $P: \mathcal{G} \rightarrow \mathbb{R}$ satisfying $P(N, v) = 0$ if $N = \emptyset$, and whenever $N \neq \emptyset$ it must hold that

$$
\sum_{i \in N} \left( P(N, v) - P(N \setminus \{i\}, v_{N \setminus \{i\}}) \right) = v(N),
$$

where as in Section 2, for every $T \subseteq N$ the restricted game $(T, v_T)$ is given by $v_T(E) = v(E)$ for all $E \subseteq T$. (As noted by Hart and Mas-Colell, in order to determine $P(N, v)$ it is sufficient that the class of games considered contains all subgames of $(N, v)$.) They show that there exists a unique potential function. Moreover, they show that the vector function $DP$ of marginals defined by $DP_i(N, v) = P(N, v) - P(N \setminus \{i\}, v_{N \setminus \{i\}})$, $i \in N$, is the Shapley value function. Using $\mu$-functions as introduced in the previous section we generalize the concept of potential function as follows.

Definition 29 Let $\mu: \mathcal{G} \rightarrow \mathbb{R}$ be given. A function $P^\mu: \mathcal{G} \rightarrow \mathbb{R}$ is a $\mu$-potential function on $\mathcal{G}$ if $P^\mu(N, v) = 0$ whenever $N = \emptyset$, and for every $(N, v) \in \mathcal{G}$ with $N \neq \emptyset$ it holds that

$$
\sum_{i \in N} \left( P^\mu(N, v) - P^\mu(N \setminus \{i\}, v_{N \setminus \{i\}}) \right) = \mu(N, v). \quad (5)
$$

Clearly, the Hart and Mas-Colell potential function is obtained by taking the Shapley $\mu$-function $\mu^S(N, v) = v(N)$. For given $\mu$-potential function $P^\mu$ on $\mathcal{G}$ we define the marginal function $DP^\mu$ on $\mathcal{G}$ by

$$
DP^\mu_i(N, v) = P^\mu(N, v) - P^\mu(N \setminus \{i\}, v_{N \setminus \{i\}}), \quad i \in N, \quad (6)
$$

and the normalized marginal function $NDP^\mu$ on $\mathcal{G}$ by

$$
NDP^\mu_i(N, v) = \begin{cases} 
\frac{DP^\mu_i(N, v)}{v(N,v)} & \text{if } \mu(N, v) \neq 0 \\
1 & \text{else} 
\end{cases}, \quad i \in N. \quad (7)
$$
We will prove that for given function $\mu$ on $\mathcal{G}$ the normalized marginal function characterizes the corresponding share function $\rho^\mu$. First, remember that by $\mathcal{G}^0$ we denote the collection of all monotone games except the null games. We now assume that $\mu(N,v) = 0$ whenever $v = v^0$. Then the next corollary follows immediately from Theorem 25. It only extends the existence of a share function to the set of null games.

**Corollary 30** Let $\mu:\mathcal{G} \to \mathbb{R}$ with $\mu(N,v) = 0$ when $v = v^0$ be additive and symmetric on $\mathcal{G}$, and positive on $\mathcal{G}^0$. Then there exists a unique share function $\rho^\mu$ on $\mathcal{G}$ with $\rho^\mu_i(N,v) = \frac{1}{|N|}$, $i \in N$, if $v = v^0$, satisfying the axioms of efficient shares, null player property on $\mathcal{G}^0$, symmetry and $\mu$-additivity.

In order to characterize share functions using normalized marginal functions we need to assume an additional property of the $\mu$-function, namely the property of null player independence stating that deleting a null player from a game does not change the value of the $\mu$-function.

**Definition 31** The function $\mu:\mathcal{G} \to \mathbb{R}$ is null player independent on $\mathcal{G}$ if for every $(N,v) \in \mathcal{G}$ and every null player $i$ in $(N,v)$ it holds that $\mu(N,v) = \mu(N \setminus \{i\}, v_{N \setminus \{i\}})$.

We now state the following result from van den Brink and van der Laan (1998).\(^7\)

**Theorem 32** Let $\mu:\mathcal{G} \to \mathbb{R}$ with $\mu(N,v) = 0$ when $v = v^0$ be additive, symmetric, and null player independent on $\mathcal{G}$ and positive on $\mathcal{G}^0$. Then there exists a unique $\mu$-potential function $P^\mu$ on $\mathcal{G}$. The corresponding normalized marginal function $NDP^\mu$ is equal to the unique share function $\rho^\mu$ satisfying the properties of Corollary 30.

**Proof** Since $P^\mu(N,v) = 0$ if $N = \emptyset$, the potential $P^\mu(N,v)$, $|N| \geq 1$, is uniquely determined by recursively using equation (5) and is given by

$$
\begin{align*}
P^\mu(N,v) &= \frac{1}{|N|} \left( \mu(N,v) + \sum_{i \in N} P^\mu(N \setminus \{i\}, v_{N \setminus \{i\}}) \right).
\end{align*}
$$

From the definition given in equation (7) it follows that for the corresponding normalized marginal function it holds that $NDP^\mu(N,v) = \rho^\mu(N,v)$ if $v = v^0$. To prove that $NDP^\mu(N,v) = \rho^\mu(N,v)$ for all $(N,v) \in \mathcal{G}$ it is sufficient to show that $NDP^\mu$ satisfies the axioms of efficient shares, null player property on $\mathcal{G}^0$, symmetry and $\mu$-additivity, since Corollary 30 says that there is only a unique function satisfying these properties.

From the equations (5), (6) and (7) it follows immediately that $NDP^\mu$ satisfies the axiom of efficient shares.

To prove the null player property, let $i \in N$ be a null player in $(N,v) \in \mathcal{G}$. First, take $N = \{i\}$ and hence $v = v^0$ because $i$ is a null player. From equation (5) and $P^\mu(\emptyset, v) = 0$ for all $v$ we obtain that

$$
DP_i^\mu(N,v) = P^\mu(\{i\}, v) - P^\mu(\emptyset, v^0) = \mu(\{i\}, v) = \mu(\{i\}, v^0) = 0.
$$

---

\(^7\)In van den Brink and van der Laan (1998) the results of this section are stated for subsets $C \subseteq \mathcal{G}$ that are subgame closed. A set $C \subseteq \mathcal{G}$ is subgame closed if for every $(N,v) \in C$ it holds that $(E,v_E) \in C$ for every $E \subseteq N$.  

15
Hence, \( DP_i^\mu(N,v) = 0 \) when \(|N| = 1 \). Proceeding by induction assume that for some given integer \( k \geq 1 \) and for any game \((N',v) \in \mathcal{G} \) with \( i \in N' \) and \(|N'| = k \), it holds that \( DP_i^\mu(N',v) = 0 \), and let \( N \) be such that \( N' \subseteq N \) and \(|N| = k + 1 \). Using the induction hypotheses with \( N' = N \setminus \{j\} \) for all \( j \in N \setminus \{i\} \) we obtain that for given \([N]-player game \((N,v)\) with the induced restricted \((|N| - 1)-player games \((N \setminus \{j\},v_{N\setminus\{j\}})\) it holds that

\[
|N|DP_i^\mu(N,v) = DP_i^\mu(N,v) + \sum_{j \in N \setminus \{i\}} DP_i^\mu(N,v) - DP_i^\mu(N \setminus \{j\},v_{N\setminus\{j\}}).
\]

\[
= \sum_{j \in N \setminus \{i\}} \left( DP_i^\mu(N,v) - DP_i^\mu(N \setminus \{j\},v_{N\setminus\{j\}}) \right)
\]

\[
= \sum_{j \in N \setminus \{i\}} \left( P^\mu(N,v) - P^\mu(N \setminus \{j\},v_{N\setminus\{j\}}) \right)
\]

\[
= \sum_{j \in N \setminus \{i\}} \left( P^\mu(N \setminus \{i\},v_{N\setminus\{i\}}) - P^\mu(N \setminus \{i,j\},v_{N\setminus\{i,j\}}) \right)
\]

\[
= \mu(N,v) - \mu(N \setminus \{i\},v_{N\setminus\{i\}}).
\]

Hence, null player independency of \( \mu \) implies that

\[
DP_i^\mu(N,v) = \frac{1}{|N|} \left( \mu(N,v) - \mu(N \setminus \{i\},v_{N\setminus\{i\}}) \right) = 0,
\]

for every \((N,v) \in \mathcal{G}\) when \( i \) is a null player in \((N,v)\), and thus \( NDP_i^\mu(N,v) = 0 \) for \((N,v) \in \mathcal{G}^0\), which shows that the null player property is true on \( \mathcal{G}^0 \).

To prove the symmetry we show that for every \((N,v) \in \mathcal{G}\) it holds that \( NDP_i^\mu(N,v) = NDP_j^\mu(N,v) \), when \( i \) and \( j \) are two symmetric players in \( N \). So, let \( i,j \in N \) be two symmetric players in \((N,v) \in \mathcal{G}\). For \( N = \{i,j\} \) it follows with \( P^\mu(0,v) = 0 \) and the symmetry of \( \mu \) that \( P^\mu(N \setminus \{i\},v_{N\setminus\{i\}}) = P^\mu(\{j\},v_{\{j\}}) = \mu(\{j\},v_{\{j\}}) = \mu(\{i\},v_{\{i\}}) = P^\mu(\{i\},v_{\{i\}}) = P^\mu(N \setminus \{j\},v_{N\setminus\{j\}}) \).

Hence, \( NDP_i^\mu(N,v) = NDP_j^\mu(N,v) \). Proceeding by induction assume that for some given integer \( k \geq 2 \) and for any game \((N',v) \in \mathcal{G} \) with \( i,j \in N' \) and \(|N'| = k \), it holds that \( P^\mu(N' \setminus \{i\},v_{N'\setminus\{i\}}) = P^\mu(N' \setminus \{j\},v_{N'\setminus\{j\}}) \) and let \( N \) be such that \( N' \subseteq N \) and \(|N| = k + 1 \). Using symmetry of \( \mu \) and the induction hypotheses with \( N' = N \setminus \{h\} \) for all \( h \in N \setminus \{i,j\} \) we obtain that for given \([N]-player game \((N,v)\) with the induced restricted \((|N| - 1)-player games \((N \setminus \{h\},v_{N\setminus\{h\}})\) it holds that

\[
P^\mu(N \setminus \{i\},v_{N\setminus\{i\}}) = \frac{1}{|N| - 1} \left( \mu(N \setminus \{i\},v_{N\setminus\{i\}}) + \sum_{h \in N \setminus \{i\}} P^\mu(N \setminus \{i,h\},v_{N\setminus\{i,h\}}) \right)
\]

\[
= \frac{1}{|N| - 1} \left( \mu(N \setminus \{j\},v_{N\setminus\{j\}}) + \sum_{h \in N \setminus \{i\}} P^\mu(N \setminus \{j,h\},v_{N\setminus\{j,h\}}) \right)
\]

\[
= P^\mu(N \setminus \{j\},v_{N\setminus\{j\}}).
\]

So, for every \((N,v) \in \mathcal{G}\) and two symmetric players \( i,j \in N \) it holds that

\[
DP_i^\mu(N,v) = P^\mu(N,v) - P^\mu(N \setminus \{i\},v_{N\setminus\{i\}}) = P^\mu(N,v) - P^\mu(N \setminus \{j\},v_{N\setminus\{j\}}) = DP_j^\mu(N,v).
\]

Hence, \( NDP_i^\mu(N,v) = NDP_j^\mu(N,v) \).

Finally we prove \( \mu \)-additivity of \( NDP^\mu \). For \((N,v),(N,w) \in \mathcal{G}\) with \(|N| = 1 \) it holds that \( P^\mu(N,v+w) = \mu(N,v+w) = \mu(N,v) + \mu(N,w) = P^\mu(N,v) + P^\mu(N,w) \) by additivity.
of \( \mu \). Proceeding by induction assume that for some given integer \( k \geq 1 \) and for any pair \((N', v), (N', w)\) \( \in \mathcal{G} \) with \( |N'| = k \), it holds \( P^\mu(N', v + w) = P^\mu(N', v) + P^\mu(N', w) \). Then for \( N \) with \( |N| = k + 1 \), it follows from the additivity of \( \mu \) and the induction hypothesis for all \( N' = N \setminus \{j\} \) for all \( j \in N \), that

\[
P^\mu(N, v + w) = \frac{1}{|N|} \left( \mu(N, v + w) + \sum_{i \in N} P^\mu(N \setminus \{i\}, (v + w)_{N \setminus \{i\}}) \right)
\]

\[
= \frac{1}{|N|} \left( \mu(N, v) + \mu(N, w) \right)
\]

\[
+ \frac{1}{|N|} \sum_{i \in N} \left( P^\mu(N \setminus \{i\}, v_{N \setminus \{i\}}) + P^\mu(N \setminus \{i\}, w_{N \setminus \{i\}}) \right)
\]

\[
= P^\mu(N, v) + P^\mu(N, w).
\]

With equation (6) it then follows that \( DP^\mu(N, v + w) = DP^\mu(N, v) + DP^\mu(N, w) \), and thus

\[
\mu(N, v + w) \cdot NDP^\mu(N, v + w) = \mu(N, v) \cdot NDP^\mu(N, v) + \mu(N, w) \cdot NDP^\mu(N, w)
\]

when both \((N, v), (N, w)\) \( \in \mathcal{G}^0 \). In case one of the games is a null game, say \( v = v^0 \), then \( v + w = w \) and \( \mu(N, v^0) = 0 \) imply that \( \mu(N, v + w) \cdot NDP^\mu(N, v + w) = \mu(N, w) \cdot NDP^\mu(N, w) = \mu(N, v) \cdot NDP^\mu(N, v) + \mu(N, w) \cdot NDP^\mu(N, w) \). Hence, \( \mu \)-additivity holds for any pair \((N, v), (N, w)\) \( \in \mathcal{G} \).

\( \square \)

Examples of additive, symmetric, and null player independent \( \mu \)-functions are the functions \( \eta^S(N, v) = v(N) \), \( \eta^B(N, v) = \frac{1}{2^{n-1}} \sum_{E \subseteq N} (2^{|E|} - n) v(E) \) and \( \eta^M(N, v) = \sum_{i \in N} m^i_N(N, v) \) as defined in the previous section. So, Theorem 32 holds for \( \rho^S \), \( \rho^B \) and \( \rho^M \). The function \( \eta^T(N, v) = \sum_{E \subseteq N} v(E) \) is additive and symmetric, but not null player independent. However, Theorem 32 also holds for \( \rho^T \) since this share function also can be obtained using the null player independent function \( \eta^T(N, v) = \frac{1}{2^{n-1}} \eta^T(N, v) \).

**Corollary 33** The share functions \( \rho^S \), \( \rho^B \), \( \rho^T \) and \( \rho^M \) are equal to the normalized marginal functions \( NDP^{nS}, NDP^{nB}, NDP^{nT} \) and \( NDP^{nM} \) respectively.

Observe that the Shapley share function \( \rho^S \) is equal to the normalized marginal function \( NDP^\mu \) on \( \mathcal{G} \) by taking \( \mu(N, v) = v(N) \) in Definition 29 of the \( \mu \)-potential function. From this it follows that for every share function corresponding to an additive, symmetric, and null player independent \( \mu \)-function it holds that the vector of shares of a game \((N, v) \in \mathcal{G} \) is equal to the vector of Shapley shares of a transformed game \((N, w^\ast)\) in which the transformation is determined by the \( \mu \)-function.

**Theorem 34** Let \( \mu : \mathcal{G} \rightarrow \mathbb{R} \) with \( \mu(N, v) = 0 \) when \( v = v^0 \) be additive, symmetric, and null player independent on \( \mathcal{G} \) and positive on \( \mathcal{G}^0 \), and let \( \rho^\mu \) be the unique share function satisfying the properties of Corollary 30. Then for every \((N, v) \in \mathcal{G}\) it holds that \( \rho^\mu(N, v) = \rho^S(N, w^\ast) \) where \((N, w^\ast) \in \mathcal{G} \) is given by \( w^\ast(E) = \mu(E, v_E) \) for all \( E \subseteq N \).
PROOF First consider the case that \((N, v)\) is a null game. Then by definition \(\rho^\mu_i(N, v) = \frac{1}{|N|}\) for all \(i \in N\). Furthermore we have that \((N, w^v)\) is a null game because \(\mu(E, v_E) = 0\) for all \(E \subset N\) and hence also \(\rho^S_i(N, w^v) = \frac{1}{|N|}\) for all \(i \in N\). To prove the theorem for all games in \(G\) we show that the share function \(\rho\) defined by \(\rho(N, v) = \rho^S(N, w^v)\) for all \(E \subset N\) satisfies efficient shares, symmetry and \(\mu\)-additivity on \(G\) and the null player property on \(G^0\). Since the share function satisfying these properties is unique we must then have that \(\rho^\mu(N, v) = \rho(N, v)\) for any \((N, v) \in G\).

The axiom of efficient shares follows immediately from the fact that \(\rho^S\) satisfies efficient shares.

Let \(i \in N\) be a null player in \((N, v) \in G^0\). Then the assumption of null player independence of \(\mu\) implies that \(i\) is a null player in \((N, w^v)\), and thus \(\rho_i(N, v) = \rho^S_i(N, w^v) = 0\).

Further, let \(i, j \in N\) be symmetric players in \((N, v) \in G\). Then symmetry of \(\mu\) implies that \(i\) and \(j\) are symmetric in \((N, w^v)\), and thus \(\rho_i(N, v) = \rho^S_i(N, w^v) = \rho^S_j(N, w^v) = \rho_j(N, v)\).

To show the \(\mu\)-additivity of \(\rho\), first consider \((N, v), (N, z) \in G^0\). By the additivity of \(\mu\) and using \(w^{v+z} = w^v + w^z\) we obtain that

\[
\mu(N, v + z)\rho(N, v + z) = \mu(N, v + z)\rho^S(N, w^{v+z}) = \mu(N, v + z)\rho^S(N, w^v + w^z).
\]

Since \(\rho^S\) satisfies \(\mu\)-additivity we obtain that

\[
\mu(N, v + z)\rho(N, v + z) = \mu(N, v + z) \left( \frac{\mu^S(N, w^v)\rho^S(N, w^v) + \mu^S(N, w^z)\rho^S(N, w^z)}{\mu^S(N, w^{v+z})} \right) = \mu(N, v + z) \left( \frac{w^v(N)\rho(N, v) + w^z(N)\rho(N, z)}{w^{v+z}(N)} \right) = \mu(N, v + z) \left( \frac{\mu(N, v)\rho(N, v) + \mu(N, z)\rho(N, z)}{\mu(N, v + z)} \right) = \mu(N, v)\rho(N, v) + \mu(N, z)\rho(N, z).
\]

In case that \(v = v^0\), then \(v + z = z\) and \(\mu(N, v) = 0\) imply that \(\mu(N, v + z)\rho(N, v + z) = \mu(N, z)\rho(N, z) = \mu(N, v)\rho(N, v) + \mu(N, z)\rho(N, z)\). Hence, \(\rho\) satisfies the axiom of \(\mu\)-additivity.

\(\square\)

We illustrate this theorem with an example.

**Example 35** Let \((N, v) \in G\) be given by \(N = \{1, 2, 3\}\) and \(v = u^{\{1,2\}} + u^{\{1,2,3\}}\), i.e.,

\[
v(E) = \begin{cases} 0 & \text{if } E \in \{\{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\} \\ 1 & \text{if } E \in \{\{1, 2\}\} \\ 2 & \text{if } E = \{1, 2, 3\}.
\end{cases}
\]

The various \(\mu\)-functions are represented in Table 3. For given \(\mu\), define the transformed game \((N, w^\mu)\) by \(w^\mu(E) = \mu(E, v_E)\) for all \(E \subset N\). Applying Theorem 34 yields

\[
\rho^S(N, v) = \frac{1}{12}(5, 5, 2).
\]
\[ J_t(E, v_E) \] 

\[ \eta^S(E, v_E) \]

\[ \eta^B(E, v_E) \]

\[ \eta^T(E, v_E) \]

\[ \eta^M(E, v_E) \]

Table 3: \( \mu \)-functions of Example 35

\[ \eta^S(E, v_E) \]

\[ \eta^B(E, v_E) \]

\[ \eta^T(E, v_E) \]

\[ \eta^M(E, v_E) \]

\[ \rho^S(N, w_{\eta^S}) = \frac{1}{4}(3, 3, 1) = \rho^B(N, v) \]

\[ \rho^S(N, w_{\eta^S}) = \frac{1}{9}(4, 4, 1) = \rho^T(N, v) \]

\[ \rho^S(N, w_{\eta^S}) = \frac{1}{5}(2, 2, 1) = \rho^M(N, v). \]

Note that \( \rho^S(N, w_{\eta^S}) = \frac{1}{18}(7, 7, 4) \neq \rho^T(N, v). \)

In Hart and Mas-Colell (1988) it is shown that the potential function \( P: G \rightarrow \mathbb{R} \) as defined in the beginning of this section is given by

\[ P(N, v) = \sum_{E \subset N} \frac{(n - |E|)!(|E| - 1)!}{n!} v(E). \]

Clearly this potential function is equal to \( P^\mu \) for the Shapley \( \mu \)-function defined by \( \eta^S(N, v) = v(N) = \sum_{i \in N} \sum_{E \subset N} \frac{(n - |E|)!(|E| - 1)!}{n!} m^i_E(N, v). \) Theorem 34 enables us to generalize the result of Hart and Mas-Colell for \( \mu \)-functions being a weighted sum of the marginal contributions with positive weights \( \omega_{|E|}, E \subset N, i.e., for functions \( \sigma^\omega: G \rightarrow \mathbb{R} \) as defined in the previous section and given by \( \sigma^\omega(N, v) = \sum_{i \in N} \sum_{E \subset N} \omega_{|E|} m^i_E(N, v) \) for a vector \( \omega \) of positive weights. Note that such a \( \sigma^\omega \)-function is additive and symmetric on \( G \) and positive on \( G^0 \) by definition. The next corollary shows that the \( \sigma^\omega \)-potential function corresponding to a null player independent function \( \sigma^\omega \) can be seen as a weighted sum of worths of coalitions with corresponding marginal contribution weights \( \omega_{|E|}, E \subset N. \) The corollary follows directly from Theorem 34 and the result of Hart and Mas-Colell.

**Corollary 36** For given vector \( \omega = (\omega_1, \ldots, \omega_n), n \in \mathbb{N}, \) of positive weights, let \( \sigma^\omega \) be given by \( \sigma^\omega(N, v) = \sum_{i \in N} \sum_{E \subset N} \omega_{|E|} m^i_E(N, v). \) If \( \sigma^\omega \) is null player independent on \( G \) then the \( \sigma^\omega \)-potential function \( P^{\sigma^\omega} \) is given by

\[ P^{\sigma^\omega}(N, v) = \sum_{E \subset N} \omega_{|E|} v(E). \]
Finally, for given function $\mu: \mathcal{G} \to \mathbb{R}$ and corresponding share function $\rho^\mu$, let the corresponding value functions $\varphi^\mu$ on $\mathcal{G}$ be given by

$$\varphi^\mu(N,v) = \mu(N,v)\rho^\mu(N,v), \text{ if } (N,v) \in \mathcal{G}^0$$

and $\varphi^\mu(N,v) = 0$, $i \in N$, if $(N,v)$ is a null game. Then Theorem 32 and Theorem 34 immediately yield the following results for the value functions.

**Corollary 37** Let $\mu: \mathcal{G} \to \mathbb{R}$ with $\mu(N,v) = 0$ when $v = v^0$ be additive, symmetric, null player independent on $\mathcal{G}$, and positive on $\mathcal{G}^0$. Then the marginal function $DP^\mu$ corresponding to the unique $\mu$-potential function $P^\mu$ is equal to $\varphi^\mu$.

**Corollary 38** Let $\mu: \mathcal{G} \to \mathbb{R}$ with $\mu(N,v) = 0$ when $v = v^0$ be additive, symmetric, null player independent on $\mathcal{G}$, and positive on $\mathcal{G}^0$. Then for every $(N,v) \in \mathcal{G}$ it holds that $\varphi^\mu(N,v) = \varphi^S(N,w^v)$ where $(N,w^v) \in \mathcal{G}$ is given by $w^v(E) = \mu(E,v E)$ for all $E \subseteq N$.

For the special case of the Banzhaf value the latter result is shown by Dragan (1996).

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