Abstract—In this paper, we study the cooperative semi-global robust output regulation problem for a class of minimum phase nonlinear uncertain multi-agent systems. We first introduce a type of distributed internal model that converts the cooperative semi-global robust output regulation problem into a cooperative semi-global robust stabilization problem of the so-called augmented system. Then we solve the semi-global stabilization problem via a distributed dynamic output control law by combining a block semi-global backstepping technique, a simultaneous high gain feedback control technique, and a distributed high gain observer technique.

I. INTRODUCTION

The cooperative output regulation problem for multi-agent systems aims to design a distributed control law such that the output of each subsystem can asymptotically track a class of reference inputs in the presence of a class of disturbances and plant parameter uncertainties. Like the classical output regulation problem, here the class of reference inputs and the class of disturbances are both generated by a differential equation called exosystem. Therefore, this problem is a generalization of the leader-following consensus problem [2, 6, 10] by viewing the plant as the follower system and the exosystem as the leader system. So far, the problem has been extensively studied for linear uncertain multi-agent systems in, say, [12], [13], [16]. More recently, the cooperative robust output regulation problem for a class of nonlinear multi-agent systems in lower triangular form was further formulated and a global solution was obtained by a distributed state feedback control law in [14]. In the special case where the number of subsystems is equal to one, the problem in [14] reduces to the conventional global robust output regulation problem as studied in [4].

However, like the conventional robust output regulation problem for a single nonlinear system in lower triangular form, the global solution cannot be obtained via an output feedback control law. Therefore, in this paper, we will study a so-called cooperative semi-global robust output regulation problem for the class of nonlinear uncertain multi-agent systems to be described in (1) via output feedback control. For the special case where the number of the subsystems is equal to one, the semi-global robust output regulation problem for various nonlinear systems was studied in [7], [8], [11].

In comparison with the semi-global robust output regulation problem for a single nonlinear system, our problem is technically more challenging in at least two ways. First, for a multi-agent system, the augmented system is a multi-input, multi-output nonlinear system. So the techniques for semi-global stabilization [15] of the single-input, single-output system is no longer applicable. We have to develop specific techniques of block backstepping method that apply to multi-input, multi-output nonlinear systems. Second, due to the communication constraint which is described by a communication graph to be introduced in Section II, we cannot use the full information of the system for feedback control, and we have to develop a distributed control law to solve the stabilization problem for the augmented system.

Notation: The symbol $\text{col}(a_1, \ldots, a_s)$ denotes a column vector $[a_1^T, \ldots, a_s^T]^T$, where $a_i$, $i = 1, \ldots, s$, are some column vectors. The symbol $\mathcal{Q}$ denotes the compact set $\{x = \text{col}(x_1, \ldots, x_s) \in \mathbb{R}^s : |x_i| \leq R, i = 1, \ldots, s\}$. Given a positive definition function $V : \mathbb{R}^s \rightarrow \mathbb{R}$, the symbol $\Omega_{\varepsilon}(V(x))$ denotes the set $\{x \in \mathbb{R}^s : V(x) \leq \varepsilon\}$ and the symbol $\Omega_{\varepsilon}(V(x))$ denotes the set $\{x \in \mathbb{R}^s : V(x) < \varepsilon\}$.

II. PROBLEM STATEMENT

We consider the following nonlinear uncertain multi-agent systems

$$
\dot{z}_i = f_{ii}(z_i, x_{11}, v, w), \\
\dot{x}_{si} = x_{(s+1)i}, s = 1, \ldots, r - 1, \\
\dot{x}_{ri} = f_{ii}(z_i, x_{11}, \ldots, x_{ri}, v, w) + b_i(w)u_i, \\
y_i = x_{1i}, i = 1, \ldots, N,
$$

where $z_i \in \mathbb{R}^{n_z}$, $x_i = \text{col}(x_{1i}, \ldots, x_{ri}) \in \mathbb{R}^r$, $y_i, u_i, v_i, w \in \mathbb{R}$ represents an uncertain constant vector. The exosystem is given by

$$
\dot{v} = Sv, \quad y_0 = g(v, w),
$$

where $y_0 \in \mathbb{R}$ is the output of the exosystem. The regulated errors for each subsystem is defined as $e_i = y_i - y_0$. We assume the functions $f_{ki}()$, $b_i()$, $k = 0, 1, i = 1, \ldots, N$, and $g()$ are sufficiently smooth functions with $f_{ki}(0, \ldots, 0, 0) = 0$, $k = 0, 1$, and $g(0, w) = 0$, for all $w \in \mathbb{R}^{n_w}$.

The plant (1) and exosystem (2) together are viewed as a multi-agent system of $N + 1$ agents with the exosystem as the leader and all the subsystems of (1) as the followers. With respect to (1) and (2), we can define a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$.
where \( \mathcal{V} = \{0, 1, \ldots, N\} \) with the node 0 associated with the exosystem and the other \( N \) nodes associated with the \( N \) followers, respectively, and \((j, i) \in \mathcal{E}, j = 0, 1, \ldots, N \) and \( i = 1, \ldots, N \), if and only if the control \( u_i \) can make use of \( y_i - y_j \) for feedback control. Thus our control law will be of the following form

\[
\begin{align*}
u_i &= k_i(\xi_i, y_i - y_j, j \in \mathcal{N}_i), \\
\dot{\xi}_i &= g_i(\xi_i, y_i - y_j, j \in \mathcal{N}_i),
\end{align*}
\]

(3)

where \( \mathcal{N}_i = \{ j : (j, i) \in \mathcal{E} \} \), \( k_i \) and \( g_i \) are sufficiently smooth functions vanishing at the origin, and \( \xi_i \in \mathbb{R}^{n_i} \) with \( n_i \) to be defined later. A control law of the form (3) is called a distributed dynamic output control law because the control of each subsystem can only take the output information of its neighbors and itself for feedback control. Our problem can be described as: Given systems (1) and (2), the digraph \( \mathcal{G} \), a real number \( R > 0 \), and compact subsets \( \mathcal{V}_0 \subseteq \mathbb{R}^n \) and \( \mathcal{W} \subseteq \mathbb{R}^m \) which contain the origins of the respective Euclidean spaces, find a control law of the form (3) such that for any \( w \in \mathcal{W}, r(0) \in \mathcal{V}_0, \) col\( (z_i(0), x_i(0)) \) \( \in \mathcal{Q}_R^{n_i} \), and \( \xi_i(0) \in \mathcal{Q}_R^{n_i} \), the trajectory of the closed-loop system composed of (1) and (3) exists and is bounded for all \( t \geq 0 \), and \( \lim_{t \to \infty} e(t) = 0 \) where \( e = \text{col}(e_1, \ldots, e_N) \).

The above problem will be called cooperative regional robust output regulation problem for the nonlinear multi-agent system (1) on the compact subset \( \mathcal{Q}_R^{n_i} \times \mathcal{V}_0 \times \mathcal{W} \), where \( n_i \) denotes the dimension of the closed-loop system. If for any \( R > 0 \), and any compact subsets \( \mathcal{V}_0 \subseteq \mathbb{R}^n \) and \( \mathcal{W} \subseteq \mathbb{R}^m \) which contain the origins of the respective Euclidean spaces, the cooperative regional robust output regulation problem for system (1) on the compact subset \( \mathcal{Q}_R^{n_i} \times \mathcal{V}_0 \times \mathcal{W} \) is solvable, then we say that the cooperative semi-global robust output regulation problem for system (1) is solvable.

### III. Problem Conversion

Let us make some standard assumptions as follows:

**Assumption 1:** The exosystem is neutrally stable, i.e., all the eigenvalues of \( S \) are semi-simple with zero real parts.

**Assumption 2:** \( b_i(v) > 0, i = 1, \ldots, N \), for all \( v \in \mathbb{R}^n \).

**Assumption 3:** There exist sufficiently smooth functions \( z_i(v, w) \) with \( z_i(0, 0) = 0 \) such that for any \( v \in \mathbb{R}^n \) and \( w \in \mathbb{R}^m \), \( \frac{\partial z_i(v, w)}{\partial v}Sv = f_{0i}(z_i(v, w), q(v, w), v, w) \).

**Remark 1:** Under Assumption 1, given any compact subset \( \mathcal{V}_0 \), there exists a compact subset \( \mathcal{V} \) such that, for any \( v(0) \in \mathcal{V}_0 \), the trajectory \( v(t) \) of the exosystem remains in \( \mathcal{V} \) for all \( t \geq 0 \). Under Assumptions 2 and 3, for \( i = 1, \ldots, N \), let \( x_i(0, v, w) = q(v, w), x_{si}(v, w) = \frac{\partial z_i}{\partial v}Sv, \) \( s = 2, \ldots, r \), \( u_i(v, w) = b_i^{-1}(v)(\frac{\partial z_i}{\partial v})Sv - f_{1i}(z_i(v, w), x_{si}(v, w), \ldots, x_{ri}(v, w), v, w) \), and \( x_i(v, w) = \text{col}(x_{11}(v, w), \ldots, x_{ri}(v, w)) \). Then it can be verified that \( x(v, w) = \text{col}(z_1(v, w), x_1(v, w), \ldots, z_N(v, w), x_N(v, w)) \), and \( u(v, w) = \text{col}(u_1(v, w), \ldots, u_N(v, w)) \) constitute the global solution of the regulator equations associated with (1) and (2) [5].

**Assumption 4:** The function \( u_i(v, w) \) is a polynomial in \( v \) with coefficients depending on \( w \).

It is known from [3] that, under Assumptions 1 to 4, there exist integers \( n_{ri}, i = 1, \ldots, N \), and real coefficients polynomials \( p_i(\lambda) = \lambda^{n_{ri}} - q_i(\lambda) \), \( q_i(\lambda) = \ldots - \eta_{ri}i^{n_{ri}-1} \) whose roots are all imaginary, such that, for all trajectories \( v(t) \) of the exosystem and all \( w \in \mathcal{W}, u_i(v, w) \) satisfy \( \frac{d^n}{dt^n}u_i(v, w) = q_{1i}u_i(v, w) + q_{2i} \frac{d}{dt}u_i(v, w) + \cdots + q_{ri}i^{n_{ri}-1}u_i(v, w) \). Now we define the dynamic compensator

\[
\hat{y}_i = M_iy_i + Q_iu_i, \quad i = 1, \ldots, N,
\]

(4)

where \( M_i \in \mathbb{R}^{n_i \times n_i} \) are any Hurwitz matrices, and \( Q_i \in \mathbb{R}^{n_i} \) are any column vectors such that the pairs \( (M_i, Q_i) \) are controllable. By Remark 3.7 in [4], under Assumptions 1 to 4, the dynamic compensator (4) is an internal model of system (1). The system composed of the plant (1) and the internal model (4) is called the augmented system [4].

For \( i = 1, \ldots, N \), let

\[
\Phi_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \\ q_{1i} & q_{2i} & \cdots & q_{ni} \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 1^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

(5)

Since \( (\Gamma_i, \Phi_i) \) is observable, \( (M_i, Q_i) \) is controllable, and the eigenvalues of \( M_i \) and \( \Phi_i \) do not coinciding, there exists a nonsingular matrix \( T_i \) that satisfies the Sylvester equation \( T_i\Phi_i - M_i\Gamma_i = Q_i\Gamma_i \) [9]. Let \( \theta_i(v, w) = T_i\text{col}(u_i(v, w), \frac{du_i(v, w)}{dt}, \ldots, \frac{d^{n_{ri}-1}u_i(v, w)}{dt^{n_{ri}-1}}) \), \( i = 1, \ldots, N \). Then by performing the coordinate and input transformation \( \bar{z}_i = z_i - z_i(v, w), \bar{x}_{si} = x_{si} - x_{si}(v, w), s = 1, \ldots, r \), \( \bar{y}_i = y_i - \theta_i(v, w), \bar{u}_i = u_i - \Gamma_iT_i^{-1}\eta_i \), we obtain that the augmented systems takes the following form:

\[
\begin{align*}
\bar{z}_i &= \bar{f}_0(\bar{z}_i, \bar{x}_{i1}, v, w) \\
\bar{x}_{si} &= \bar{f}_1(\bar{z}_i, \bar{x}_{i1}, \ldots, \bar{x}_{ri}, v, w) + b_i(v)\Gamma_iT_i^{-1}\eta_i + b_j(v)\bar{u}_i \\
\bar{y}_i &= (M_i + Q_i\Gamma_iT_i^{-1})\bar{y}_i + Q_i\bar{u}_i
\end{align*}
\]

(6)

where

\[
\begin{align*}
\bar{f}_0(\bar{z}_i, \bar{x}_{i1}, v, w) &= \bar{f}_0(\bar{z}_i + z_i(v, w), \bar{x}_{i1} + x_{i1}(v, w), v, w) \\
\bar{f}_1(\bar{z}_i, \bar{x}_{i1}, \ldots, \bar{x}_{ri}, v, w) &= \bar{f}_1(\bar{z}_i + z_i(v, w), \bar{x}_{i1} + x_{i1}(v, w), \ldots, \bar{x}_{ri} + x_{ri}(v, w), v, w)
\end{align*}
\]

The augmented system (6) has the property that the origin is an equilibrium point for all \( v \) and \( w \), and \( e \) is identically zero at the origin. As a result, it is possible to solve the cooperative semi-global robust output regulation problem for system (1) by stabilizing the augmented system (6) semi-globally via a distributed dynamic output feedback control law. To elaborate this point further, for \( i = 1, \ldots, N \), let \( e_{vi} = \sum_{j=0}^{n_{ri}}a_{ij}(y_i - y_j) = \sum_{j=0}^{n_{ri}}a_{ij}(\bar{x}_{i1} - \bar{x}_{1j}) \), where \( \bar{x}_{10} \equiv 0 \), and the parameters \( a_{ij} \) are chosen so that \( a_{ii} = 0, a_{ij} > 0 \) if \((j, i) \in \mathcal{E} \) and \( a_{ij} = 0 \) otherwise, \( i = 1, \ldots, N \),
Then we will consider a class of dynamic output feedback controller of the form
\[
\bar{u}_i = \bar{k}_i(\zeta_i, \ldots, \varepsilon_i),
\]
where \(\zeta_i \in \mathbb{R}^{n_c_i}\) for some integer \(n_c_i\), and \(\bar{k}_i, \varphi_i\) are globally defined sufficiently smooth functions that vanish at the origin. It can be seen that \(\bar{u}_i\) depends on \((y_i - y_j)\) if and only if \(j \in N_i\). Thus this control law is indeed a distributed control law. We now further define the cooperative regional robust stabilization problem of system (6) as: Given a real number \(R > 0\), and some compact subsets \(\mathbb{V} \subseteq \mathbb{R}^q\) and \(\mathbb{W} \subseteq \mathbb{R}^{n_w}\), which contain the origins of the respective Euclidean spaces, find a control law of the form
\[
\nu_i = \nu_i(0) + \Gamma_i q_i + \bar{F}_i + \bar{G}_i Y_i,
\]
\(\nu_i(0) \in \mathbb{W}\), and all \(\bar{F}_i, \bar{G}_i, \bar{Q}_i\) are smooth with \(\varphi_i(0, v, w) = 0\) and \(\varphi_2(0, v, w) = 0\) for all \(\cal{Q}(v, w) \in \mathbb{V} \times \mathbb{W}\). Assume that for any given compact set \(\Xi_1 \subseteq \mathbb{R}^n\), there exist positive real numbers \(c_1 > 0\) and \(\varepsilon_1 \geq 1\), and a \(C^2\) positive definite function \(U_0(x_1)\) such that, \(\Xi_1 \subseteq \Omega_{c_1}(U_0(x_1))\), and for all \(\cal{Q}(v, w) \in \mathbb{V} \times \mathbb{W}\), and all \(\chi_1 \in \Omega_{c_1 + \varepsilon_1}(U_0(x_1)), U_0(x_1) \big|_{|x_1|^2} \leq -\alpha(|x_1|^2)\), where \(\alpha\) is a positive real number. Define the \(C^2\) function
\[
V(x_1, x_2) = -\frac{\varepsilon_1 U_0(x_1)}{c_1} + \nu^T P_2 x_2,
\]
where \(P\) is a positive definite solution to the inequality \(M^T P + P M \leq -I_{n_2}\). Then \(V(x_1, x_2)\) is positive definite defined on \(\Omega_{c_1 + \varepsilon_1}(U_0(x_1)) \times \mathbb{R}^{n_2}\). Furthermore, for any other given compact set \(\Xi_2 \subseteq \mathbb{R}^{n_2}\), there exist \(c^* > 0\) and \(\nu > 0\), such that, \(\Xi_1 \times \Xi_2 \subseteq \Omega_{c^*}(V(x_1, x_2))\), and for all \(\cal{Q}(v, w) \in \mathbb{V} \times \mathbb{W}\), and all \(\chi_1, \chi_2 \in \Omega_{c^* + \varepsilon_1}(V(x_1, x_2)), V(x_1, x_2) \big|_{(|x_1|^2 + |x_2|^2)} \leq -\beta \|\chi_1, \chi_2\|^2\), where \(\beta\) is a positive real number and \(\varepsilon^* \geq 1\) can be arbitrarily chosen.

\section{IV. MAIN RESULT}

By Lemma 1, we only need to focus on solving the cooperative semi-global stabilization problem for system (6).
For this purpose, we further need two more assumptions.

Assumption 5: For the ith subsystem of the augmented system (6), there exists a $C^2$ positive definite and proper function $V_{0i} : \mathbb{R}^{n_{i+1}}$ into $\mathbb{R}$, such that for all $v_i, w_i \in V \times W$, $V_{0i}(\tilde{z}_i)$ along the trajectory of the zero dynamics, that is, $\dot{\tilde{z}}_i = f_{0i}(\tilde{z}_i, 0, v, w), \dot{v}_i = \alpha_{0i}, i = 1, \ldots, N$ are some known positive real numbers.

Assumption 6: Every node $i = 1, \ldots, N$ is reachable from the node 0 in the digraph $\tilde{G}$.

Remark 3: Let $H = [h_{ij}]_{N \times N}, H(w) = \text{diag}\{h_1(w), \ldots, h_N(w)\}$. Then, by Lemma 4 in [2] or Lemma 1 in [12], we can conclude that $H(w)$ has all the eigenvalues with positive real parts for all $w \in \mathbb{W}$ if and only if Assumption 6 is satisfied.

We further define the following coordinate transformation for each subsystem in (6),

$$\hat{x}_i = \frac{\tilde{x}_i}{\gamma_i}, \quad s = 1, \ldots, r - 1$$

$$\hat{v}_i = \tilde{v}_i + g^{r-1} \gamma_i x_{(r+1)i}, \quad i = 1, \ldots, N$$

where the number $g$ and the coefficients $\gamma_i, s = 1, \ldots, r - 1$ are to be determined later. Let $\tilde{x}_i = \text{col}(\tilde{x}_{1i}, \ldots, \tilde{x}_{ri-1i}), i = 1, \ldots, N$. Then, under the coordinate transformation (11) to (13), the augmented system (6) takes the following form:

$$\dot{\tilde{z}}_i = f_{0i}(\tilde{z}_i, \hat{x}_i, v, w)$$

$$\dot{\hat{x}}_i = gA_i \hat{x}_i + B_i(v)\hat{v}_i$$

$$\dot{\hat{v}}_i = \hat{M}_i \hat{\theta}_i + \hat{f}_i, i = 1, \ldots, N$$

and the coefficients $\gamma_j, j = 1, \ldots, r - 1$, and $\delta_k, k = 1, \ldots, r$, are chosen so that the polynomials $P_2(s) = s^{r-1} + \gamma_{r-1}s^{r-2} + \ldots + \gamma_1$ and $P_3(s) = s^r + \delta_1s^{r-1} + \ldots + \delta_1s + \delta_0$ are stable, and $g$, $K$, and $h$ are positive numbers to be determined. Then the closed-loop system composed of (14) and (15) takes the following form:

$$\dot{\xi}_i = \hat{f}_0(\tilde{z}_i, \hat{x}_i, v, w)$$

$$\hat{\theta}_i = gA_i \hat{x}_i + B_i(v)\hat{v}_i$$

$$\dot{\hat{v}}_i = \hat{M}_i \hat{\theta}_i + \hat{f}_i, i = 1, \ldots, N$$

Then for any $i = 1, \ldots, N$,

$$\psi_i = h(\psi_{(i+1)i}) - \delta_{i-1}\psi_{si}, \quad s = 1, \ldots, r - 1$$

where $(H\hat{\theta})_i$ denotes the $i$th component of vector $H\hat{\theta}$, and $[A_{i-1}]_{r \times 1}$ denotes the $(r-1)$th row of the matrix $A_i$. Let $z = \text{col}(\tilde{z}_1, \ldots, \tilde{z}_N), \bar{z} = \text{col}(\tilde{x}_1, \ldots, \tilde{x}_N)$, $\bar{v} = \text{col}(\tilde{v}_1, \ldots, \tilde{v}_N), \theta = \text{col}(\hat{\theta}_1, \ldots, \hat{\theta}_N)$, $\psi_i = [\psi_1, \ldots, \psi_N]^T, \psi = [\psi_1, \ldots, \psi_N]^T$, and $D_h = \text{diag}(h_1^{r-1}, \ldots, h_1)$.

Then system (6) is equivalent to the following block lower triangular form

$$\tilde{z} = F_0(\tilde{z}, \hat{x}_a, v, w)$$

$$\hat{x}_a = g(I_N \otimes A_0)\hat{x}_a + (1_N \otimes B_0(v))\hat{v}_a$$

$$\hat{\theta} = M\hat{\theta} + F_1(\tilde{z}, \hat{x}_a, \hat{v}_a, v, w)$$

$$\psi = h(I_N \otimes A_0(1))\psi + F_2(\tilde{z}, \hat{x}_a, \hat{v}_a, \kappa(\psi, h), v, w, g)$$

where $1_N$ denotes an $N \times 1$ column vector whose elements are all one, $M = \text{diag}(M_1, \ldots, M_N)$, $\kappa(\psi, h) = (I_N \otimes D_h)^T\psi, F_0 = \text{col}(f_{01}, \ldots, f_{0N}), F_1 = \text{col}(f_{11}, \ldots, f_{1N}), F_2 = \text{col}(f_{21}, \ldots, f_{2N})$, $F_3 = \text{col}(F_31, \ldots, F_3N)$, and $F_3 = \text{col}(0, \ldots, 0, (H\hat{\theta})_i - g[A_{i-1}]_{r \times 1}(H\hat{\theta}_a))$. Let $n_2 = \sum_{i=1}^N n_{2i}, n_{2x} = (r - 1)N, n_\eta = \sum_{i=1}^N n_{\eta i}, n_\varphi = N, \varphi = n_N$. Then we have the following theorem.

Theorem 1: Under Assumptions 2, 5 and 6, given any arbitrarily large number $\bar{R} > 0$, there exist sufficiently large positive real numbers $g$, $K$ and $h$, which depend on $\bar{R}$, such that the equilibrium at the origin of the closed-loop system composed of (6) and (15) is locally asymptotically stable with its domain of attraction containing $\tilde{Q}_R^{\bar{R}}$, where
Note that it has the form of (10) with \( \chi_1 = \alpha(\bar{z}, \bar{x}_a, \bar{y}), \chi_2 = \rho = K, \) and \( \tau(\chi_2, \rho) = \chi_2 = \bar{\vartheta} \). Here \( \tau(\chi_2, \rho) \) is independent on \( \rho \). By (12), for any \( \xi \in Q_{R_1}^\infty \) and any \( \bar{z}, \bar{x}_a, \bar{y} \in Q_{R_1}^{n+\nu} \times Q_{R_2}^{\nu} \), there exists \( \bar{R}_3 > 0 \) such that \( \bar{z} \in Q_{R_1}^\infty \). By (22), applying Lemma 4 with \( \Xi = \bar{Q}_{R_1}^{n+\nu} \times \bar{Q}_{R_2}^{\nu} \) and \( \Xi_2 = \bar{Q}_{R_4}^{\nu} \), there exist a sufficient large positive number \( K \), a positive real number \( c_3 \), a \( C^2 \) function \( V_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}, w) \), positive definite functions \( \bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \) and \( \bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \), and a positive real number \( \alpha_3 \), such that \( \bar{Q}_{R_1}^{n+\nu} \times \bar{Q}_{R_2}^{\nu} \times \bar{Q}_{R_4}^{\nu} \subseteq \Omega_c(\bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta})), \) and for all \( \col(v, w) \in \mathbb{V} \times \mathbb{W} \), and all \( \col(v, w) \in \mathbb{V} \times \mathbb{W} \),

\[
V_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \leq V_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}, w) \leq V_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta})
\]

where \( \bar{z}_3 \) is chosen so that

\[
\bar{z}_3 \geq \max_{(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \in \Omega_c(\bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}))} \|\bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta})\|.
\]

Step-4: consider the \( \col(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \) subsystem of (18) when \( \psi = 0 \), that is,

\[
\bar{z} = F_0(\bar{z}, \bar{x}_a, v, w) \quad \dot{\bar{x}}_a = g(I_N \otimes A_\bar{c} \bar{x}_a + (1_N \otimes B_\bar{c}) \bar{y}) \quad \dot{\bar{y}} = -K(H(w) \bar{y} + F_2(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}, 0, v, w, g))
\]

where \( \bar{z}_3 \) is chosen so that

\[
\bar{z}_3 \geq \max_{(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \in \Omega_c(\bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}))} \|\bar{V}_3(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta})\|.
\]

By Lemma 4, from (27) to (29), any trajectory of \( \col(\bar{z}, \bar{x}_a, \bar{y}, \bar{\vartheta}) \) starting in compact set \( \bar{Q}_{R_1}^{n+\nu} \times \bar{Q}_{R_2}^{\nu} \times \bar{Q}_{R_4}^{\nu} \) converges to the origin asymptotically as \( t \to \infty \). Then by (11) to (13), and noting that \( \bar{x}_c \in \bar{Q}_{R_3}^{\nu} \) implies...
col(\hat{z}, \hat{x}_a, \hat{\eta}, \hat{\psi}) ∈ \tilde{Q}^{n_z + n_x} \times G^{n_x} \times \tilde{Q}^{n_{\eta}} \times G^{n_{\eta}} \times \tilde{Q}^{n_{\psi}} \times G^{n_{\psi}}
\text{any trajectory of } \hat{x}_c \text{ starting in compact set } \tilde{Q}_R^{n_c} \text{ converges to the origin asymptotically as } t → \infty. \text{ Thus, the equilibrium at the origin of the closed-loop system composed of (6) and (15) is asymptotically stable with its domain of attraction containing } \tilde{Q}_R^{n_c}. \text{ The proof is thus completed.}

By Lemma 1 and Theorem 1, we have the following result.

Theorem 2: Under Assumptions 1 to 6, given any \( R > 0 \), and any compact subsets \( \mathbb{V}_0 \subseteq \mathbb{R}^q \) and \( \mathbb{W} \subseteq \mathbb{R}^{n_w} \), there exist sufficiently large numbers \( g > 0, K > 0, \hat{h} > 0 \), which depend on \( R \), such that the cooperative regional output regulation problem of system (1) on the compact set \( \tilde{Q}_R^{n_c} \), where \( n_c = 2rN + \sum_{i=1}^{N}(n_{z_i} + n_{n_i}) \), is solved by a distributed dynamic output feedback controller

\[
\begin{align*}
\dot{u}_i &= -K \dot{\hat{e}}_i + \Gamma \Phi^{-1} \hat{e}_i \\
\dot{\hat{e}}_i &= \zeta_i + g \gamma_{i-1} \hat{e}_{i-1} + \cdots + g^{-2} \gamma_2 \hat{e}_{2i} + g^{-1} \gamma_1 \hat{e}_{1i} \\
\hat{z}_i &= A_i(h) \hat{z}_i + B_i(h) e_{oi} \\
\dot{\hat{y}}_i &= A_i \hat{y}_i + Q_i u_i
\end{align*}
\]

Thus, the cooperative semi-global robust output regulation problem of system (1) is solvable.

V. AN EXAMPLE

Consider a group of van der Pol oscillators as follows

\[
\begin{align*}
\dot{x}_{1i} &= x_{2i} \\
\dot{x}_{2i} &= -x_{1i} + \mu_i(w)x_{2i}(1 - x_{1i}^2) + b_i(w) u_i \\
y_i &= x_{1i}, \quad i = 1, \ldots, 5,
\end{align*}
\]

where the coefficients \( \mu_i(w) = \tilde{\mu}_i + w^\mu_i \) and \( b_i(w) = \tilde{b}_i + w^b_i \) with \( \tilde{\mu}_i \) and \( \tilde{b}_i \) being their nominal values, respectively, and \( w = \text{col}(w_1, w_2, \ldots, w_N, w_N^b) \) being the uncertainty vector. We assume \( \mathbb{W} \) is some known compact set containing the origin such that \( \mathbb{W} \subset \{ w \in \mathbb{R}^l : \hat{b}_i + w^b_i > 0, i = 1, \ldots, 5 \} \). The exosystem is an unforced harmonic oscillator \( \hat{v}_1 = v_2, \hat{v}_2 = -v_1, y_0 = v_1 \). The digraph \( \hat{G} \) is given such that the parameters \( a_{ij} \) are chosen as \( a_{10} = a_{20} = a_{31} = a_{42} = a_{45} = a_{53} = a_{54} = 1 \), and zero for the others.

It is noted that, even for the special case where \( N = 1 \), the output regulation problem for van der Pol system does not have a global solution with output feedback control. However, it can be verified that system (31) has the form of (1) with \( r = 2 \), and satisfies Assumptions 1 to 6. Therefore, applying Theorem 2, it is possible to solve the cooperative semi-global robust output regulation problem for this system via output feedback control. Due to space limit, we omit the detailed design process, and only show the simulation result in Figure 1. It can be seen that the tracking errors of all the followers converge to zero asymptotically. Thus, the performance of the controller is quite satisfactory.

VI. CONCLUSION

In this paper, we have studied the cooperative semi-global robust output regulation problem for a class of minimum phase nonlinear multi-agent systems, which generalizes the semi-global output regulation problem from a single nonlinear system to a multi-agent system. Future work will focus on the same problem subject to the time-varying network topologies.

REFERENCES