A Sketch of Menshikov's Theorem

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Abstract

Let $\overrightarrow{\Lambda}$ be an infinite, locally finite oriented multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected, and let $p < p_H^s(\overrightarrow{\Lambda})$. Menshikov's theorem states that there exists an $\alpha > 0$ s.t.

$$\mathbb{P}_p^s(x \overset{n}{\to}) \leq exp(-\alpha n/(\log n)^2)$$

for all sites x and $n \geq 2$.

In this paper, we begin backwards, explaining the importance of this theorem and then giving a sketch of the proof, emphasizing the intuition and motivations behind the methods used. Unfortunately, due to constraints on space, we'll sometimes simply summarize a statement and defer its rigorous proof to the text, when the proof is relatively short or simple or when the proof is too technical for an article.

1 Introduction

Physicists have long believed that $P_T = P_H$. However, only until the 1986 paper by Menshikov has this been proven. In fact Menshikov's theorem is proven only for site percolation on a locally finite oriented graph.

However, as **Theorem 1** of section 4.2 in [1] reveals, site percolation on finite oriented graphs is nearly equivalent to that of bond percolation. Similarly, we can take an unoriented graph and create an equivalent oriented graph by replacing any bond undirected bond xy, with two directed ones $\overrightarrow{xy}$ and $\overleftarrow{xy}$, so that the probability measures are the same.

Nonetheless, there is still one more snag to get around. The way Menshikov's theorem implies $P_T = P_H$ requires a very mild additional assumption on the number of elements exactly n away from the center of the cluster (see **Theorem 7** on page 100 of [1]).

But, as before, further work extends Menshikov's work on the almost exponential decay of the radius to purely exponential decay (see section 4.4 of [1]). Finally, this can be generalized even further to exponential decay of the volume (see section 4.5 of [1]).

2 Preliminaries

We will try to follow the book's notation as much as possible, so that one can flip as easily as possible between the book and this paper. Also, in order to save space, rather than writing out all of the notation, simply consult the textbook's list of notation on page 323.

That being said, anything not explained in the textbook's list of notation, I will explain as they come up. In the event that I accidentally ommitted a definition, it can be found in section 4.3.

Finally, I will be skipping the first part of section 4.3, which rigorously shows that the ordered set of pivot points $\{b_1, b_2, ..., b_r\}$ for a path P from x to $S_n^+(x)$ remains in the same order regardless of which path P we choose.

Now, before we begin, I highly suggest taking a quick look at **Theorem 7** of section 4.3, to see how exactly Menshikov's theorem implies $P_T = P_H$. The proof is quite simple and short. Anyways, let's begin.

3 Menshikov's theorem & some intuition behind it

Menshikov's Theorem: Let $\overrightarrow{\Lambda}$ be an infinite, locally finite oriented multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected, and let $p < p_H^s(\overrightarrow{\Lambda})$. Then there exists an $\alpha > 0$ s.t.

$$\mathbb{P}_p^s(x \xrightarrow{n}) \leq exp(-\alpha n/(\log n)^2)$$

for all sites x and $n \geq 2$.

Now before we dive right in let's think about how we might get to this theorem. Notice the theorem is a statement about rates, so it is natural to look towards the Margulis-Russo formula. This naturally leads us to think about the expected number of pivots.

Now, remember that Menshikov's theorem is ultimately used to prove that $P_H = P_T$. That is, when we take $p = P_H - \epsilon$, we want to show $\mathbb{E}_p(|C_x|) < \infty$ in contrast with $\mathbb{E}_{P_H}(|C_x|) = \infty$. So naturally, we should hope that the rate of change is large (otherwise $\mathbb{E}_p(|C_x|)$ remains infinite, rather than finite). So, we want the expected number of pivots to be large.

In addition, Menshikov is also a statement only for $p < p_H^s(\overrightarrow{\Lambda})$, that is when $\theta(p) = lim_{n\to\infty}\rho_n(x,p) = 0$, where $\rho_n(x,p) = \mathbb{P}_p^s(R_n(x))$. Thus, we should make use of our choice of p and that $\rho_n(x,p) \to 0$. So we seek an inequality which gives us a lower bound for the expected number of pivots (which by the Margulis Russo formula is the rate of change) in terms of $\rho_n(x,p)$.

Lemma 5: Let x be a site of an oriented graph $\overrightarrow{\Lambda}$, and let n and k be positive integers. Then

$$\mathbb{E}_p\left((N(R_n(x))|R_n(x)\right) \geq \lfloor n/k \rfloor \left(1 - sup_y\, \rho_k(y,p)\right)^{\lfloor n/k \rfloor}$$

where y is any site and $N(E)$ denotes the number of pivotal sites for an event E.

Lemma 5 will indeed turn out to be crucial, but it actually follows quite nicely from:

Lemma 4: Let x be a site of an oriented graph $\overrightarrow{\Lambda}$, and let n, r, and d_i, $1 \leq i < r$, be positive integers. For $1 \leq k \leq n - \sum_{i=1}^{r-1} d_i$, we have

$$\mathbb{P}_p^s(D_r > k|E) \leq sup\{\rho_k(y,p) : y \in \overrightarrow{\Lambda}\},$$

where E is the event

$$E = R_n(x) \cap \{D_1 = d_1, D_2 = d_2, ..., D_{r-1} = d_{r-1}\}$$

where D_i is the distance of the shortest open path from b_{i-1} to b_i, where b_i is the ith pivot point (as mentioned in the preliminaries).

Let's first partition our event E, so we have a little bit more information to start with. Since we primarily care about D_r, then let's partition it by the location of the $r-1th$ pivot, call it y, and I, an interior cluster consistent with $E \cap \{b_{r-1} = y\}$. In other words:

$$E_{y,I} = R_n(x) \cap \{y = b_{r-1}\} \cap \{\mathbf{I} = I\}.$$

Now let's say $E_{y,I}$ holds. Then so does $F_{y,I}$, the event that all sites in $I \cup \{y\}$ are open, and $\partial I \backslash \{y\}$, the set of outneighbors of I not including $\{y\}$, are closed. (Otherwise, if $p \in \partial I \backslash \{y\}$ was open, $p \in I$, since it's an open site that is adjacent to an element in I.) Further notice, that $E_{y,I}$ holds if and only if $F_{y,I}$ holds and there is an open path P from a neighbor of y to $S_n^+(x)$ disjoint from $I \cup \partial I$. (Note that $y \in \partial I$.) Let us write $G_{y,I}$ for the event that there is such a path P.

Let's $X = (I \cup \partial I)^c$, and let $\mathbb{P}_p^x$ denote the product probbility measure in which each site of X is open independently with probability p.

Now take a look at figure 10 page 94. Notice that we made sure that $\partial I \backslash \{y\}$ was closed. This allows us to "look" at $F_{y,I}$ without "peeking" at $G_{y,I}$. In fact, we can easily define such an "exploration algorithm," where we look at the open neighbors of x, then the new open neighbors of those neighbors, and so on, until we either have no new open neighbors or we hit y (we first chose $y = b_{r-1}$ when we created the partition $E_{y,I}$ so we know its location without looking at $G_{y,I}$). So $G_{y,I}$ and $F_{y,I}$ occur independently, and by our earlier if and only if statement

$$(i)\ \mathbb{P}_p(E_{y,I}) = \mathbb{P}_p(F_{y,I}) \mathbb{P}_p^X(G_{y,I}).$$

Now suppose again that $E_{y,I}$ holds and that $D_r > k$. Since $y = b_{r-1}$, the next pivot, b_r, will be greater than k away from y. That is, b_r lies outside of the sphere $S_k(y)$. Since b_r is a pivot, then there must be two open paths from y to b_r, completely disjoint from one another except at y and b_r. Since b_r lies outside the sphere $S_k(y)$, there must be a path from a neighbor of y to $S_k^+(y)$ that is completely disjoint from $G_{y,I}$. Thus, letting $H_{y,I}$ denote the event that X contains an open path from a neighbour of y to $S_k^+(y)$, $G_{y,I} \Box H_{y,I}$ holds.

Since $E_{y,I} \cap \{D_r > k\}$ is a subset of $F_{y,I}$ and we've just shown that when it does hold $G_{y,I} \Box H_{y,I}$ also holds,

$$(ii)\ \mathbb{P}_p(E_{y,I} \cap \{D_r > k\}) \leq \mathbb{P}_p(F_{y,I}) \mathbb{P}_p^X(G_{y,I} \Box H_{y,I}).$$

Now by the van den Berg-Kesten inequality

$$\mathbb{P}_p^X(G_{y,I} \Box H_{y,I}) \leq \mathbb{P}_p^X(G_{y,I}) \mathbb{P}_p^X(H_{y,I}).$$

Combining the three relations above

$$\mathbb{P}_p(E_{y,I} \cap \{D_r > k\}) \leq \mathbb{P}_p(E_{y,I}) \mathbb{P}_p^X(H_{y,I}).$$

Since $\mathbb{P}_p^X(H_{y,I}) \leq \mathbb{P}_p(R_k(y)) = \rho_k(y,p)$, then

$$\mathbb{P}_p(E_{y,I} \cap \{D_r > k\}) \leq \mathbb{P}_p(E_{y,I}) \rho_k(y,p) \Rightarrow \mathbb{P}_p(D_r > k) | E_{y,I}) \leq \rho_k(y,p).$$

Since the events $E_{y,I}$ partition E, Lemma 4 follows.

So to summarize, we first partitioned the event E by the location of $y = b_{r-1}$. Then we noted that $F_{y,I}$ and $G_{y,I}$ occur iff $E_{y,I}$ holds, and used the fact that we can "look at" $F_{y,I}$ without "peeking" at $G_{y,I}$ to get (i). Then we added the additional condition that $D_r > k$, which implied b_r lies outside of $S_k(y)$. Since there must be two disjointpaths from a neighbor of y to a neighbor of $b_r \notin S_k(y)$ (otherwise b_r wouldn't be the next pivot), $G_{y,I} \square H_{y,I}$ holds. Using this observation, we got (ii), and the rest followed from van den Berg-Kesten and stacking our inequalities.

As mentioned in the abstract, due to constraints on space please see page 95 for the proof of Lemma 5 from Lemma 4. (It's relatively short and follows almost immediately)

We are now ready for:

Menshikov's Theorem: Let $\overrightarrow{\Lambda}$ be an infinite, locally finite oriented multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected, and let $p < p^s_H(\overrightarrow{\Lambda})$. Then there is an $\alpha > 0$ s.t.

$$\mathbb{P}^s_p(x \xrightarrow{n}) \leq exp(-\alpha n/(log\, n)^2)$$

for all site x and integers $n \geq 2$.

Fix $p < p_H(\overrightarrow{\Lambda})$. For any up-event E, by Margulis Russo, we have:

$$\frac{d}{dp}\mathbb{P}_p(E) = \mathbb{E}_p(N(E)) \geq \mathbb{E}_p(N(E)\mathbf{1_E}) = \mathbb{E}_p(N(E)|E)\mathbb{P}_p(E),$$

where $\mathbf{1_E}$ is the characteristic function for the set E. Notice that E could not occur but still have 1 or more pivots, so that the inequality above is not just a stylistic choice. Thus,

$$\frac{d}{dp}log\mathbb{P}_p(E) = \frac{1}{\mathbb{P}_p(E)}\frac{d}{dp}\mathbb{P}_p(E) \geq \mathbb{E}_p(N(E)|E)$$

Now let $\rho_n(p) = sup_{x \in \overrightarrow{\Lambda}} \rho_n(x,p)$. Taking $E = R_n(x)$ and applying Lemma 5 to the inequality above

$$\frac{d}{dp}log\; \rho_n(x,p) \geq \lfloor n/k \rfloor (1 - \rho_k(p))^{\lfloor n/k \rfloor}$$

Fix $p_- < p_+$. Since $\rho_k(p)$ monotonically increases with p, for any $p \in [p_-, p_+]$,

$$\frac{d}{dp}log\; \rho_n(x,p) \geq \lfloor n/k \rfloor (1 - \rho_k(p))^{\lfloor n/k \rfloor} \geq \lfloor n/k \rfloor (1 - \rho_k(p_+))^{\lfloor n/k \rfloor}.$$

$$Thus: \; log\; \rho_n(x,p_+) = log\; \rho_n(x,p_-) + \int_{p_-}^{p_+} log\; \rho_n(x,p)dp \geq log\; \rho_n(x,p_-) + \int_{p_-}^{p_+} \lfloor n/k \rfloor (1 - \rho_k(p_+))^{\lfloor n/k \rfloor}dp$$

$$\Rightarrow log\frac{\rho_n(x,p_-)}{\rho_n(x,p_+)} \leq -(p_+ - p_-)\lfloor n/k \rfloor (1 - \rho_k(p_+))^{\lfloor n/k \rfloor} \Rightarrow$$

$$because,\; e^x \; is \; monotonic: \; \frac{\rho_n(x,p_-)}{\rho_n(x,p_+)} \leq exp(-(p_+ - p_-)\lfloor n/k \rfloor (1 - \rho_k(p_+))^{\lfloor n/k \rfloor}).$$

Finally, as the upper bound does not depend on x,

$$(\dagger)\; \rho_n(p_-) \leq \rho_n(p_+) exp(-(p_+ - p_-)\lfloor n/k \rfloor (1 - \rho_k(p_+))^{\lfloor n/k \rfloor}).$$

At this point, we reach the toughest part of the proof. How do we go from the above inequality to Menshikov's theorem, which depends soley on n and not the value of ρ_n for some p? You would think we require some additional work on how fast $\rho_n(p_+) \to 0$, but we don't. Instead, we will pass through to a sequence that allows us to create a constant polynomial bound and then refine this to get the desired result.

Fix p, p_0, s.t. $p < p_0 < p_H$. Since $\rho_n(p_0) \to 0$. Writing $\rho_0 = \rho_{n_0}(p_0)$ we may fix n_0, sufficiently large s.t. $\rho_0 \leq 1/100$ and $\rho_0 \log(1/\rho_0) \leq (p_0 - p)/6$ (Letting $y = 1/x$, we see $lim_{x \to 0}\, x \log(1/x) = lim_{y \to \infty} \frac{\log y}{y} = 0$ by l'hopital's).

Now, let us inductively define two sequences, $n_0 \leq n_1 \leq ...$, and $p_0 \geq p_1 \geq ....$:

$$(i)\ n_{i+1} = n_i \lceil 1/\rho_i \rceil \ \ and\ p_{i+1} = p_i - \rho_i log(1/\rho_i).$$

Amazingly this sequence has the following key properties:

$$(ii)\ \rho_{i+1} := \rho_{n_{i+1}}(p_{i+1}) \leq \rho_i^{(4/3)}\ \ and\ p_i > p' := (p_0 + p)/2\ \forall i.$$

The proof of the above properties relies on two facts, (†) and $\rho_n(p_0) \to 0$, as $n \to \infty$. Because the proof is fairly technical though short, we will skip ahead. (The proof is on page 98.)

Now let $s_i = \lceil 1/\rho_i \rceil$, so $n_i = n_0 \prod_{j<i} s_j$, and $s_0 = \lceil 1/\rho_0 \rceil \geq 100$. Since $s_i \geq s_0 \geq 100$, we will simply ignore the rounding. By (ii), $s_{i+1} = \left\lceil 1/\rho_i^{4/3} \right\rceil \approx s_i^{4/3}$. Thus, for $1 \leq j \leq i$, $s_{i-j} \approx s_i^{(3/4)^j}$, giving

$$(iii)\ \prod_{j<i} s_j = \prod_{1 \leq j \leq i} s_{i-j} \lesssim s_i^{\sum_{j \geq 1} (3/4)^j} = s_i^3,$$

so by (i),

$$(iv)\ n_{i+1} = n_i s_i = n_0 (\prod_{j<i} s_j) s_i \lesssim n_0 s_i^4.$$

Let $n \geq n_0$ be arbitrary. Then $\exists i$, s.t.

$$(v)\ n_i \leq n \leq n_{i+1} \lesssim n_0 s_i^4.$$

Thus,

$$\rho_n(p') \leq \rho_{n_i}(p') \leq \rho_{n_i}(p_i) = \rho_i \approx 1/s_i \lesssim (n_0/n)^{1/4} = (n/n_0)^{-1/4},$$

with the second inequality following from (ii) and the final inequality following from (v).

Since n_0 is fixed, we can take $c = 1/n_0^{-1/4}$

$$(\dagger\dagger)\ \rho_n(p') \lesssim cn^{-1/4}.$$

To summarize, we cleverly defined s_i to make use of the bound in (ii). Then we ingeniously derived (iii) by noting the symmetry to get the first equality. After that, (v) naturally followed, allowing us to bound $\rho_n(p')$ in terms of n. The key was to somehow bound s_i, or $1/s_i$, in terms of n and n_0, which all started with (ii).

We're still far from the theorem, but we're close. All that remains is to apply (†) repeatedly to (††).

In the interest of keeping things in line with the book, we'll realx (††) to get

$$\rho_n(p') \leq cn^{-1/5}\ \ (*).$$

Fix p'' with $p < p'' < p'$. for $n \geq 1$, let $k_1 = k_1(n) = \lfloor n^{5/6} \rfloor$. Again, let us ignore rounding, so $\lfloor n/k_1 \rfloor \approx n^{1/6}$, and, from $(*)$, $\rho_{k_1}(p') \lesssim cn^{-1/6}$. Then $1 - \rho_{k_1}(p') \gtrsim 1 - cn^{-1/6}$, so

$$(1 - \rho_{k_1}(p'))^{\lfloor n/k_1 \rfloor} \gtrsim (1 - cn^{-1/6})^{1/6} \to e^{-c}$$

as $n \to \infty$. Since convergent sequences are bounded, it follows that $(1-\rho_{k_1}(p'))^{\lfloor n/k_1 \rfloor}$ is bounded away from 0, by some constant, Ω. Applying (†) with $k = k_1(n), p_+ = p'$ and $p_- = p''$, we get

$$\rho_n(p'') \lesssim \rho_n(p') exp(-(p_+ - p_-)n^{1/6}\Omega),$$

since $\rho_n(p') < 1$, and $p_+ - p_-$is constant and can be absorbed by Ω, then

$$\rho_n(p'') \lesssim exp(-\Omega n^{1/6})$$

Now, fix p''' with $p < p''' < p''$, and let $k_2 = k_2(n) = \lfloor (\log n)^7 \rfloor$ (for $n \geq 2$). By the inequality above,

$$\rho_{k_2}(p''') \lesssim exp(-\Omega(\log n)^{7/6}) = o(n^{-1})$$

for some constant c, since $exp(-\Omega(\log n)^{7/6})$ goes to 0 much faster than n^{-1}. ($f = o(g)$, if $f/g \to 0$ as $n \to \infty$.)

Thus, $(1-\rho_{k_2}(p''))^{\lfloor n/k_2 \rfloor} \to 1$, since we've just shown $\rho_{k_2}(p''') = o(n^{-1})$. Applying (†) with $k = k_2(n)$, and using the convergence towards 1, we get

$$\rho_n(p''') \leq exp(-\Omega(n/(\log n)^7).$$

(We again ignore $\rho_n(p_+)$ since it is less than 1, so the inequality still holds.

Iterating again, $k_3 = \log n(\log\log n)^8$, it follows that $\rho_{k_3}(p''') = o(n^{-1})$, and applying (†) again, as we've done, we find

$$\rho_n(p) = exp(-\Omega(n/(\log n(\log\log n)^8))),$$

and the theorem follows.

References

[1] B. Bollobás and O. Riordan (2006), *Percolation*, Cambridge U. Press, Cambridge UK.