

ON THE CONNECTED COMPONENT OF COMPACT COMPOSITION OPERATORS ON THE HARDY SPACE

EVA A. GALLARDO-GUTIÉRREZ, MARÍA J. GONZÁLEZ, PEKKA J. NIEMINEN,
AND EERO SAKSMAN

ABSTRACT. We show that there exist non-compact composition operators in the connected component of the compact ones on the classical Hardy space \mathcal{H}^2 . This answers a question posed by Shapiro and Sundberg in 1990. We also establish an improved version of a theorem of MacCluer, giving a lower bound for the essential norm of a difference of composition operators in terms of the angular derivatives of their symbols. As a main tool we use Aleksandrov–Clark measures.

1. INTRODUCTION

Let \mathbb{D} denote the open unit disc of the complex plane and \mathcal{H}^2 the classical Hardy space, that is, the space of analytic functions f on \mathbb{D} for which the norm

$$\|f\|_2 = \left(\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2}$$

is finite. By a variant of Fatou’s theorem, any Hardy function f has non-tangential limits on the boundary of the unit disc except on a set Lebesgue measure zero (see [8], for instance). Moreover, $\|f\|_2$ equals the L^2 -norm of the boundary function. Throughout this work, $f(e^{i\theta})$ will denote the non-tangential limit of f at $e^{i\theta}$.

If φ is an analytic map which takes \mathbb{D} into itself, a result proved by Littlewood in 1925 ensures that the composition operator induced by φ ,

$$C_\varphi f = f \circ \varphi,$$

is always a bounded linear operator on \mathcal{H}^2 . The properties of such operators on \mathcal{H}^2 and many other function spaces have been studied extensively during the past few decades.

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We refer the reader to the monographs [7, 25] for an overview of the field as of the early 1990s.

Starting from Earl Berkson's pioneering work [3], many authors have focused attention on the topological structure of the set $\text{Comp}(\mathcal{H}^2)$ of all composition operators on \mathcal{H}^2 . Here $\text{Comp}(\mathcal{H}^2)$ is usually equipped with the metric induced by the operator norm. A remarkable contribution in this area is due to Joel H. Shapiro and Carl Sundberg [27], who provided several results and examples to describe the isolated members of $\text{Comp}(\mathcal{H}^2)$. Towards the end of their paper, they also raised the general problem of determining the connected components of $\text{Comp}(\mathcal{H}^2)$, and suggested the following conjecture:

(A) *C_φ and C_ψ lie in the same component of $\text{Comp}(\mathcal{H}^2)$ if and only if $C_\varphi - C_\psi$ is compact.*

The most important special case of this conjecture, mentioned explicitly in [27], states that the compact composition operators themselves form a component in $\text{Comp}(\mathcal{H}^2)$. In fact, Shapiro and Sundberg observed that the collection of the compact composition operators on \mathcal{H}^2 is arcwise connected, so the remaining question can be stated as follows:

(B) *Let $\text{Comp}_K(\mathcal{H}^2)$ be the component of $\text{Comp}(\mathcal{H}^2)$ that contains all the compact composition operators. Does any non-compact composition operator belong to $\text{Comp}_K(\mathcal{H}^2)$?*

The general form (A) of the Shapiro–Sundberg conjecture has recently been answered negatively by Moorhouse and Toews [18] and Bourdon [4]. They have provided fairly simple and concrete examples of symbols φ and ψ such that the operators C_φ and C_ψ lie in the same component of $\text{Comp}(\mathcal{H}^2)$ but have a non-compact difference. However, in those examples both operators are non-compact, leaving question (B) unanswered.

In this work, we will show that the special case of the Shapiro–Sundberg conjecture fails, too. That is, we will give an affirmative answer to question (B).

Main Theorem. *For $0 \leq t \leq 1$ there are analytic maps $\varphi_t: \mathbb{D} \rightarrow \mathbb{D}$ such that $t \mapsto C_{\varphi_t}$ is a continuous map from $[0, 1]$ into $\text{Comp}(\mathcal{H}^2)$, where C_{φ_0} is compact and C_{φ_1} is non-compact on \mathcal{H}^2 .*

Let us point out an important result of Barbara MacCluer [13] which states that if two composition operators belong to the same component in $\text{Comp}(\mathcal{H}^2)$, then their symbols must have the same angular derivative (possibly infinity) at each point of the unit circle $\mathbb{T} = \partial\mathbb{D}$. Hence any symbol that induces an operator belonging to $\text{Comp}_K(\mathcal{H}^2)$ cannot have a finite angular derivative at any point of \mathbb{T} . This indicates that the construction of the map φ_1 above is probably not an elementary task. In particular, since non-existence of finite angular derivatives characterizes compact composition operators induced by finitely valent symbols, the valence of φ_1 has to be infinite.

As a main tool in the proof of Main Theorem we will employ Aleksandrov–Clark measures. These measures, associated to any analytic self-map of the unit disc, have lately

found several applications in the study of composition operators (see Section 2). The essence of our argument comprises a construction of a family of certain continuously singular measures on \mathbb{T} , one for each point of $[0, 1]$, which are then used to define the maps φ_t in terms of their Aleksandrov–Clark measures.

The rest of the paper is organized as follows. In Section 2, we collect some preliminaries on Aleksandrov–Clark measures and composition operators. In Section 3, we revisit the theorem of MacCluer cited above and strengthen it slightly. This result will provide a clue for part of the proof of our Main Theorem, which is then carried out in Section 4 (see, in particular, Remark 4.4). Finally, in Section 5 we make some additional observations related to Main Theorem and also present some open questions that arise from our work.

We finally remark that the questions raised by Shapiro and Sundberg have been studied in many classical function spaces besides the original \mathcal{H}^2 . See, for example, [14, 1, 11, 10, 17, 12]. In most cases the situation seems to be considerably easier than in the setting of \mathcal{H}^2 . In particular, for the standard Bergman space \mathcal{A}^2 , MacCluer’s theory shows that the compact composition operators do form a component of $\text{Comp}(\mathcal{A}^2)$ (see Remark 3.2). Also, in the setting of \mathcal{H}^∞ , the space of bounded analytic functions, a complete description of the component structure of $\text{Comp}(\mathcal{H}^\infty)$ was found in [14].

2. ALEKSANDROV–CLARK MEASURES

In this section we collect some preliminaries and background on Aleksandrov–Clark measures and their relation to composition operators. For more information on these measures and their applications in other areas of analysis, we refer the reader to the lecture notes [21], the book [6] and the surveys [16, 20].

2.1. Definition. Let φ be an analytic self-map of \mathbb{D} . For any $\alpha \in \mathbb{T}$, the real part of the function $(\alpha + \varphi)/(\alpha - \varphi)$ is positive and harmonic in \mathbb{D} , so it may be expressed as the Poisson integral of a positive Borel measure $\tau_{\varphi, \alpha}$ supported on \mathbb{T} . That is,

$$\operatorname{Re} \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int_{\mathbb{T}} P_z d\tau_{\varphi, \alpha},$$

where $P_z(\zeta) = (1 - |z|^2)/|\zeta - z|^2$ is the Poisson kernel for $z \in \mathbb{D}$. The family of measures $\{\tau_{\varphi, \alpha} : \alpha \in \mathbb{T}\}$ are called the *Aleksandrov–Clark measures* associated to φ .

For any Borel measure τ on \mathbb{T} , we write $\tau = \tau^a dm + \tau^s$ for the Lebesgue decomposition of τ , so that τ^a is the density of the absolutely continuous part, m is the normalized Lebesgue measure on \mathbb{T} and τ^s is singular. It follows from the basic properties of Poisson integrals that

$$\tau_{\varphi, \alpha}^a(\zeta) = \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2}.$$

Furthermore, $\tau_{\varphi, \alpha}^s$ is carried by the set where $\varphi(\zeta) = \alpha$.

2.2. Angular derivatives. Recall that if the quotient $(\varphi(z) - \eta)/(z - \zeta)$ has a finite non-tangential limit at $\zeta \in \mathbb{T}$ for some $\eta \in \mathbb{T}$, then this limit is called the *angular derivative* of φ at ζ and denoted by $\varphi'(\zeta)$. It satisfies $\varphi'(\zeta) = |\varphi'(\zeta)|\bar{\zeta}\eta$ where $\eta = \varphi(\zeta)$. A nice feature of the Aleksandrov–Clark measures is that their discrete parts (i.e. mass points, or atoms) have a perfect correspondence with the finite angular derivatives of φ :

- *The map φ has a finite angular derivative at $\zeta \in \mathbb{T}$ if and only if there is $\alpha \in \mathbb{T}$ such that $\tau_{\varphi, \alpha}(\{\zeta\}) > 0$. In that case $\varphi(\zeta) = \alpha$ and $|\varphi'(\zeta)| = \tau_{\varphi, \alpha}(\{\zeta\})^{-1}$.*

For the proof of this result convenient references are [6, 21], where it is established in conjunction with the classical Julia–Carathéodory theorem.

2.3. Relation to composition operators. To bring Aleksandrov–Clark measures into the theory of composition operators, we follow Sarason’s [22] idea of describing composition operators as integral operators acting on the unit circle. Let us denote by \mathcal{M} the space of all complex Borel measures on \mathbb{T} endowed with the total variation norm. Then, if $\mu \in \mathcal{M}$ is given, the Poisson integral $u(z) = \int_{\mathbb{T}} P_z d\mu$ defines a harmonic function on \mathbb{D} . Consequently the function $v = u \circ \varphi$ is also harmonic, and it follows easily that v is the Poisson integral of a unique measure $\nu \in \mathcal{M}$. Thus it makes sense to define $C_\varphi \mu = \nu$. One can show that $C_\varphi: \mathcal{M} \rightarrow \mathcal{M}$ is bounded and, furthermore, that C_φ restricts to a bounded operator $L_p \rightarrow L_p$, where $L_p = L_p(\mathbb{T}, m)$ for $1 \leq p \leq \infty$. Moreover, the restriction of C_φ to the analytic Hardy spaces \mathcal{H}^p (viewed as subspaces of L^p) coincides with the standard definition of C_φ .

By the definition of the Aleksandrov–Clark measures we see that $\tau_{\varphi, \alpha} = C_\varphi \delta_\alpha$, where δ_α is the δ -Dirac measure at α . In addition, the correspondence $C_\varphi \mu = \nu$ can be written as

$$(2.1) \quad \int_{\mathbb{T}} f d\nu = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f d\tau_{\varphi, \alpha} \right) d\mu(\alpha)$$

for a suitable class of functions f . Indeed, if f is a Poisson kernel P_z , this follows directly from the definitions. The case of continuous f is then obtained by approximating with linear combinations of Poisson kernels. Finally one may invoke a further approximation argument (e.g. a monotone class theorem; cf. [6, Sec. 9.4]) to establish (2.1) for all bounded Borel functions f on \mathbb{T} .

In [22] Sarason characterized those composition operators C_φ that are compact on \mathcal{M} and L^1 by a condition which says that $\tau_{\varphi, \alpha}^s = 0$ for all $\alpha \in \mathbb{T}$; that is, the Aleksandrov–Clark measures of φ are required to be absolutely continuous. Later Shapiro and Sundberg [26] observed that Sarason’s criterion is equivalent to Shapiro’s [24] characterization of compact composition operators on \mathcal{H}^p , $1 \leq p < \infty$, involving the Nevanlinna counting function. Moreover, Cima and Matheson [5] have shown that the essential norm (i.e. distance, in the operator norm, from the compact operators) of any C_φ acting on \mathcal{H}^2 equals $\sup_\alpha \|\tau_{\varphi, \alpha}^s\|^{1/2}$. In particular, a necessary condition for the compactness of C_φ on all the

spaces mentioned is that the symbol φ has no finite angular derivative at any point of \mathbb{T} . This condition, however, is not sufficient unless φ is of finite valence (see e.g. [25]).

Aleksandrov–Clark measures have also been used to study differences and more general linear combinations of composition operators in [12, 19, 23]. In particular, a characterization for compact differences of composition operators on \mathcal{M} and L^1 was found in [19].

3. EXTENSION OF MACCLUER’S THEOREM

In 1989 Barbara MacCluer obtained the following result concerning differences of composition operators on \mathcal{H}^2 .

Theorem 3.1 (MacCluer [13]). *Assume that $\varphi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ are analytic maps and φ has a finite angular derivative at $\zeta \in \mathbb{T}$. Then, unless $\psi(\zeta) = \varphi(\zeta)$ and $\psi'(\zeta) = \varphi'(\zeta)$, one has*

$$\|C_\varphi - C_\psi\|_e^2 \geq \frac{1}{|\varphi'(\zeta)|},$$

where $\|\cdot\|_e$ denotes the essential norm of an operator on \mathcal{H}^2 .

The relationship between angular derivatives and the atoms of the Aleksandrov–Clark measures (see Sec. 2.2) allows us to restate Theorem 3.1 as follows:

- Assume that $\tau_{\varphi,\alpha}(\{\zeta\}) > 0$ for some $\alpha \in \mathbb{T}$. Then, unless $\tau_{\psi,\alpha}(\{\zeta\}) = \tau_{\varphi,\alpha}(\{\zeta\})$, one has $\|C_\varphi - C_\psi\|_e^2 \geq \tau_{\varphi,\alpha}(\{\zeta\})$.

Theorem 3.1 implies that, for each $\zeta \in \mathbb{T}$ and $d \neq 0$, the set of all C_φ with $\varphi'(\zeta) = d$ is both open and closed in $\text{Comp}(\mathcal{H}^2)$ (even in the topology induced by the essential norm). Hence a necessary condition for two composition operators to lie in the same component (or essential component) of $\text{Comp}(\mathcal{H}^2)$ is that the angular derivatives of their symbols coincide. In particular, it follows that if C_φ belongs to $\text{Comp}_K(\mathcal{H}^2)$, the component containing all compact composition operators, then φ has no finite angular derivative at any point of \mathbb{T} — or, equivalently, the Aleksandrov–Clark measure $\tau_{\varphi,\alpha}$ has no atoms for any $\alpha \in \mathbb{T}$.

Remark 3.2. MacCluer’s work was actually carried out in a general context of weighted Dirichlet (or Bergman) spaces \mathcal{D}_β , $\beta \geq 1$, which includes as special cases the Hardy space \mathcal{H}^2 ($\beta = 1$) as well as the standard Bergman space \mathcal{A}^2 ($\beta = 2$). For $\beta > 1$ it is known that the non-existence of finite angular derivatives is both necessary and sufficient for the compactness of a composition operator on \mathcal{D}_β (see [15] or [7]). So, in these spaces, MacCluer’s theorem implies (e.g. by the argument at the beginning of the preceding paragraph) that the compacts indeed form a connected component of $\text{Comp}(\mathcal{D}_\beta)$.

In another direction, Kriete and Moorhouse [12] have recently obtained various interesting refinements of MacCluer’s results. In particular, they establish a version of Theorem 3.1 for higher-order boundary data of the symbols.

In this section we will provide a slight improvement of Theorem 3.1. Our lower bound will involve the whole discrete part of the Aleksandrov–Clark measure at α . This result yields some heuristics for our construction in the proof of our Main Theorem in Section 4 (see Remark 4.4).

Theorem 3.3. *Let $\varphi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ be analytic maps and $\alpha \in \mathbb{T}$. Write*

$$Z = \{\zeta \in \mathbb{T} : 0 < \tau_{\varphi, \alpha}(\{\zeta\}) \neq \tau_{\psi, \alpha}(\{\zeta\})\}.$$

Then

$$\|C_\varphi - C_\psi\|_e^2 \geq \tau_{\varphi, \alpha}(Z).$$

In the proof of Theorem 3.3 we will use as test functions the normalized reproducing kernels

$$f_w(z) = \frac{\sqrt{1 - |w|^2}}{1 - \overline{w}z}.$$

They have the property that $\|f_w\|_2 = 1$ for all $w \in \mathbb{D}$ and $f_w \rightarrow 0$ weakly as $|w| \rightarrow 1$, whence $\|C_\varphi - C_\psi\|_e \geq \limsup_{|w| \rightarrow 1} \|(C_\varphi - C_\psi)f_w\|_2$. We will borrow MacCluer's idea of letting w approach α along a curve which makes almost right angle with the radius to α . However, instead of considering the adjoints of C_φ and C_ψ as in [13] and [12], we will deal with the composition operators themselves. The following lemma contains the estimates crucial for our argument.

Lemma 3.4. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be analytic and fix $a > 0$. For $\delta, \kappa, \lambda, r > 0$, write*

$$I(\delta, \kappa, \lambda, r) = \frac{1}{2\pi} \int_{\kappa r a - \lambda r}^{\kappa r a + \lambda r} |C_\varphi f_{(1-r)e^{i\kappa r}}((1-\delta r)e^{it})|^2 dt.$$

(1) *If $\tau_{\varphi, 1}(\{1\}) = a$, then*

$$\lim_{r \rightarrow 0} I(\delta, \kappa, \lambda, r) = \frac{a \cdot c(\delta, \lambda)}{1 + \delta/a},$$

where $0 < c(\delta, \lambda) < 1$ and $\lim_{\lambda \rightarrow \infty} c(\delta, \lambda) = 1$ for all $\delta > 0$.

(2) *If $\tau_{\varphi, 1}(\{1\}) \neq a$, then*

$$\lim_{r \rightarrow 0} I(\delta, \kappa, \lambda, r) = \varepsilon(\delta, \kappa, \lambda),$$

where $\lim_{\kappa \rightarrow \infty} \varepsilon(\delta, \kappa, \lambda) = 0$ for all $\delta, \lambda > 0$.

Proof. Let us fix $\delta, \kappa, \lambda > 0$, and write $w_r = (1-r)e^{i\kappa r}$ and $z_r(t) = (1-\delta r)e^{it}$. Then

$$(3.1) \quad I(\delta, \kappa, \lambda, r) = \frac{2r - r^2}{2\pi} \int_{\kappa r a - \lambda r}^{\kappa r a + \lambda r} \frac{dt}{|1 - \overline{w}_r \varphi(z_r(t))|^2}.$$

We first consider the case when $\tau_{\varphi, 1}(\{1\}) = b$ for some $b > 0$. That is, $\varphi(1) = 1$ and φ has a finite angular derivative equal to $1/b$ at 1. Note that the points $z_r(t)$ involved

in (3.1) for $0 < r < 1$ all lie in a non-tangential approach region for the point 1 (whose opening angle depends on δ , κ , a , and λ). Therefore, for these $z_r(t)$ we have

$$1 - \varphi(z_r(t)) = b^{-1}(1 - z_r(t)) + r\varepsilon_r(t),$$

uniformly in t . Here and elsewhere in this proof we use ε_r (with or without additional parameters) to denote a quantity which tends to zero as $r \rightarrow 0$. With this notation, we may also write $1 - \overline{w}_r = r + i\kappa r + r\varepsilon_r$ and $1 - z_r(t) = \delta r - it + r\varepsilon_r(t)$. Consequently,

$$\begin{aligned} 1 - \overline{w}_r \varphi(z_r(t)) &= (1 - \overline{w}_r) + \{1 - \varphi(z_r(t))\} + r\varepsilon_r(t) \\ &= r(1 + \delta/b) + i(\kappa r - t/b) + r\varepsilon_r(t). \end{aligned}$$

We substitute this expression into the integrand in (3.1) and perform the change of variables $u = t/ra - \kappa$ to get

$$\begin{aligned} I(\delta, \kappa, \lambda, r) &= \frac{(2r - r^2)ra}{2\pi} \int_{-\lambda/a}^{+\lambda/a} \frac{du}{|r(1 + \delta/b) + i(\kappa r - \kappa ra/b - rau/b) + r\varepsilon_r(u)|^2} \\ &= \frac{(2 - r)a}{2\pi} \int_{-\lambda/a}^{+\lambda/a} \frac{du}{|(1 + \delta/b) + i((1 - a/b)\kappa - au/b) + \varepsilon_r(u)|^2}. \end{aligned}$$

Hence

$$(3.2) \quad \lim_{r \rightarrow 0} I(\delta, \kappa, \lambda, r) = \frac{a}{\pi} \int_{-\lambda/a}^{+\lambda/a} \frac{du}{(1 + \delta/b)^2 + ((1 - a/b)\kappa - au/b)^2}.$$

If $b = a$, this limit equals

$$\frac{a}{\pi} \int_{-\lambda/a}^{+\lambda/a} \frac{du}{(1 + \delta/a)^2 + u^2},$$

which is of the desired form $ac(\delta, \lambda)/(1 + \delta/a)$. On the other hand, if $b \neq a$, then the integrand in (3.2) tends to zero as $\kappa \rightarrow \infty$, uniformly in u . So, in this case (3.2) goes to zero as $\kappa \rightarrow \infty$.

Finally assume that $\tau_{\varphi,1}(\{1\}) = 0$, so φ has no finite angular derivative at 1 or $\varphi(1) \neq 1$. By the Julia–Carathéodory theorem, we now have $(1 - \varphi(z))/(1 - z) \rightarrow \infty$ as $z \rightarrow 1$ non-tangentially. By considerations similar to those in the first part of the proof, this implies that $\{1 - \overline{w}_r \varphi(z_r(t))\}/r \rightarrow \infty$ as $r \rightarrow 0$, uniformly in t , and hence $I(\delta, \kappa, \lambda, r) \rightarrow 0$ as $r \rightarrow 0$. We leave the details to the reader. \square

Proof of Theorem 3.3. Without loss of generality, we may take $\alpha = 1$. We first treat the case of a single mass point and then indicate the general argument. Let us assume that $\tau_{\varphi,1}(\{1\}) = a \neq \tau_{\psi,1}(\{1\})$ for some $a > 0$. Then, for $\delta, \kappa, \lambda > 0$ and small enough $r > 0$, we have

$$\begin{aligned} \|(C_\varphi - C_\psi)f_{(1-r)e^{i\kappa r}}\|_2 &\geq \left(\frac{1}{2\pi} \int_{\kappa ra - \lambda r}^{\kappa ra + \lambda r} |(C_\varphi - C_\psi)f_{(1-r)e^{i\kappa r}}((1 - \delta r)e^{it})|^2 dt \right)^{1/2} \\ &\geq I_\varphi(\delta, \kappa, \lambda, r)^{1/2} - I_\psi(\delta, \kappa, \lambda, r)^{1/2}, \end{aligned}$$

where I_φ and I_ψ refer to the integrals of Lemma 3.4 corresponding to φ and ψ , respectively. Passing to the limit as $r \rightarrow 0$, we then get the following type of lower bound for the essential norm of $C_\varphi - C_\psi$:

$$\|C_\varphi - C_\psi\|_e \geq \left(\frac{a \cdot c(\delta, \lambda)}{1 + \delta/a} \right)^{1/2} - \varepsilon(\delta, \kappa, \lambda)^{1/2}.$$

Letting $\kappa \rightarrow \infty$, $\lambda \rightarrow \infty$ and $\delta \rightarrow 0$ now yields $\|C_\varphi - C_\psi\|_e \geq a^{1/2}$ as desired.

To prove the theorem in full (assuming still $\alpha = 1$), we observe that the above reasoning is local in the sense that the interval $[\kappa r a - \lambda r, \kappa r a + \lambda r]$ shrinks to 0 as $r \rightarrow 0$. Let $Z_0 = \{\zeta_1, \dots, \zeta_n\}$ be any finite subset of the (possibly infinite) set Z , where $\zeta_k \neq \zeta_l$ for $k \neq l$. Write $t_k = \arg \zeta_k$ and $a_k = \tau_{\varphi,1}(\{\zeta_k\})$. We proceed as above, just integrating over the union of the intervals $[t_k + \kappa r a_k - \lambda r, t_k + \kappa r a_k + \lambda r]$, $k = 1, \dots, n$. Since these are disjoint for small r , we get, after passing to the appropriate limits as above,

$$\|C_\varphi - C_\psi\|_e \geq \left(\sum_{k=1}^n \tau_{\varphi,1}(\{\zeta_k\}) \right)^{1/2} = \tau_{\varphi,1}(Z_0)^{1/2}.$$

Finally, if Z is infinite, we take the supremum over all finite subsets $Z_0 \subset Z$ to complete the proof of the theorem. \square

4. PROOF OF MAIN THEOREM: NON-COMPACT COMPOSITION OPERATORS IN THE COMPONENT OF COMPACTS

In this section we will establish our Main Theorem, giving a positive answer to the question (B) stated in Section 1. We will actually find a continuous path that connects compact composition operators to a non-compact one. Moreover, the same construction turns out to work for a variety of spaces in addition to \mathcal{H}^2 .

Main Theorem. *For $0 \leq t \leq 1$ there are analytic maps $\varphi_t: \mathbb{D} \rightarrow \mathbb{D}$ such that C_{φ_0} is compact and C_{φ_1} is non-compact on X , and $t \mapsto C_{\varphi_t}$ is continuous from $[0, 1]$ into $\text{Comp}(X)$, where X is any of the spaces \mathcal{M} , L^p or \mathcal{H}^p with $1 \leq p < \infty$.*

We begin with some preliminary observations and lemmas. First of all, it is enough to deal with the case $X = \mathcal{M}$. Indeed, as we pointed out in Section 2.3, the compactness of composition operators is equivalent in any two of the spaces mentioned. Furthermore, we may apply interpolation between L^1 (a subspace of \mathcal{M}) and L^∞ to conclude that for any $1 \leq p < \infty$ and $s, t \in [0, 1]$,

$$\begin{aligned} & \|C_{\varphi_s} - C_{\varphi_t}: L^p \rightarrow L^p\| \\ & \leq \|C_{\varphi_s} - C_{\varphi_t}: L^1 \rightarrow L^1\|^{1/p} \|C_{\varphi_s} - C_{\varphi_t}: L^\infty \rightarrow L^\infty\|^{1-1/p} \\ & \leq 2^{1-1/p} \|C_{\varphi_s} - C_{\varphi_t}: \mathcal{M} \rightarrow \mathcal{M}\|^{1/p}. \end{aligned}$$

(See e.g. [2, Sec. 4.1] for the classical Riesz–Thorin interpolation theorem.)

Throughout the proof we will utilize Sarason's way of viewing composition operators as acting on the unit circle (cf. Sec. 2.3). If φ is an analytic self-map of \mathbb{D} and $E \subset \mathbb{T}$ is a Borel set, we let $\chi_E C_\varphi$ denote the restriction of C_φ to E . More precisely, if $\mu \in \mathcal{M}$ and $C_\varphi \mu = \nu$, then $\chi_E C_\varphi \mu$ refers to the Borel measure $B \mapsto \nu(E \cap B)$ on \mathbb{T} . For functions $f \in L^1$, this simply means that $\chi_E C_\varphi f(\zeta) = \chi_E(\zeta) f(\varphi(\zeta))$ for m -a.e. $\zeta \in \mathbb{T}$. In this context, equation (2.1) easily yields that

$$(4.1) \quad \|\chi_E C_\varphi: \mathcal{M} \rightarrow \mathcal{M}\| = \sup\{\tau_{\varphi, \alpha}(E) : \alpha \in \mathbb{T}\}.$$

We will also need a tool to estimate the size of the difference of two composition operators in terms of the boundary values of their symbols. We use ρ to denote the hyperbolic distance in \mathbb{D} ; it is the conformally invariant metric induced by the arc length element $2|dz|/(1 - |z|^2)$ (see e.g. [9, Sec. I.1]). When working with hyperbolic distances, it is often convenient to shift to the right half-plane $\mathbb{H} = \{z' : \operatorname{Re} z' > 0\}$, where the hyperbolic metric $\rho_{\mathbb{H}}$ is induced by the arc length element $|dz'|/\operatorname{Re} z'$. For any $\alpha \in \mathbb{T}$, this is accomplished through the Möbius transformation $z' = (\alpha + z)/(\alpha - z)$, which takes \mathbb{D} onto \mathbb{H} isometrically relative to ρ and $\rho_{\mathbb{H}}$. This transformation we have already encountered in the definition of Aleksandrov–Clark measures.

Lemma 4.1. *Let $\varphi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ be analytic, and let $E \subset \mathbb{T}$ be a Borel set such that $\tau_{\varphi, \alpha}(\partial E) = \tau_{\psi, \alpha}(\partial E) = 0$ for all $\alpha \in \mathbb{T}$. Also let $0 < \varepsilon < 1$. Suppose that for m -a.e. $\zeta \in E$ the following holds: if one of $\varphi(\zeta)$ and $\psi(\zeta)$ is unimodular, then $\varphi(\zeta) = \psi(\zeta)$, and otherwise $\rho(\varphi(\zeta), \psi(\zeta)) \leq \varepsilon$. Then*

$$\|\chi_E(C_\varphi - C_\psi): \mathcal{M} \rightarrow \mathcal{M}\| \leq C\varepsilon/(1 - |\varphi(0)|),$$

where $C > 0$ is a universal constant.

Proof. We first note that the Poisson kernel functions P_z satisfy the following estimate: for all $z, w \in \mathbb{D}$ with $\rho(z, w) \leq 1$ and $\alpha \in \mathbb{T}$,

$$(4.2) \quad |P_z(\alpha) - P_w(\alpha)| \leq C\rho(z, w) P_z(\alpha),$$

where $C > 0$ is a universal constant. In fact, one may use the transformation $z' = (\alpha + z)/(\alpha - z)$ to pass to the right half-plane where (4.2) becomes

$$|\operatorname{Re}(z' - w')| \leq C\rho_{\mathbb{H}}(z', w') \operatorname{Re} z',$$

which is easy to verify by geometric reasoning.

Now fix $\alpha \in \mathbb{T}$ and $0 < r < 1$. Since $\rho(r\varphi(\zeta), r\psi(\zeta)) \leq \varepsilon$ for m -a.e. $\zeta \in E$, we get by (4.2) that

$$\int_E \left| \frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2} - \frac{1 - |r\psi|^2}{|\alpha - r\psi|^2} \right| dm \leq C\varepsilon \int_E \frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2} dm \leq C\varepsilon \frac{1 - |r\varphi(0)|^2}{|\alpha - r\varphi(0)|^2}.$$

The last inequality was obtained by extending the integral over the whole circle \mathbb{T} and using the harmonicity of the integrand. The definition of the Aleksandrov–Clark measures

implies that the absolutely continuous measure $(1 - |r\varphi|^2)/|\alpha - r\varphi|^2 dm$ converges to $\tau_{\varphi,\alpha}$ weak* as $r \rightarrow 1$. Similarly $(1 - |r\psi|^2)/|\alpha - r\psi|^2 dm$ converges to $\tau_{\psi,\alpha}$. Therefore, the preceding chain of inequalities yields, as $r \rightarrow 1$,

$$|\tau_{\varphi,\alpha} - \tau_{\psi,\alpha}|(E) \leq C\varepsilon \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2}.$$

(Here we needed the assumption that $\tau_{\varphi,\alpha}$ and $\tau_{\psi,\alpha}$ both assign measure zero to the boundary of E .) Hence

$$\begin{aligned} \|\chi_E(C_\varphi - C_\psi): \mathcal{M} \rightarrow \mathcal{M}\| &= \sup\{|\tau_{\varphi,\alpha} - \tau_{\psi,\alpha}|(E) : \alpha \in \mathbb{T}\} \\ &\leq \frac{2C\varepsilon}{1 - |\varphi(0)|}, \end{aligned}$$

and the proof is complete. \square

We are now in a position to define the maps φ_t . Recall from Section 2.3 that a composition operator C_φ is non-compact on any of the spaces mentioned in Main Theorem if and only if at least one of the Aleksandrov–Clark measures $\tau_{\varphi,\alpha}$ fails to be absolutely continuous. On the other hand, if C_φ is required to belong to the component of compact composition operators, MacCluer’s theorem implies that none of $\tau_{\varphi,\alpha}$ may have atoms. That is why we have to consider Aleksandrov–Clark measures with continuous singularity.

Let λ be any nontrivial, positive and finite continuously singular Borel measure on the unit circle \mathbb{T} . For $0 \leq t \leq 1$, let

$$(4.3) \quad \tau_{t,1} = m + \chi_{I(0,t)}\lambda,$$

where $I(0,t) \subset \mathbb{T}$ is the closed arc connecting the point 1 to $e^{2\pi it}$ in the positive direction and, as before, m denotes the normalized Lebesgue measure. We consider the Herglotz integral of $\tau_{t,1}$,

$$H\tau_{t,1}(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\tau_{t,1}(\zeta),$$

and define the map φ_t by

$$(4.4) \quad \frac{1 + \varphi_t}{1 - \varphi_t} = H\tau_{t,1}, \quad \text{that is,} \quad \varphi_t = \frac{H\tau_{t,1} - 1}{H\tau_{t,1} + 1}.$$

Since the real part of $H\tau_{t,1}$ is the Poisson integral of $\tau_{t,1}$, we see that $\tau_{t,1}$ becomes the Aleksandrov–Clark measure of φ_t at 1. Moreover, since this Poisson integral is ≥ 1 everywhere on \mathbb{D} , it follows that φ_t either takes \mathbb{D} into the open disc $\{w : |w - \frac{1}{2}| < \frac{1}{2}\}$ or is constant 0 (for small t). In general, we let $\tau_{t,\alpha}$ denote the Aleksandrov–Clark measure of φ_t at $\alpha \in \mathbb{T}$.

The compactness statements of Main Theorem are now immediate. Since $\tau_{1,1} = m + \lambda$ is not absolutely continuous, the operator C_{φ_1} is non-compact. On the other hand, $\varphi_0 \equiv 0$, so C_{φ_0} is clearly compact.

The hard part of the proof consists of showing that the map $t \mapsto C_{\varphi_t}$ is indeed continuous. This will be based on the following two lemmas.

Lemma 4.2. *Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $I \subset \mathbb{T}$ is an arc with $m(I) \leq \delta$, then the Aleksandrov–Clark measures of the maps φ_t satisfy $\tau_{t,\alpha}(I) \leq \varepsilon$ for all $t \in [0, 1]$ and $\alpha \in \mathbb{T}$. In particular, none of $\tau_{t,\alpha}$ have atoms.*

Proof. First of all we note that all the measures $\tau_{t,\alpha}$ are indeed continuous, i.e. have no atoms. For $\alpha = 1$ this is clear from (4.3). For $\alpha \neq 1$ we need to note that since the image of φ_t does not touch α , the harmonic function

$$(4.5) \quad \operatorname{Re} \frac{\alpha + \varphi_t(z)}{\alpha - \varphi_t(z)} = \int_{\mathbb{T}} P_z d\tau_{t,\alpha},$$

is bounded and hence $\tau_{t,\alpha}$ is absolutely continuous.

Using (4.3) and (4.4) one can easily show that the left-hand side of (4.5) is continuous as a function of the pair (t, α) in $[0, 1] \times \mathbb{T}$. Since linear combinations of Poisson kernels are dense among the continuous functions on \mathbb{T} , it follows that the map $(t, \alpha) \mapsto \tau_{t,\alpha}$ is continuous in the weak* sense.

Now assume that the claim of the lemma fails. Then there are arcs $I_n \subset \mathbb{T}$ and points $t_n \in [0, 1]$ and $\alpha_n \in \mathbb{T}$ such that $\tau_{t_n, \alpha_n}(I_n) > \varepsilon$ for all $n \geq 1$ while $m(I_n) \rightarrow 0$. By passing to a subsequence we may assume that the intervals I_n (i.e. their endpoints) converge to a point $\zeta_0 \in \mathbb{T}$ and also that $t_n \rightarrow t_0$ and $\alpha_n \rightarrow \alpha_0$. Now for each $\eta > 0$ we have $\tau_{t_n, \alpha_n}(I(e^{-i\eta}\zeta_0, e^{i\eta}\zeta_0)) > \varepsilon$ whenever n is large enough. Since the map $(t, \alpha) \mapsto \tau_{t,\alpha}$ is weak* continuous, it follows that $\tau_{t_0, \alpha_0}(I(e^{-i\eta}\zeta_0, e^{i\eta}\zeta_0)) \geq \varepsilon$ for all $\eta > 0$, and hence $\tau_{t_0, \alpha_0}(\{\zeta_0\}) \geq \varepsilon$. This is a contradiction since we observed that τ_{t_0, α_0} cannot have atoms. \square

Lemma 4.3. *Fix $t_0 \in [0, 1]$ and let $I_0 \subset \mathbb{T}$ be an arc whose midpoint is $e^{2\pi i t_0}$. If $\varepsilon > 0$ is given, there exists $\delta > 0$ such that*

$$\rho(\varphi_{t_0}(\zeta), \varphi_t(\zeta)) \leq \varepsilon \quad \text{for } \zeta \in \mathbb{T} \setminus I_0$$

whenever $|t_0 - t| \leq \delta$.

Proof. Assume that $|t_0 - t|$ is so small that the distance of the point $e^{2\pi i t}$ to the set $\mathbb{T} \setminus I_0$ is greater than a positive constant c . Then $H\tau_{t,1} = H\tau_{t_0,1} \pm H(\chi_{J_t}\lambda)$, where $J_t \subset \mathbb{T}$ is the arc connecting $e^{2\pi i t_0}$ to $e^{2\pi i t}$. Moreover, for $\zeta \in \mathbb{T} \setminus I_0$ we have

$$|H(\chi_{J_t}\lambda)(\zeta)| = \left| \int_{J_t} \frac{\xi + \zeta}{\xi - \zeta} d\lambda(\xi) \right| \leq \frac{2}{c} \lambda(J_t).$$

Since this upper bound tends to zero as $t \rightarrow t_0$ and $\operatorname{Re} H\tau_{t_0,1} \geq 1$, we see that the distance between $H\tau_{t,1}(\zeta)$ and $H\tau_{t_0,1}(\zeta)$ in the hyperbolic metric of the right half-plane tends to zero as $t \rightarrow t_0$, uniformly for $\zeta \in \mathbb{T} \setminus I_0$. In view of (4.4) and the conformal invariance of

the hyperbolic metric, the same conclusion holds true for the distance of $\varphi_t(\zeta)$ and $\varphi_{t_0}(\zeta)$ in the metric ρ . \square

We are now ready to prove the continuity of the map $t \mapsto C_{\varphi_t}$ with respect to the operator norm on \mathcal{M} . Let $0 < \varepsilon < 1$. By Lemma 4.2 we can find $\delta > 0$ such that $\tau_{t,\alpha}(I) \leq \varepsilon$ for all $t \in [0, 1]$ and $\alpha \in \mathbb{T}$ whenever $I \subset \mathbb{T}$ is an arc with $m(I) \leq \delta$. For all such I , equation (4.1) yields the estimate

$$(4.6) \quad \|\chi_I C_{\varphi_t}\| \leq \varepsilon.$$

(Here and throughout the rest of the proof $\|\cdot\|$ refers to the operator norm on \mathcal{M} .)

Now fix $t_0 \in [0, 1]$ and pick an arc $I_0 \subset \mathbb{T}$ with $m(I_0) \leq \delta$ whose midpoint is $e^{2\pi i t_0}$. By Lemma 4.3 there exists $\eta > 0$ such that if $|t_0 - t| \leq \eta$, then $\rho(\varphi_{t_0}(\zeta), \varphi_t(\zeta)) \leq \varepsilon$ for all $\zeta \in \mathbb{T} \setminus I_0$. Hence Lemma 4.1 shows that

$$(4.7) \quad \|\chi_{\mathbb{T} \setminus I_0}(C_{\varphi_{t_0}} - C_{\varphi_t})\| \leq C\varepsilon/(1 - |\varphi_{t_0}(0)|)$$

whenever $|t_0 - t| \leq \eta$. To finish the argument we just write

$$C_{\varphi_{t_0}} - C_{\varphi_t} = \chi_{I_0} C_{\varphi_{t_0}} - \chi_{I_0} C_{\varphi_t} + \chi_{\mathbb{T} \setminus I_0}(C_{\varphi_{t_0}} - C_{\varphi_t})$$

and, when $|t_0 - t| \leq \eta$, invoke estimates (4.6) and (4.7) to conclude that

$$\|C_{\varphi_{t_0}} - C_{\varphi_t}\| \leq \varepsilon + \varepsilon + C\varepsilon/(1 - |\varphi_{t_0}(0)|).$$

Since $\varepsilon > 0$ was arbitrary, this clearly shows that the norm of $C_{\varphi_{t_0}} - C_{\varphi_t}$ on \mathcal{M} tends to zero as $t \rightarrow t_0$.

The proof of Main Theorem is now complete.

Remark 4.4. We try to describe the heuristics behind the above construction. First of all, one can easily show that if a continuous path (C_{φ_t}) yielding the desired example exists, then one may assume that the image of each map φ_t is contained in the disc $\{w : |w - \frac{1}{2}| \leq \frac{1}{2}\}$. Then $\tau_{1,1}$ is necessarily of the form $g dm + \lambda$ where $g \geq 1$ and λ is non-trivial and continuously singular. One may also assume that $\varphi_0 \equiv 0$. The central issue then is to find the intermediate maps φ_t for $0 < t < 1$. A seemingly natural choice might be $\varphi_t = (1-t)\varphi_0 + t\varphi_1$, but this obviously fails to work since each such map is compact. On the other hand, in certain applications to spectral theory one proceeds by considering the maps corresponding to the Aleksandrov–Clark measures $\tau_{t,1} = (1-t)\tau_{0,1} + t\tau_{1,1}$. However, Theorem 3.3 suggests that this approach might not work either. Namely, in the case of a discrete singular part, Theorem 3.3 shows that if one makes a simultaneous change—no matter how small—to all the mass points of the singular part, then this induces a big difference in the corresponding composition operator. These considerations were behind our actual choice (4.3), where the singularity λ is continuously “wiped off” in such a way that the change in $\tau_{t,1}$ is strictly local at every instant t .

5. FURTHER REMARKS AND OPEN PROBLEMS

After the work of Section 4 it is natural to search for a larger class of composition operators that could be continuously joined to the compacts. For instance, one might be tempted to expect a positive answer to the following question:

- Assume that φ and $\alpha_0 \in \mathbb{T}$ are such that the measure τ_{φ, α_0} has no atoms and, for all $\alpha \neq \alpha_0$, the measure $\tau_{\varphi, \alpha}$ is absolutely continuous. Does it follow that C_φ belongs to $\text{Comp}_K(\mathcal{H}^2)$?

The answer to this question is, however, negative.

Example 5.1. *There is a symbol ψ such that C_ψ is isolated in $\text{Comp}(\mathcal{H}^2)$ and the following properties hold: $\tau_{\psi, 1}$ has a continuous non-trivial singular part while all the other measures $\tau_{\psi, \alpha}$ are absolutely continuous. In fact, one may choose $\psi = \varphi \circ \sigma$, where σ is an inner function and φ is a conformal map from \mathbb{D} onto a region $\Omega \subset \mathbb{D}$ with $\overline{\Omega} \cap \mathbb{T} = \{1\}$.*

The above example is based on a construction of Shapiro and Sundberg [27]. We first recall some terminology. Shapiro and Sundberg call a continuous and 2π -periodic function $\kappa: \mathbb{R} \rightarrow [0, 1)$ a *contact function* if it is increasing and positive on $(0, \pi]$, decreasing and positive on $[-\pi, 0)$ and vanishes at the origin. Such a function determines an approach region

$$\Omega(\kappa) = \{re^{i\theta} : 0 \leq r < 1 - \kappa(\theta)\},$$

whose boundary is a Jordan curve in $\overline{\mathbb{D}}$ that meets the unit circle only at the point 1. In this setting Shapiro and Sundberg prove the following (see Theorem 4.1 and Remark 5.1 of [27]).

Theorem 5.2 (Shapiro–Sundberg). *Suppose κ is a C^2 contact function and φ is a conformal map from \mathbb{D} onto $\Omega(\kappa)$. If*

$$(5.1) \quad \int_0^\pi \log \kappa(\theta) d\theta = -\infty,$$

then C_φ is (essentially) isolated in $\text{Comp}(\mathcal{H}^2)$.

We observe that this theorem can be extended as follows.

Proposition 5.3. *Let φ be a function given by Theorem 5.2, and let σ be an inner function with $\sigma(0) = 0$. Put $\psi = \varphi \circ \sigma$. Then C_ψ is (essentially) isolated in $\text{Comp}(\mathcal{H}^2)$.*

Let us note that an analytic self-map of \mathbb{D} is an inner function if and only if any (or all) of its Aleksandrov–Clark measures is singular. Therefore, to produce the symbol needed for Example 5.1, we choose any inner function σ vanishing at the origin whose Aleksandrov–Clark measure $\tau_{\sigma, 1}$ is continuously singular. We then apply Proposition 5.3 with the additional requirement that $\varphi(1) = 1$. It is relatively easy to check that $\psi = \varphi \circ \sigma$ has the required properties; in particular, $\tau_{\psi, 1}$ cannot have atoms.

Proof of Proposition 5.3. We start by recalling some ideas from the proof of Theorem 5.2. Write $\Omega = \Omega(\kappa)$ for the image of φ . A crucial part of Shapiro's and Sundberg's argument is the construction of a sequence of test functions $f_n \in \mathcal{H}^2$ which converges to zero weakly in \mathcal{H}^2 . Their functions satisfy the following properties: $|f_n|^2 \geq c/m(J_n)$ on Γ_n , where $\Gamma_n \subset \partial\Omega$ are arcs converging to 1 and $J_n = \varphi^{-1}(\Gamma_n)$; and $|f_n| \leq 1$ on $\mathbb{D} \setminus T_n$, where $T_n \subset \mathbb{D}$ is a set containing Γ_n whose diameter is roughly twice the length of Γ_n . Now suppose that $\eta: \mathbb{D} \rightarrow \mathbb{D}$ is any analytic map different from φ . Shapiro and Sundberg consider the sets $E_n = \{\zeta \in J_n : |\varphi(\zeta) - \eta(\zeta)| \geq c_n\}$ where c_n is approximately twice the diameter of T_n . They observe that for $\zeta \in E_n$ one has $\varphi(\zeta) \in \Gamma_n$ and $\eta(\zeta) \in \mathbb{D} \setminus T_n$. Therefore $|f_n \circ \varphi - f_n \circ \eta|^2 \geq c/m(J_n)$ on E_n . Since $f_n \rightarrow 0$ weakly, this yields the estimate

$$\|C_\varphi - C_\eta\|_e^2 \geq c \limsup_{n \rightarrow \infty} \frac{m(E_n)}{m(J_n)}.$$

Finally Shapiro and Sundberg show that $\limsup m(E_n)/m(J_n) = 1$, based simply on the fact that $\int_{\mathbb{T}} \log|\varphi - \eta| dm > -\infty$.

Our argument is just a minor adaptation of the one explained above. Suppose that $\eta: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic map different from ψ , and put $J'_n = \psi^{-1}(\Gamma_n)$ and $E'_n = \{\zeta \in J'_n : |\psi(\zeta) - \eta(\zeta)| \geq c_n\}$. Then $J'_n = \sigma^{-1}(J_n)$, and since σ is an inner function fixing the origin, we have $m(J'_n) = m(J_n)$. Thus, using the test functions f_n as before, we arrive at the estimate

$$\|C_\psi - C_\eta\|_e^2 \geq c \limsup_{n \rightarrow \infty} \frac{m(E'_n)}{m(J'_n)}.$$

The proof is now completed by using the same argument as Shapiro and Sundberg to show that the limit superior here equals 1. \square

Given the above example, it seems appropriate to close this section with the following general open problem.

Problem 5.4. *Determine all the non-compact composition operators in $\text{Comp}_K(\mathcal{H}^2)$.*

This problem might be quite hard. As a first step one could try to describe interesting subsets of $\text{Comp}_K(\mathcal{H}^2)$ that are larger than those provided by obvious modifications of our construction presented in Section 4. For instance, it would be instructive to know if the extremality condition (5.1) that was essential for the example provided by Proposition 5.3 can be relaxed.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE ZARAGOZA, PLAZA SAN FRANCISCO s/n,
50009 ZARAGOZA, SPAIN.

E-mail address: `eva@unizar.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE CÁDIZ, APARTADO 40, 11510 PUERTO REAL
(CÁDIZ), SPAIN.

E-mail address: `majose.gonzalez@uca.es`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI, PO BOX 68, FI-
00014 HELSINKI, FINLAND.

E-mail address: `pjniemin@cc.helsinki.fi`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF JYVÄSKYLÄ, PO BOX 35, FI-
40014 JYVÄSKYLÄ, FINLAND.

E-mail address: `saksman@maths.jyu.fi`