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MEAN-SQUARE AND ASYMPTOTIC STABILITY OF THE STOCHASTIC THETA METHOD*

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Abstract. Stability analysis of numerical methods for ordinary differential equations (ODEs) is motivated by the question "for what choices of stepsize does the numerical method reproduce the characteristics of the test equation?" We study a linear test equation with a multiplicative noise term, and consider mean-square and asymptotic stability of a stochastic version of the theta method. We extend some mean-square stability results in [Saito and Mitsui, *SIAM. J. Numer. Anal.*, 33 (1996), pp. 2254–2267]. In particular, we show that an extension of the deterministic A-stability property holds. We also plot mean-square stability regions for the case where the test equation has real parameters. For asymptotic stability, we show that the issue reduces to finding the expected value of a parametrized random variable. We combine analytical and numerical techniques to get insights into the stability properties. For a variant of the method that has been proposed in the literature we obtain precise analytic expressions for the asymptotic stability region. This allows us to prove a number of results. The technique introduced is widely applicable, and we use it to show that a fully implicit method suggested by [Kloeden and Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, 1992] has an asymptotic stability extension of the deterministic A-stability property. We also use the approach to explain some numerical results reported in [Milstein, Platen, and Schurz, *SIAM J. Numer. Anal.*, 35 (1998), pp. 1010–1019.]

Key words. asymptotic stability, mean-square stability, multiplicative noise, random sequence, theta method

AMS subject classifications. 60H10, 65C20, 65U05, 65L20

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1. Introduction and motivation.

Our aim is to study the stability of numerical methods for stochastic differential equations (SDEs) by generalizing concepts from the well-established deterministic theory. We consider an autonomous scalar Itô SDE,

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t), \quad X(0) = X_0, \tag{1.1}$$

driven by the standard Wiener process $W(t)$ [5, 13]. The following numerical method computes approximations $X_i \approx X(i\Delta t)$:

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta\Delta t f(X_{n+1}) + \Delta t^{\frac{1}{2}} g(X_n)V_n. \tag{1.2}$$

Here, $\Delta t > 0$ is the constant stepsize, $\theta \in [0,1]$ is a fixed parameter, and each V_n is an independent Normal$(0,1)$ random variable. In the deterministic case, $g \equiv 0$, (1.2) is called the theta method (TM) and the choice $\theta = 0$ gives Euler's method, $\theta = \frac{1}{2}$ gives the trapezoidal rule, and $\theta = 1$ gives the implicit, or backward, Euler method. The method (1.2) is discussed in [18], where it is called the semi-implicit Euler method. In particular, taking $\theta = 0$ gives the widely used Euler–Maruyama method. Because of its natural connection with the TM, we will refer to (1.2) as the stochastic theta method (STM). For details of the concepts of order of convergence for (1.2) and other numerical methods for SDEs, see, for example, [10]. In this work we are concerned with the linear stability properties of the STM.

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In section 2 we summarize the idea of linear (or absolute) stability for the deterministic TM. Here the underlying idea is one that has proved valuable throughout many areas of numerical analysis—study a numerical method on a test problem which is simple enough to allow analysis to be performed, but which retains features present in more general problems of interest. In this context the test problem is linear, scalar, and autonomous, and the property under consideration is stability of the trivial solution. The key question is then

$$\begin{array}{l}\text{for what stepsizes } \Delta t \text{ does the numerical method} \\ \text{share the stability property of the underlying test problem?}\end{array} \tag{1.3}$$

In section 3 we introduce a linear stochastic test equation. We consider two natural, but distinct, definitions of stability for this test equation—mean-square stability and asymptotic stability. In sections 4 and 5 we study the corresponding stability properties for the STM. We show that a mean-square generalization of the deterministic A-stability property holds for $\frac{1}{2} \leq \theta$, and for $0 \leq \theta < \frac{1}{2}$ we give a stability bound for the stepsize. For real values of the test equation parameters, we characterize and plot the stability regions. In the case of asymptotic stability, we do not find it possible to derive neat characterizations of the stability regions, and hence we rely upon a mixture of analysis and numerical computation. In section 6 we consider a variant of the STM for which a more detailed asymptotic stability analysis is possible. In this case we are able to characterize the stability regions precisely and draw conclusions in the spirit of (1.3). Some numerical simulations with the STM are presented in section 7. These tests indicate that the linear stability theory has relevance to the behavior near equilibrium on a nonlinear problem. In subsection 8.1 we apply our asymptotic stability analysis to two other methods from the literature. We show that a fully implicit method proposed by Kloeden and Platen [10] has an extremely desirable asymptotic stability extension of A-stability. We also confirm the numerical evidence reported by Milstein, Platen, and Schurz [14], which shows that a particular balanced method has good asymptotic stability. Subsection 8.2 discusses related work and possible extensions.

Our approach was heavily influenced by the ideas of Mitsui and his coworkers [11, 17, 18]. In [11, 18] the concept of mean-square stability with respect to a linear test equation was studied. In particular, the condition (4.2) below that characterizes mean-square stability of the STM was derived in [18]. Our contribution to mean-square stability comes from studying this condition with (1.3) in mind. A similar approach was taken in [19]. In [17], the authors introduced the concept of T-stability for the Euler–Maruyama scheme based on a two point or three point random variable, applied to a test equation with real coefficients. This stability condition arises by averaging the stability function. Our analysis in section 6 coincides with the T-stability approach in the case $\theta = 0$. However, we prefer to use the term asymptotic stability for this property, since, as we show in sections 5 and 6, it can be motivated from the analogous concept of asymptotic stability for the underlying SDE. Lemma 5.1 shows that studying asymptotic stability reduces to studying the expected value of a random variable with one complex parameter. In this way, we are able to investigate stability for the Normal$(0, 1)$ sampling methods.

2. Deterministic ODEs: Linear stability concepts. There is a vast literature on the stability of numerical methods for deterministic ordinary differential equations (ODEs); see, for example, [7] for a review. In this section we summarize well-known results for the TM in a way that helps to motivate the SDE analysis. Linear stability (or absolute stability) is based on the scalar test equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \lambda X(t), \quad t > 0, \quad X(0) = X_0 \neq 0, \tag{2.1}$$

where $\lambda \in \mathbb{C}$ is a constant. For this equation,

$$\lim_{t\to\infty} X(t) = 0 \quad \Leftrightarrow \quad \lambda \in \mathbb{C}^-, \tag{2.2}$$

where $\mathbb{C}^-$ denotes the left-half complex plane. The TM applied to (2.1) produces the recurrence

$$X_{n+1} = X_n + (1-\theta)\Delta t\lambda X_n + \theta\Delta t\lambda X_{n+1}, \tag{2.3}$$

which is well defined for $1 - \theta\lambda\Delta t \neq 0$. This recurrence depends on the parameter θ and the product $\Delta t\lambda \in \mathbb{C}$. For a given θ, we let S_θ denote the *stability region* for the TM; that is, the set of points $\Delta t\lambda \in \mathbb{C}$ for which $X_n \to 0$ as $n \to \infty$. If $\mathbb{C}^- \subseteq S_\theta$, then the method is said to be *A-stable*. In other words the method is A-stable if and only if the following statement is true:

> Whenever the test problem (2.1) is such that $\lim_{t\to\infty} X(t) = 0$, the method (2.3) gives $\lim_{n\to\infty} X_n = 0$ for all $\Delta t > 0$.

It is easily shown that the region S_θ has the following form:

$$\begin{aligned} 0 \leq \theta < \tfrac{1}{2}, &\quad S_\theta = \text{ interior of } B(\tfrac{-1}{1-2\theta}, \tfrac{1}{1-2\theta}), \\ \theta = \tfrac{1}{2}, &\quad S_\theta = \mathbb{C}^-, \\ \tfrac{1}{2} < \theta \leq 1, &\quad S_\theta = \text{ exterior of } B(\tfrac{1}{2\theta-1}, \tfrac{1}{2\theta-1}), \end{aligned}$$

where $B(c, r)$ denotes the disc with center $c \in \mathbb{C}$ and radius r. It follows that the TM is A-stable if and only if $\frac{1}{2} \leq \theta$. For $\theta = \frac{1}{2}$ the method has a stability region that matches exactly the region for the test problem. We also note that for $0 \leq \theta < \frac{1}{2}$, since the relevant disc intersects the imaginary axis tangentially at the origin, for $\lambda \in \mathbb{C}^-$ with small real part the method is stable only for very small Δt. In later sections we will be concerned with similar effects; hence we formalize this result as a lemma.

LEMMA 2.1. *Given $\epsilon > 0$ and $0 \leq \theta < \frac{1}{2}$, there exists $\lambda = \lambda(\epsilon, \theta) \in \mathbb{C}^-$, normalized to $|\lambda| = 1$, such that the TM with $\Delta t = \epsilon$ gives $\lim_{n\to\infty} |X_n| = \infty$.*

Proof. The condition for stability may be written

$$\Delta t < \frac{-2\Re\{\lambda\}}{(1-2\theta)|\lambda|^2}. \tag{2.4}$$

Choosing λ so that $\Re\{\lambda\} = -\min\{(1-2\theta)\epsilon/4, 1/2\}$ and $|\lambda| = 1$, we see that $\Delta t = \epsilon$ violates (2.4). ∎

3. Stochastic ODEs: Linear stability concepts. A stochastic analogue of (2.1) is the problem

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad t > 0, \quad X(0) = X_0, \tag{3.1}$$

where $\lambda, \mu \in \mathbb{C}$ are constants and where $X_0 \neq 0$ with probability 1. Because of the factor $X(t)$ in the second term on the right-hand side of (3.1), this is said to be a linear equation with *multiplicative* noise. This equation has been considered by a number of authors [4, 8, 11, 12, 15, 17, 18, 19], in some cases with the restriction $\lambda, \mu \in \mathbb{R}$.

Solutions of (3.1) have the following properties [18]:

$$\lim_{t\to\infty} \mathsf{E}(|X(t)|^2) = 0 \Leftrightarrow \Re\{\lambda\} + \tfrac{1}{2}|\mu|^2 < 0, \tag{3.2}$$

$$\lim_{t\to\infty} |X(t)| = 0, \text{with probability } 1 \Leftrightarrow \Re\{\lambda - \tfrac{1}{2}\mu^2\} < 0, \tag{3.3}$$

where $\mathsf{E}(\cdot)$ denotes the expected value. The property on the left-hand side of (3.2) is known as *mean-square stability*, whereas the left-hand side of (3.3) defines *asymptotic stochastic stability (in the large)*. See, for example, [5, 13] for further details of stability concepts for SDEs. Note that for $\mu = 0$ these stability conditions collapse to the deterministic one, $\Re\{\lambda\} < 0$.

Applying the STM (1.2) to the problem (3.1) produces

$$X_{n+1} = \left(\frac{1 + (1-\theta)\Delta t\lambda + \Delta t^{\frac{1}{2}}\mu V_n}{1 - \theta\Delta t\lambda} \right) X_n, \tag{3.4}$$

where we recall that each V_n is an independent Normal$(0, 1)$ random variable. The recurrence (3.4) may be regarded as a *stochastic difference equation* or a *Markov chain with uncountable state space*. For a particular θ and Δt, the general form of the recurrence is

$$X_{n+1} = (a + bV_n)X_n, \tag{3.5}$$

where

$$a := \frac{1 + (1-\theta)\Delta t\lambda}{1 - \theta\Delta t\lambda} \quad \text{and} \quad b := \frac{\Delta t^{\frac{1}{2}}\mu}{1 - \theta\Delta t\lambda} \tag{3.6}$$

are independent of n.

In order to study the stability properties of the STM, we must therefore study the long term behavior of random variables of the form (3.5). In section 4 we consider mean-square stability, and in section 5 we consider asymptotic stability.

4. Mean square stability. By analogy with the definition for the SDE (3.1), we will say that the sequence (3.5) is *mean-square stable* if $\lim_{n\to\infty} \mathsf{E}(|X_n|^2) = 0$. Note that the STM depends upon the problem parameters λ and μ, and the method parameters θ and Δt. For a particular choice of parameters, we will say that the STM is mean-square stable if it produces a mean-square stable sequence. Our interest lies in finding the parameter values for which the STM is stable, and comparing results with the region $\Re\{\lambda\} + \frac{1}{2}|\mu|^2 < 0$ in (3.2) for the underlying SDE.

To analyze mean-square stability of the STM, we see from (3.5) that

$$|X_{n+1}|^2 = \left(|a|^2 + (a\bar{b} + \bar{a}b)V_n + |b|^2 V_n^2\right) |X_n|^2.$$

Taking expected values gives

$$\mathsf{E}(|X_{n+1}|^2) = \left(|a|^2 + |b|^2\right) \mathsf{E}(|X_n|^2),$$

and hence

$$\lim_{n\to\infty} \mathsf{E}(|X_n|^2) = 0 \Leftrightarrow |a|^2 + |b|^2 < 1. \tag{4.1}$$

It follows from (4.1) that the STM is mean-square stable if and only if

$$\frac{|1 + (1-\theta)\Delta t\lambda|^2 + \Delta t|\mu|^2}{|1-\theta\Delta t\lambda|^2} < 1. \tag{4.2}$$

We see from (4.2) that mean-square stability is dependent upon the quantities θ, $\Delta t\lambda$ and $\Delta t|\mu|^2$. Saito and Mitsui [18] derived the condition (4.2). They plotted domains of stability for various slices through the parameter values. However, in the spirit of (1.3), it is natural to *compare* the mean-square stability of the SDE (3.1) and the method (3.4). We now proceed in this manner.

For convenience, we let S_{SDE} and $S_{\mathrm{STM}}(\theta, \Delta t)$ denote the sets of pairs of complex problem parameters for which the problem and method are stable, respectively; that is,

$$\begin{aligned} S_{\mathrm{SDE}} &:= \{\lambda, \mu \in \mathbb{C} : \Re\{\lambda\} + \tfrac{1}{2}|\mu|^2 < 0\}, \\ S_{\mathrm{STM}}(\theta, \Delta t) &:= \{\lambda, \mu \in \mathbb{C} : (4.2) \text{ holds}\}. \end{aligned}$$

The following result is immediate from (4.2).

THEOREM 4.1. *For all $\Delta t > 0$ we have*

$$\begin{aligned} &S_{\mathrm{STM}}(\theta, \Delta t) \subset S_{\mathrm{SDE}} \quad \text{for } 0 \le \theta < \tfrac{1}{2}, \\ &S_{\mathrm{STM}}(\tfrac{1}{2}, \Delta t) \equiv S_{\mathrm{SDE}}, \\ &S_{\mathrm{STM}}(\theta, \Delta t) \supset S_{\mathrm{SDE}} \quad \text{for } \tfrac{1}{2} < \theta. \end{aligned}$$

For $0 \le \theta < \frac{1}{2}$, given $(\lambda, \mu) \in S_{\mathrm{SDE}}$, the STM is mean-square stable if and only if

$$\Delta t < \frac{-2(\Re\{\lambda\} + \frac{1}{2}|\mu|^2)}{|\lambda|^2(1-2\theta)}. \tag{4.3}$$

In the case $\theta \ge \frac{1}{2}$, Theorem 4.1 shows that whenever the SDE is stable then so is the STM for any $\Delta t > 0$, an observation that is also made in [19]. This is a direct generalization of the deterministic A-stability property. For $0 \le \theta < \frac{1}{2}$, Theorem 4.1 shows that (a) if the SDE is unstable, then so is the STM for all Δt, and (b) if the SDE is stable, then so is the STM for sufficiently small Δt. However, in the latter case, as we have already seen from Lemma 2.1, the resulting stepsize restriction can be arbitrarily severe, even when λ and μ are bounded.

Since mean-square stability of the STM depends upon $\lambda\Delta t$ and $|\mu|^2\Delta t$, in the general case $\lambda, \mu \in \mathbb{C}$ it is not easy to visualize regions of stability. However, if we restrict attention to $\lambda, \mu \in \mathbb{R}$ then we may draw pictures in an appropriate real plane. By analogy with the standard practice for deterministic stability regions [7], we will draw these regions in the x-y plane, where $x = \Delta t\lambda$ and $y = \Delta t\mu^2$. In this way, given problem parameters λ and μ, varying Δt corresponds to moving along a ray that passes through the origin and (λ, μ^2). The next result is immediate from (4.2).

LEMMA 4.2. *Suppose $\lambda, \mu \in \mathbb{R}$ and let $x = \Delta t\lambda$, $y = \Delta t\mu^2$. The STM is mean-square stable if and only if $y < (2\theta - 1)x^2 - 2x$.*

The upper left-hand and right-hand pictures in Figure 4.1 illustrate the cases $\theta = 0$ and $\theta = 0.25$, respectively. The mean-square stability region is shown with vertical hashing. Superimposed on the pictures with horizontal hashing is the region

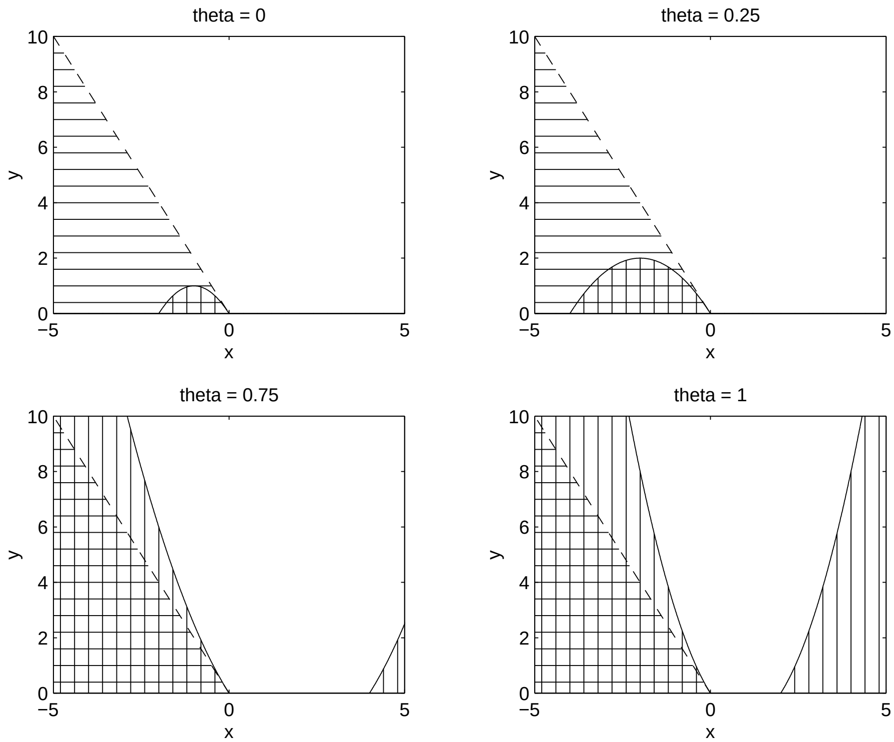

FIG. 4.1. *Real mean-square stability regions for the* STM *(vertical hashing) and the underlying* SDE *(horizontal hashing). Top left, $\theta = 0$; top right, $\theta = 0.25$; bottom left, $\theta = 0.75$; bottom right, $\theta = 1$.*

$y < -2x$ where the underlying SDE (3.1) is stable. The lower left-hand and right-hand pictures in Figure 4.1 illustrate the cases $\theta = 0.75$ and $\theta = 1$, respectively. We remark that general set relations in Theorem 4.1 are seen to be true in these special cases.

The following straightforward consequence of Lemma 4.2 shows that the type of stability region shrinkage for $0 \le \theta < \frac{1}{2}$ outlined in Lemma 2.1 for the deterministic case with $\lambda \in \mathbb{C}$ arises in the mean-square stability case even when only *real* parameters are allowed.

THEOREM 4.3. *Given $\epsilon > 0$ and $0 \le \theta < \frac{1}{2}$, there exist $\lambda = \lambda(\epsilon, \theta) \in \mathbb{R}$ and $\mu = \mu(\epsilon, \theta) \in \mathbb{R}$ with $\lambda + \frac{1}{2}\mu^2 < 0$, normalized to $\lambda^2 + \mu^2 = 1$, such that the* STM *with $\Delta t = \epsilon$ is not mean-square stable.*

5. Asymptotic stability. By analogy with the definition for SDEs, we will say that the sequence (3.5) is *asymptotically stable* if $\lim_{n\to\infty} |X_n| = 0$, with probability 1. For a particular choice of λ, μ, θ and Δt, we will say that the STM is asymptotically stable if it produces an asymptotically stable sequence. As in the previous section, we seek to characterize those $\lambda, \mu, \theta, \Delta t$ for which the STM is stable, and then to compare the results with the corresponding constraint for the underlying test problem.

The following lemma is useful for our purposes.

LEMMA 5.1. *Given a sequence of real-valued, nonnegative, independent, and identically distributed random variables $\{Z_n\}$, consider the sequence of random variables*

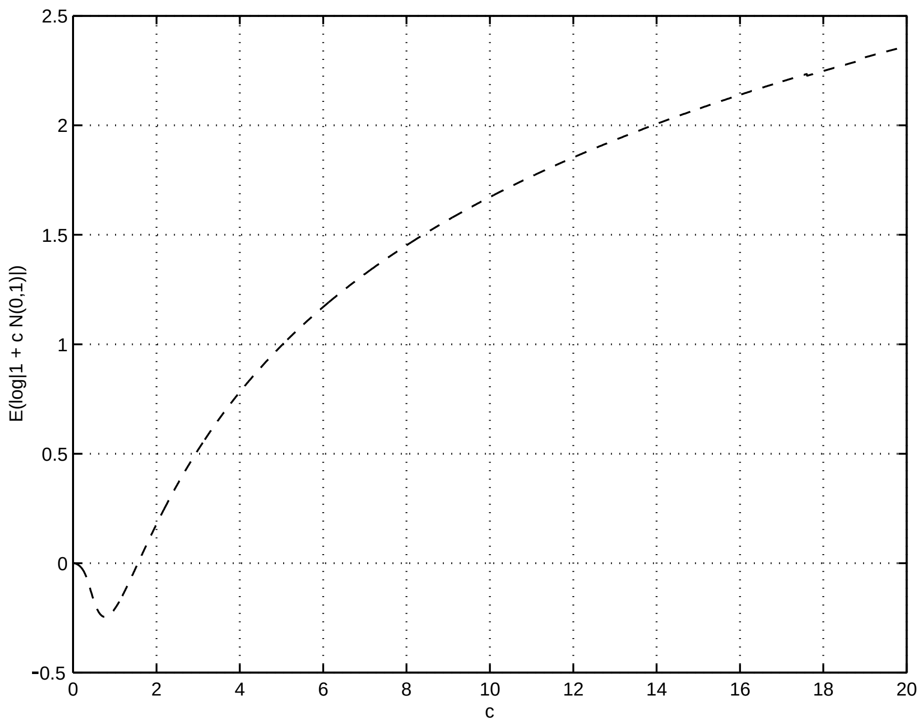

FIG. 5.1. *Plot of* $\gamma(c) := \mathsf{E}(\log|1 + cV_i|)$ *against* c.

$\{Y_n\}_{n\geq 1}$ *defined by*

$$Y_n = \left(\prod_{i=0}^{n-1} Z_i\right) Y_0, \tag{5.1}$$

where $Y_0 \geq 0$ *and where* $Y_0 \neq 0$ *with probability* 1*. Suppose that the random variables* $\log(Z_i)$ *are square-integrable. Then*

$$\lim_{n\to\infty} Y_n = 0, \text{ with probability } 1 \Leftrightarrow \mathsf{E}(\log(Z_i)) < 0.$$

Proof. Taking logs in (5.1) gives

$$\log(Y_n) = \sum_{i=0}^{n-1} \log(Z_i) + \log(Y_0). \tag{5.2}$$

Now let $\mu := \mathsf{E}(\log(Z_i))$ and $S_n := \sum_{i=0}^{n-1} \log(Z_i)$. Since $\log(Z_i)$ is integrable, the strong law of large numbers [1] shows that

$$\lim_{n\to\infty} \left(\frac{S_n}{n} - \mu\right) = 0, \quad \text{with probability 1.} \tag{5.3}$$

If $\mu < 0$, then it follows from (5.3) that $\lim_{n\to\infty} S_n = -\infty$ with probability 1. Hence, in (5.2), $\lim_{n\to\infty} Y_n = 0$ with probability 1.

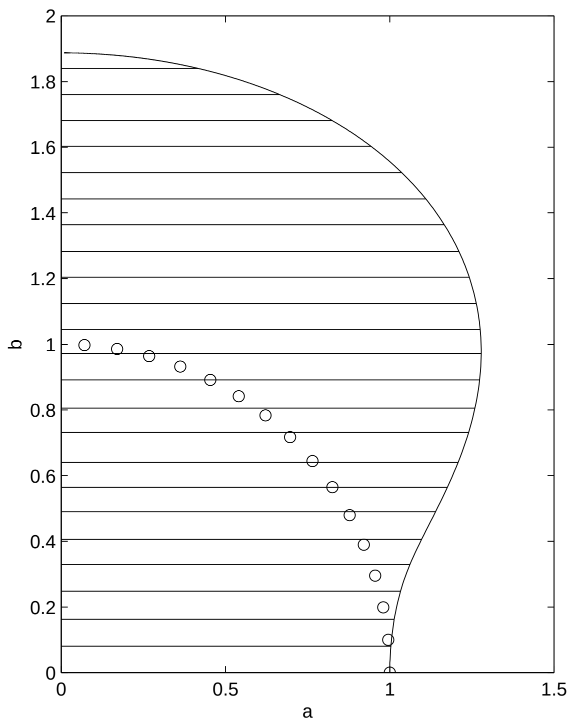

FIG. 5.2. *Real asymptotic stability region for* (3.5).

Similarly, if $\mu > 0$ then it follows from (5.3) that $\lim_{n\to\infty} S_n = \infty$ and $\lim_{n\to\infty} Y_n = \infty$ with probability 1.

If $\mu = 0$ we may appeal to the law of the iterated logarithm [1], which shows that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = \sigma, \quad \text{with probability 1},$$

where σ is the variance of $\log(Z_i)$. Hence, in this case it is not true that $\lim_{n\to\infty} Y_n = 0$ with probability 1. □

We remark that in the proof of Lemma 5.1 the strong law of large numbers was used to show that $\mathsf{E}(\log(Z_i)) \le 0$ and $\mathsf{E}(\log(Z_i)) < 0$ are necessary and sufficient, respectively, for asymptotic stability. This result requires only the assumption that $\mathsf{E}(|\log(Z_i)|)$ is finite. Applying the law of the iterated logarithm to get the full result required the stronger assumption that $\mathsf{E}((\log(Z_i))^2)$ is finite.

In order to apply Lemma 5.1 to (3.5) we take $Y_n := |X_n|$ and $Z_i := |a + bV_i|$, where we recall that $a, b \in \mathbb{C}$ are constants and V_i is Normal$(0,1)$. The condition determining asymptotic stability is then $\mathsf{E}(\log|a + bV_i|) < 0$. Note that, for $a \neq 0$,

$$\mathsf{E}(\log|a + bV_i|) = \log|a| + \mathsf{E}(\log|1 + cV_i|), \quad \text{where } c := b/a. \tag{5.4}$$

Hence, the stability issue reduces to the study of the expected value of a random variable with one complex parameter. In the case where V_i is Normal$(0,1)$ it appears not to be possible to find a simple analytical formula for $\mathsf{E}(\log|1 + cV_i|)$ in terms of c. The symbolic algebra facility in Mathematica [21] shows that the required integral can be expressed in terms of Meijer's G-function [6]. However, this does not provide

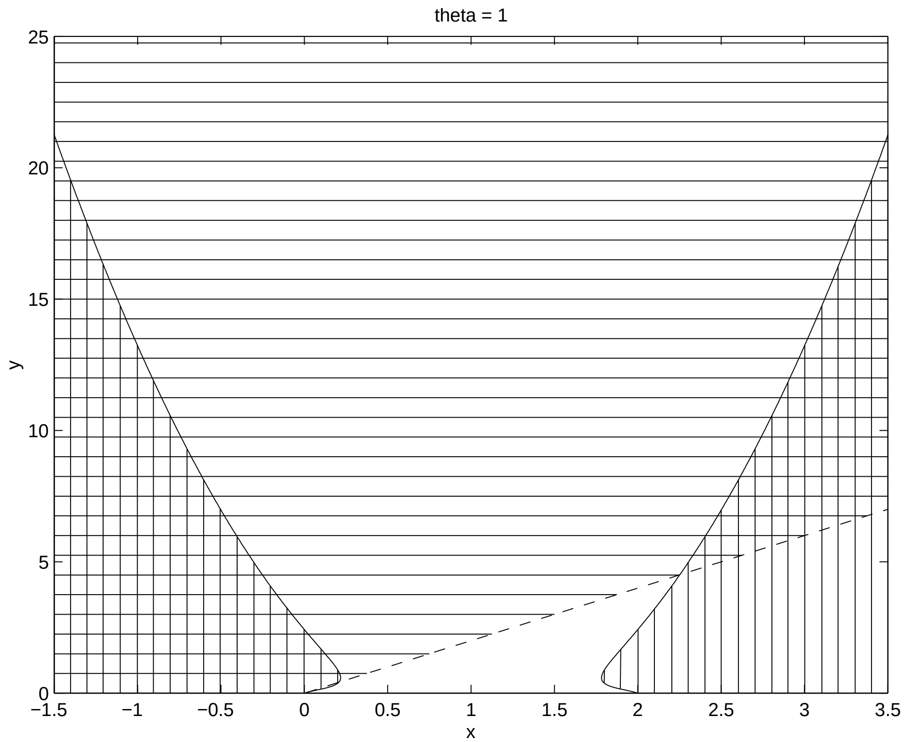

FIG. 5.3. *Real asymptotic stability region for* STM *with* $\theta = 1$ *(vertical hashing) and the underlying* SDE *(horizontal hashing).*

a straightforward means for analytically determining stability regions of the STM. In section 6 we show that further analysis is possible for as variant of the STM where the Normal(0, 1) distribution is replaced by a two point distribution. In the remainder of this section, we consider the case where $\lambda, \mu \in \mathbb{R}$, and hence $a, b, c \in \mathbb{R}$, and focus on numerical computations.

In Figure 5.1 we plot $\gamma(c) := \mathsf{E}(\log |1 + cV_i|)$ for real $c > 0$. Note that a symmetry argument shows that the stability is unchanged if $a \mapsto -a$ or $b \mapsto -b$, so the restriction to $c > 0$ is not significant. The plots were computed using numerical quadrature to approximate the relevant integrals.

Using Lemma 5.1 and the information in Figure 5.1, for any a and b we may now check whether the sequence X_n in (3.5) is asymptotically stable. We used this approach to produce Figure 5.2, which shows the positive a and b for which (3.5) is asymptotically stable. The boundary of this region has the parametric form $a = \exp(-\gamma(c))$, $b = c\exp(-\gamma(c))$, for $c > 0$. The boundary of the unit circle is superimposed using the symbol $\circ$. From (4.1), this is the region where the sequence is mean-square stable.

It is interesting to note the "bulge" in the asymptotic stability region in Figure 5.2 as b increases from zero. For example, fixing $a = 1.1$, we see that $b = 0.2$ gives instability, $b = 1.0$ gives stability, and $b = 1.8$ gives instability. In other words, a small amount of noise fails to stabilize the iteration, a larger amount of noise induces stability, and an even larger amount of noise destabilizes. This property arises as a consequence of the fact that $\gamma(c)$ dips below zero for small c in Figure 5.1.

In order to test whether the STM is asymptotically stable for a particular choice of $\Delta t\lambda$ and $\Delta t\mu^2$, we may check whether a and b in (3.6) lie in the stability region in Figure 5.2. In this manner, for each θ we could compute (a finite portion of) the stability region in the x-y plane, where $x := \Delta t\lambda$ and $y := \Delta t\mu^2$. In the case where $\theta = 1$ it is possible to visualize this region directly from Figure 5.1. Here, $a = 1/(1-x)$ and $b = y^{\frac{1}{2}}/(1-x)$, so that $c = b/a = y^{\frac{1}{2}}$. Hence, using Lemma 5.1 and (5.4) it follows that the stability region is defined by $x < 1 - \exp(\gamma(y^{\frac{1}{2}}))$ and $x > 1 + \exp(\gamma(y^{\frac{1}{2}}))$. The result is shown with vertical hashing in Figure 5.3. The region $y > 2x$ where the SDE is asymptotically stable is shown with horizontal hashing.

Although it does not seem possible to obtain simple analytical formulas describing stability regions of the STM, we can prove that an extension of A-stability does not hold. This is essentially a consequence of the fact that the recurrence (3.5) has a pole at $x = 1/\theta$, whereas the test equation (3.1) can be made stable at this value.

THEOREM 5.2. *For each* $\theta \geq \frac{1}{2}$*, there exist* $\lambda, \mu \in \mathbb{R}$ *and* $\Delta t > 0$ *such that* $\lambda - \frac{1}{2}\mu^2 < 0$ *but the resulting* STM *is not asymptotically stable.*

Proof. With the notation $x = \Delta t\lambda$ and $y = \Delta t\mu^2$ let $x = (1-\delta)/\theta$, where δ is a small parameter to be determined, and let $y = 4x = 4(1-\delta)/\theta$. Then, from Lemma 5.1, the condition for asymptotic stability of the STM becomes

$$\log\left|1 + \frac{(1-\theta)}{\theta}(1-\delta)\right| + \log\left|\frac{1}{\delta}\right| + \mathsf{E}\left(\log\left|1 + 2\sqrt{\frac{1-\delta}{\theta}}\frac{V_i}{1+(1-\theta)\delta/\theta}\right|\right) < 0.$$

Clearly, by choosing δ sufficiently small we can violate this condition. ∎

6. The weak stochastic theta method. Some authors have analyzsed numerical methods in which the samples from a Normal$(0, 1)$ distribution are replaced by samples from a simpler distribution [2, 10, 17]. This allows cheaper simulations when only weak convergence is required. From the point of view of our study, sampling from a simpler distribution gives the advantage of allowing analytical expressions to be derived for regions of asymptotic stability. The numerical method that we consider in this section has the form

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta\Delta t f(X_{n+1}) + \Delta t^{\frac{1}{2}} g(X_n)\widetilde{V}_n. \tag{6.1}$$

Here each $\widetilde{V}_n$ comes from a two point distribution: $P(\widetilde{V}_n = 1) = P(\widetilde{V}_n = -1) = \frac{1}{2}$. Note that (6.1) differs from (1.2) only in the choice of the "noise" factor. In the case $\theta = 0$, this method appears in [2, 17] and [10, p. 328], and is known to have weak order of convergence equal to one. We therefore refer to the method (6.1) as the weak stochastic theta method (WSTM).

Initially, it may seem strange to analyze asymptotic stability for a method that does not make use of the underlying Wiener path, and hence is only weakly convergent. However, we feel that there are good reasons for this pursuit. First, studying the numerical solution on a linear test equation may be equally well regarded as studying *error propagation* in the method and we may wish for the error to decay with probability one. Second, time averages equal ensemble averages for ergodic problems, and so paths of the WSTM might be used in this context. Third, analytical insight about the method (6.1) gives an indication of what to expect from the Normal$(0, 1)$ sampling method (1.2), for which analysis is harder.

Applying the WSTM to the test problem (3.1) produces the recurrence

$$X_{n+1} = (a + b\widetilde{V}_n)X_n, \tag{6.2}$$

where a and b are defined in (3.6). The relation (4.1) remains valid, and we see that the mean-square stability of the WSTM is identical to that of the corresponding STM. To study asymptotic stability, a straightforward application of Lemma 5.1 gives the following result.

LEMMA 6.1. *For $a, b \in \mathbb{C}$, the recurrence* (6.2) *satisfies*

$$\lim_{n\to\infty} |X_n| = 0, \text{ with probability } 1 \Leftrightarrow \left|a^2 - b^2\right| < 1. \tag{6.3}$$

Note that if we sample the recurrence (6.2), then the growth factor taking us from X_n to X_{n+1} is equally likely to be $a+b$ or $a-b$. Lemma 6.1 shows that asymptotic stability requires the "geometric mean growth factor," $\sqrt{(a+b)(a-b)}$, to have modulus less than unity. Saito and Mitsui [17] used this averaged growth factor approach to define T-stability for the Euler–Maruyama method (that is, the STM with $\theta = 0$) with a two point and three point noise sample. In this case, the definition coincides with our definition of asymptotic stability.

For the case where λ and μ are real, the following result is immediate from Lemma 6.1.

COROLLARY 6.2. *For $\lambda, \mu \in \mathbb{R}$ the* WSTM (6.1) *is asymptotically stable if and only if*

$$\left|(1+(1-\theta)x)^2 - y\right| < (1-\theta x)^2, \tag{6.4}$$

where $x = \Delta t\lambda$ and $y = \Delta t\mu^2$.

Using

$$RAS_{\text{SDE}} := \{(x,y) : x, y \in \mathbb{R}, y \geq 0 \text{ and } y > 2x\},$$
$$RAS_{\text{WSTM}}(\theta) := \{(x,y) : x, y \in \mathbb{R}, y \geq 0 \text{ and (6.4) holds}\}$$

to denote the real asymptotic stability regions of the SDE and the WSTM, respectively, the following further corollaries can be deduced directly from Corollary 6.2.

COROLLARY 6.3. *For $0 \leq \theta < \frac{1}{2}$,*

(a) *$RAS_{\text{WSTM}}(\theta) \subset RAS_{\text{SDE}}$, and*

(b) *given any $\epsilon > 0$, there exist $x = x(\epsilon, \theta)$ and $y = y(\epsilon, \theta)$ with $(x,y) \in RAS_{\text{SDE}}$ such that $x^2 + y^2 < \epsilon$ but $(x,y) \notin RAS_{\text{WSTM}}(\theta)$.*

Part (a) of Corollary 6.3 shows that, for $0 \leq \theta < \frac{1}{2}$, the WSTM always reflects instability in the test equation (3.1). Part (b) shows that the range of scaled stepsizes for which the WSTM reflects stability in the test equation may be made arbitrarily small by appropriate choice of λ and μ. This is analogous to the circumstance outlined in Lemma 2.1 for the deterministic problem with complex parameters.

COROLLARY 6.4. *For $\theta = \frac{1}{2}$,*

(a) *$RAS_{\text{WSTM}}(\frac{1}{2}) \subset RAS_{\text{SDE}}$, and*

(b) *$(x,y) \in RAS_{\text{SDE}}$ and $y < 2 \Rightarrow (x,y) \in RAS_{\text{WSTM}}(\frac{1}{2})$.*

Corollary 6.4 shows that for $\theta = \frac{1}{2}$, the method always reflects instability and will preserve stability if $\Delta t < 2/\mu^2$.

COROLLARY 6.5. *For $\frac{1}{2} < \theta \leq 1$,*

(a) *$RAS_{\text{WSTM}}(\theta) \not\subseteq RAS_{\text{SDE}}$ and $RAS_{\text{SDE}} \not\subseteq RAS_{\text{WSTM}}(\theta)$, and*

(b) *$(x,y) \in RAS_{\text{SDE}}$ and $y < 1/(\theta^2 + (1-\theta)^2) \Rightarrow (x,y) \in RAS_{\text{WSTM}}(\theta)$.*

Part (a) of Corollary 6.5 shows that, for $\frac{1}{2} < \theta \leq 1$, the WSTM does not always reflect stability or instability of the test problem. However, part (b) shows that stability is preserved if $\Delta t < 1/(\mu^2(\theta^2 + (1-\theta)^2))$, and hence if $\Delta t < 1/\mu^2$.

Our analysis of the case $\lambda, \mu \in \mathbb{R}$ shows that the WSTM does not possess the asymptotic stability extension of A-stability for $\frac{1}{2} \leq \theta \leq 1$. However, Corollary 6.4 part (b) and Corollary 6.5 part (b) show that the range of stepsizes for which the method is stable can be bounded away from zero over all $\lambda \in \mathbb{R}$ and all normalized $\mu \in \mathbb{R}$ for which the SDE is stable. Hence, the stability properties do not degenerate to the extent illustrated by Lemma 2.1 for the deterministic TM with $0 \leq \theta < \frac{1}{2}$.

We now consider $\lambda, \mu \in \mathbb{C}$. Part (b) of Theorem 6.6 below shows that in this case stability region shrinkage of the type indicated in Lemma 2.1 does occur. Note that the result below is more negative than that in Lemma 2.1 in the sense that the parameters λ and μ may be chosen independently of θ. On the other hand, part (a) of Theorem 6.6 shows that given fixed λ and μ for which the SDE is stable, a nonempty range of stable stepsizes exists.

THEOREM 6.6. *Consider the* WSTM *applied to* (3.1).

(a) *Given* $\lambda, \mu \in \mathbb{C}$ *such that* $\Re\{\lambda - \frac{1}{2}\mu^2\} < 0$, *so that the* SDE (3.1) *is asymptotically stable, there exists* $\Delta t^\star = \Delta t^\star(\lambda, \mu) > 0$ *such that the* WSTM *is asymptotically stable for any* $0 \leq \theta \leq 1$ *and* $0 < \Delta t < \Delta t^\star$.

(b) *Given any* $\epsilon > 0$, *there exist* $\lambda, \mu \in \mathbb{C}$ *with* $\Re\{\lambda - \frac{1}{2}\mu^2\} < 0$, *normalized to* $|\lambda|^2 + |\mu|^2 = 1$, *such that the* WSTM *is not asymptotically stable when* $\Delta t = \epsilon$ *for any* $0 \leq \theta \leq 1$.

Proof. For a and b defined in (3.6), the condition $|a^2 - b^2| < 1$ in Lemma 6.1, which determines asymptotic stability for the WSTM, may be written

$$\begin{aligned} 4\Re\{\lambda - \tfrac{1}{2}\mu^2\} + \Delta t[2(1-\theta)^2\Re\{\lambda^2\} + \big|2(1-\theta)\lambda - \mu^2 \\ + (1-\theta)^2\lambda^2\Delta t\big|^2 - 2\theta^2(|\lambda|^2 + 2\Re\{\lambda\}^2)] \\ + 4\Delta t^2\theta^3\Re\{\lambda\}|\lambda| - \theta^4\Delta t^3|\lambda|^4 < 0. \end{aligned} \tag{6.5}$$

If $\Re\{\lambda - \frac{1}{2}\mu^2\} < 0$, this has the form

$$|\Re\{\lambda - \tfrac{1}{2}\mu^2\}| > \Delta t g_1(\theta, \lambda, \mu^2) + \Delta t^2 g_2(\theta, \lambda, \mu^2) + \Delta t^3 g_3(\theta, \lambda, \mu^2), \tag{6.6}$$

where g_1, g_2, and g_3 are smooth, real-valued functions of θ, λ and μ^2. Since the left-hand side of (6.6) is positive, for each θ we may find $\widehat{\Delta t}$ such that (6.6) holds for $0 < \Delta t < \widehat{\Delta t}$, and we may obtain $\Delta t^\star$ by minimizing over $0 \leq \theta \leq 1$. This proves part (a).

To prove part (b) we consider the case $\lambda = -\delta$ and $\mu^2 = (1 - \delta^2)i$, where $\delta > 0$ is a small parameter to be determined. When $\Delta t = \epsilon$, the stability criterion (6.5) for this λ and μ becomes

$$\begin{aligned} -4\delta + \epsilon[2(1-\theta)^2\delta^2 + \big|-2(1-\theta)\delta - (1-\delta^2)i + (1-\theta)^2\delta^2\epsilon\big|^2 \\ -6\theta^2\delta^2] + 4\epsilon^2\theta^3\delta^3 - \theta^4\epsilon^3\delta^4 < 0. \end{aligned} \tag{6.7}$$

For fixed ϵ, the left-hand side of (6.7) tends to ϵ as $\delta \to 0$. Hence, by choosing δ sufficiently small, we can violate the stability criterion. ∎

7. Numerical tests. For deterministic ODEs, linear stability theory for numerical methods is known to be relevant to behavior around fixed points for systems [9], as well as forming a starting point for more general theories on nonlinear problem classes and partial differential equations. How much of this theory carries through to the SDE setting? We do not attempt to answer that question in any generality. Instead we focus on behavior around a scalar fixed point and perform numerical tests.

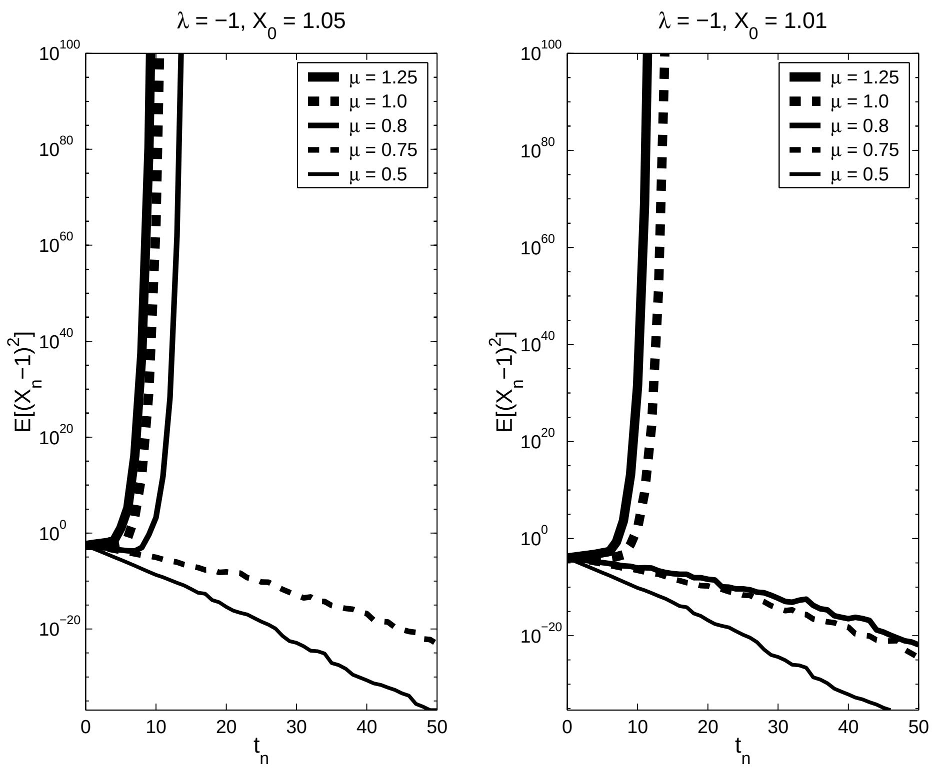

FIG. 7.1. $\theta = 0$ *method: sample approximation to* $\mathsf{E}\left((X_n - 1)^2\right)$ *against* t_n.

Our approach is to solve a nonlinear problem and compare the behavior to that predicted for the corresponding linearized version. We wish to check whether the linear theory gives a reasonable guide to the general behavior as problem parameters are varied. Note that for numerical simulations in a stochastic setting it is not realistic to attempt to determine parameter values that give a precise cut-off between stability and instability, especially in the asymptotic stability case. Hence, we are looking for a reasonable match between the linear theory and the numerical results, rather than a precise connection.

For the tests, we take the SDE

$$\mathrm{d}X(t) = -\lambda X(t)\,(1 - X(t))\,\mathrm{d}t - \mu X(t)\,(1 - X(t))\,\mathrm{d}W(t), \tag{7.1}$$

which is a normalized version of a population model in [5, equation (2.7)]. Note that linearizing about the fixed point $X(t) \equiv 1$ leads to the linear test equation (3.1). We use constant positive initial conditions throughout the tests.

We first test the mean-square stability behavior of the $\theta = 0$ (Euler–Maruyama) method. We take $\Delta t = 1$ and solve over $0 \leq t \leq 50$. To estimate $\mathsf{E}\left((X_n - 1)^2\right)$, we average over 10^5 numerically generated paths. We fix $\lambda = -1$ and use $\mu = 0.5, 0.75, 0.8, 1.0, 1.25$. From Lemma 4.2 (or Figure 4.1), with $\lambda = -1$ and $\Delta t = 1$, the linear problem is mean-square stable for $\mu^2 < 1$. The left-hand picture in Figure 7.1 plots the sampled approximation to $\mathsf{E}\left((X_n - 1)^2\right)$ against t_n for $X_0 = 1.05$. We see that the $\mu = 0.5, 0.75$ computations appear to be mean-square stable with the remainder unstable. For the right-hand picture in Figure 7.1 we use $X_0 = 1.01$. We

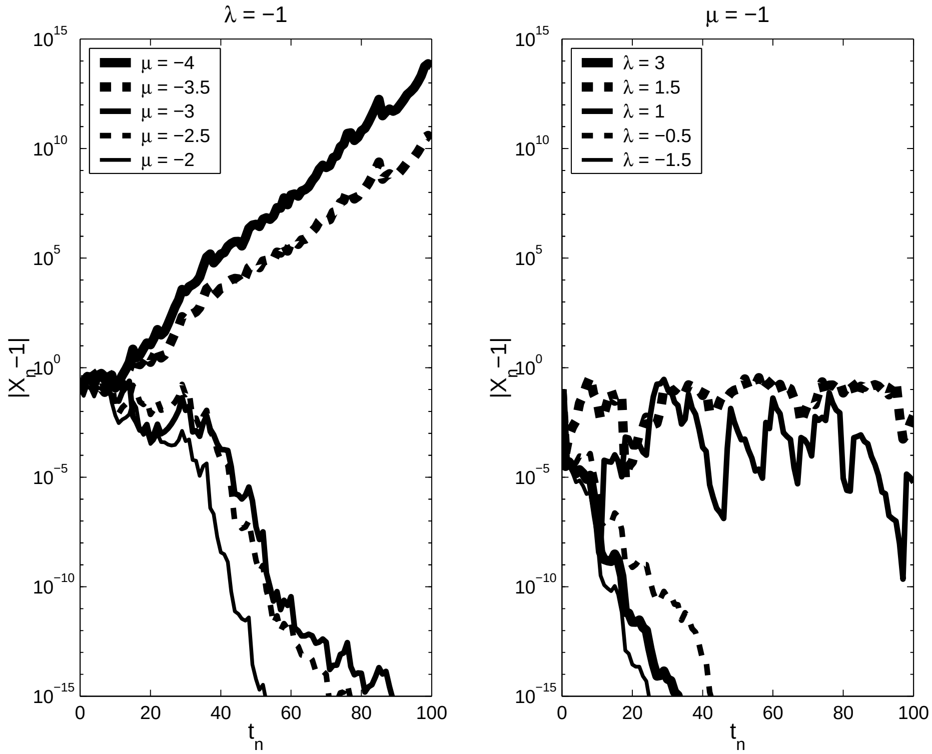

FIG. 7.2. $\theta = 1$ *method:* $|X_n - 1|$ *against* t_n *for one path.*

see that taking the initial condition closer to the fixed point has made the $\mu = 0.8$ computation stable.

Next we test asymptotic stability by applying the $\theta = 1$ method with $\Delta t = 1$, $X_0 = 1.1$ and $0 \le t \le 100$. In order to apply the method (1.2) to (7.1) we must solve a quadratic equation on each step. For these tests, we compute the absolute values of the roots of the quadratic and use the one closest to X_n as X_{n+1}. All tests use the same path (generated by `randn('state',100); dW = randn(100,1);` in Matlab version 5.3 [20]). First, we set $\lambda = -1$. Examining the stability region boundary in Figure 5.3 we find that with $\Delta t = 1$ and $\lambda = -1$ the $\theta = 1$ method is asymptotically linearly stable for μ^2 values between 0 and (to two significant digits) 13. For the test, we take $\mu = -2, -2.5, -3, -3.5, -4$. The left-hand plot in Figure 7.2 shows $|X_n - 1|$ against t_n. We see that the $\mu = -2, -2.5, -3$ paths appear to give asymptotic stability. For a second test, we set $\mu = -1$ and vary λ. For $\Delta t = 1$ and $\mu = -1$ examining the data in Figure 5.3 shows that the method is asymptotically linearly stable for $\lambda < 0.2$ and $\lambda > 1.8$. The right-hand plot in Figure 7.2 shows $|X_n - 1|$ against t_n for $\lambda = -1.5, -0.5, 1, 1.5, 3$. We see that the $\lambda = -1.5, -0.5, 3$ paths appear to be stable.

Overall, we conclude that in these tests the appropriate linear stability theory gives a good guide to the behavior of the STM when the solution is close to equilibrium. Investigating whether rigorous results of this nature can be proved is clearly of fundamental importance in this area.

8. Further remarks.

8.1. Other numerical methods. The techniques for analyzing stability developed in this work can be applied to a wide range of numerical methods for SDEs. We illustrate this point by studying two methods from the literature that are fully implicit; that is, implicit in both the deterministic and the stochastic terms. These methods were intended to have good stability properties, and we show here that this is indeed the case.

First, we consider the method from [10, p. 337]. When applied to the test equation (3.1) this method takes the form

$$X_{n+1} = X_n + \Delta t(\lambda - \mu^2)X_{n+1} + \mu X_{n+1}\Delta t^{\frac{1}{2}}\widetilde{V}_n,$$

where $\widetilde{V}_n$ has the two point distribution discussed in section 6. Note that for $\mu = 0$ this coincides with the deterministic implicit Euler method ($\theta = 1$ in section 2), which is A-stable. If we consider the case where λ and μ are real, and let $x = \Delta t\lambda$ and $y = \Delta t\mu^2$, then the method may be written in the form

$$X_{n+1} = \left(\frac{1}{1 - x + y - y^{\frac{1}{2}}\widetilde{V}_n}\right) X_n.$$

From Lemma 5.1, this sequence is asymptotically stable if and only if

$$\mathsf{E}\left(\log\left|\frac{1}{1 - x + y - y^{\frac{1}{2}}\widetilde{V}_n}\right|\right) < 0.$$

This condition may be rearranged to

$$|1 + x^2 + (y - 2x)(1 + y)| > 1. \tag{8.1}$$

Now we recall that the condition for asymptotic stability of the underlying test equation is $y > 2x$. Hence we see that (8.1) holds whenever the SDE is stable. This shows that, for real parameters in the test equation, the method has the extension of deterministic A-stability to the case of stochastic asymptotic stability. To our knowledge, this is the first time that such a property has been identified. (We emphasize, however, that the method is not designed to give strong convergence.)

Milstein, Platen, and Schurz [14] looked at the construction of implicit methods for SDEs with good linear stability properties and proposed a class of methods that are implicit in the stochastic term. They motivated their work by considering a special case of the linear test equation (3.1) where $\lambda = 0$ and $\mu \in \mathbb{R}$. Note from (3.2) and (3.3) that this equation is not mean-square stable but is asymptotically stable for any $\mu \neq 0$. As a prototype of the class of balanced methods, the method

$$X_{n+1} = X_n + \mu X_n \Delta t^{\frac{1}{2}} V_n + \mu(X_n - X_{n+1})\Delta t^{\frac{1}{2}}|V_n|$$

was proposed in [14] for this test equation. Numerical evidence in [14] suggested that this method has good asymptotic stability properties. Writing the recurrence as

$$X_{n+1} = \left(1 + \frac{\Delta t^{\frac{1}{2}}\mu V_n}{1 + \Delta t^{\frac{1}{2}}\mu|V_n|}\right) X_n, \tag{8.2}$$

we see from Lemma 5.1 that the condition for asymptotic stability is

$$\mathsf{E}\left(\log\left|1 + \frac{\Delta t^{\frac{1}{2}}\mu V_n}{1 + \Delta t^{\frac{1}{2}}\mu|V_n|}\right|\right) < 0. \tag{8.3}$$

Numerical tests indicate that (8.3) is true for any $\mu \neq 0$ and $\Delta t > 0$, and hence the method reproduces the asymptotic stability of the test problem.

In the case where the Normal(0, 1) random variable V_n in (8.2) is replaced by the two point random variable $\widetilde{V}_n$ discussed in section 6, it follows from Lemma 5.1 that the method is asymptotically stable for all $\mu > 0$.

We also note that the Euler–Maruyama method, that is, the STM with $\theta = 0$, is applied to this test equation in [14]. On the basis of numerical simulations, the authors suggest that given μ, there is a critical Δt beyond which "the global error explodes in practice and the scheme becomes useless." Using Lemma 5.1 we may analyze the recurrence from the point of view of asymptotic stability. We see that there is indeed a critical stepsize limit; namely, $\Delta t < (\widehat{c}/\mu)^2$, where $\widehat{c} \approx 1.6$ is the nonzero root of $\gamma(c)$ in Figure 5.1.

8.2. Related work and extensions. This work has focused on the linear test equation (3.1) with *multiplicative* noise, which has also been discussed in [4, 8, 11, 12, 15, 17, 18, 19]. Other authors have studied the behavior of numerical methods for the *additive* noise case

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu \mathrm{d}W(t). \tag{8.4}$$

In this case the stability properties are strongly tied to those of the underlying deterministic method—that is, the method remaining when $\mu = 0$ in (8.4). See [16] for a review.

Hernandez and Spigler [8, section 4] introduced a different concept of stability for numerical methods applied to the test equation (3.1) and gave a technique for computing the resulting stability regions for a general class of numerical methods. This approach is similar in spirit to the T-stability idea of Saito and Mitsui [17], which is mentioned in section 6. The idea in [8] is to ask for the growth factor over a single step to be bounded above by a threshold less than 1 with high probability. Plotting stability regions in this manner is highly computer-intensive and the approach has the drawbacks of (a) requiring arbitrary tolerance parameters to be specified and (b) having no direct connection to a stability property of the underlying SDE. A more systematic approach to investigating asymptotic stability that is also based on computer simulation can be found in [3, section 5.6].

As a final remark, we mention that the extent to which this scalar linear stability theory is relevant to systems, linearized problems, fully nonlinear problems and partial differential equations is an open question of great importance.

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