Self-duality and parity in non-abelian Lubin–Tate theory

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ABSTRACT. We give a geometric proof of a "parity-switching" phenomenon that occurs when applying the local Langlands and Jacquet–Langlands correspondence to a self-dual supercuspidal representation of $\mathrm{GL}(n)$ over a nonarchimedean local field. This turns out to reflect a duality property on the self-dual part of the ℓ-adic étale cohomology of the Lubin–Tate tower.

1. INTRODUCTION

Let F be a nonarchimedean local field and W_F' its Weil group. The local Langlands correspondence for GL_n in this setting ([LRS93], [HT01], [Hen00]) says that irreducible smooth admissible complex representations π of $\mathrm{GL}_n(F)$ are parameterized by representations $\mathrm{rec}_F(\pi)$ of $W_F := W_F' \times \mathrm{SL}_2(\mathbf{C})$ called Langlands parameters. The bijection rec_F is known to respect a number of natural properties; in particular, if $\pi^\vee$ is the dual representation of π, then $\mathrm{rec}_F(\pi^\vee) = \mathrm{rec}_F(\pi)^\vee$, where $\mathrm{rec}_F(\pi)^\vee$ is the dual of the Langlands parameter $\mathrm{rec}_F(\pi)$.

Smooth representations π of $\mathrm{GL}_n(F)$ that are self-dual—that is, representations such that $\pi \cong \pi^\vee$, sometimes called "self-contragredient" in the literature—-play a distinguished role in the geometric construction of global Galois representations and their relation to automorphic representations. In particular, most constructions realize such representations in the middle-dimensional étale cohomology of certain Shimura varieties and so such representations are self-dual with respect to the intersection pairing on cohomology (cf. [Tay04], introductions of [CHT08] and [CG]). Almost all known instances of the global Langlands correspondence for GL_n over number fields rely on such geometric methods and thus usually only apply over totally real or CM number fields and require strong self-duality or "polarization" conditions, such as being self-dual up to a twist by a character in the totally real case or being conjugate self-dual in the CM case. Local components of self-dual automorphic representations of GL_n are precisely the smooth self-dual representations of GL_n over various completions of the number field in question, and one can often use information gleaned from a local component at a prime v of a global field to extract v-local arithmetic information from the corresponding Galois representation. Indeed, better understanding this connection is a primary motivation for this article.

Self-dual representations ρ of any group on a vector space V over $\mathbf{C}$ preserve nondegenerate symmetric or alternating bilinear forms

$$B : V \times V \to \mathbf{C}.$$

If ρ is irreducible, the form B is unique up to scaling, and so we define the **sign** (a.k.a **parity**) of an irreducible self-dual representation ρ to be

$$c(\rho) = \begin{cases} +1, & \text{if } B \text{ is symmetric;} \\ -1, & \text{if } B \text{ is alternating;} \end{cases}$$

and say such a ρ is **orthogonal** if $c(\rho) = +1$ and **symplectic** if $c(\rho) = -1$. (For finite-dimensional representations of compact groups, $c(\rho)$ is often called the Frobenius–Schur indicator of ρ.)

If π is a self-dual smooth representation of $GL_n(F)$, then its Langlands parameter $\mathrm{rec}_F(\pi)$ is a self-dual representation of W_F. To ensure that $c(\mathrm{rec}_F(\pi))$ is well-defined, we need $\mathrm{rec}_F(\pi)$ to be irreducible; in other words, π needs to be a discrete series representation. Having assumed this, we ask the following natural question.

> **Question.** What is the relation between the signs $c(\pi)$ and $c(\mathrm{rec}_F(\pi))$? For example, given $c(\pi)$, can we determine $c(\mathrm{rec}_F(\pi))$?

D. Prasad and D. Ramakrishnan discovered that the relation is, surprisingly, controlled by the Jacquet–Langlands correspondence [PR12, Thm. B], as previously conjectured in [PR95]. Such a result is especially useful because the self-dual representations of division algebras over non-archimedean local fields are well-understood and have properties that make their signs easy to calculate [Pra99]. They proved their theorem by appealing to certain globalization theorems for smooth local representations of GL_n and by applying some general correspondences in the theory of automorphic representations. In contrast, our goal is to (i) give a "purely local" proof of this fact without recourse to such global methods, and (ii) give a geometric interpretation for the result. To state our results precisely, we first establish some additional notation.

Fix an integer $n \geq 1$. Let D be the central division algebra of invariant $1/n$ over our nonarchimedean local field F. To any irreducible discrete series representation π of $GL_n(F)$, the local Jacquet–Langlands correspondence attaches an irreducible smooth representation $JL(\pi)$ of $D^\times$ (cf. [Rog83], [Bad02]).

We can now state our main result.

Theorem 1.1. *Let π be an irreducible self-dual supercuspidal representation of $GL_n(F)$, where F is a nonarchimedean local field of arbitrary characteristic.*

(i) *If n is odd, then $\mathrm{rec}_F(\pi)$ is orthogonal.*

(ii) *If n is even, then*

$$c(\mathrm{rec}_F(\pi)) = -c(\mathrm{JL}(\pi)).$$

In other words, $\mathrm{rec}_F(\pi)$ is symplectic (resp. orthogonal) if and only if $\mathrm{JL}(\pi)$ is orthogonal (resp. symplectic).

Part (i) of the theorem is immediate, since skew-symmetric bilinear forms cannot exist on odd-dimensional vector spaces. However, the "parity flipping" result of part (ii) requires proof.

The proof is performed in the framework of non-abelian Lubin–Tate theory [Car90]. Roughly speaking, non-abelian Lubin–Tate theory says that the local Langlands and Jacquet–Langlands correspondences for supercuspidal representations appear in the ℓ-adic étale cohomology of a geometric object called the Lubin–Tate tower—a projective system of universal deformation spaces of a one-dimensional formal $\mathcal{O}_F$-module of height n with suitable choices of level structures. By using the cup product on cohomology and applying a result of Mieda that characterizes the degree of cohomology in which supercuspidal representations appear [Mie10], we obtain a perfect pairing

$$(\mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)) \times (\mathrm{JL}(\pi^\vee) \boxtimes \mathrm{rec}_F(\pi^\vee)) \to \mathbf{C}$$

for an irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$, which allows us to compare the signs $c(\mathrm{rec}_F(\pi))$ and $c(\mathrm{JL}(\pi))$ when π is self-dual and ultimately deduce our theorem.

The theorem here is less general than that of Prasad and Ramakrishnan in characteristic zero, since working in our geometric setting requires us to assume that our division algebras D have invariant $1/n$ and that we restrict ourselves to supercuspidal representations. On the other hand, the geometric method has the benefit of establishing the result in positive characteristic.

The corresponding result in the case of conjugate self-dual supercuspidals over quadratic extensions of nonarchimedean local fields that are at worst tamely ramified has already been done by Y. Mieda [Mie16]. Additionally, the idea of proving results like this via non-abelian Lubin–Tate theory seems to have been known to L. Fargues. For example, the statement of the theorem appears without proof in [Far06, §5] as part of a much larger program for studying the interactions between different kinds of dualities that arise in this context. Our goals are much more modest: we concentrate on our particular result, developing only enough machinery to give a proof of our main theorem.

Notation and Conventions. For a nonarchimedean local field F, we let $\mathcal{O}_F$ denote the ring of integers and $\mathfrak{m}_F$ its maximal ideal. We assume that the residue field $\mathcal{O}_F/\mathfrak{m}_F$ is of characteristic $p > 0$. All representations of groups are assumed to be on vector spaces over $\mathbf{C}$.

2. RESULTS FROM NON-ABELIAN LUBIN–TATE THEORY.

Our main theorem rests on results from non-abelian Lubin–Tate theory, which gives a geometric realization of the local Langlands correspondence for $GL_n(F)$ as well as the Jacquet–Langlands correspondence to $D^\times = GL_1(D)$ for the central division algebra D over F of invariant $1/n$, provided that we restrict ourselves to supercuspidal representations of $GL_n(F)$. We only cite the theorems needed for our proof and refer the reader to [Car90], [Boy99], [HT01] for details. In our exposition, we follow the abbreviated presentation in [Mie16, §3.1]. Let F be a non-archimedean field and let ϖ be a uniformizer. Fix a separable closure $\overline{F}$ of F and let F^{ur} be the maximal unramified extension of F in $\overline{F}$. Write $\breve{F}$ for the completion of F^{ur}.

Remark 2.1. In our proof, we use the tower constructed from the "Lubin–Tate" moduli problem, as opposed to the "Drinfeld" moduli problem, where the roles of $GL_n(F)$ and $D^\times$ are switched. Both towers recover essentially identical results at the level of representations, but must appeal to different cohomology theories to linearize the group actions (see e.g. [Dat07, §1] for a discussion of the precise relation between the two).

2.1. **A summary of non-abelian Lubin–Tate theory.** The Lubin–Tate moduli problem describes the deformations of a 1-dimensional height n formal $\mathcal{O}_F$-module over an algebraic closure of the residue field $\mathcal{O}_F/\mathfrak{m}_F$ to $\breve{F}$. It is represented by a formal power series, which when viewed as a formal scheme has a rigid analytic generic fiber whose cohomology comes with an action of a trio of groups that decompose the cohomology into components that correspond to specific cases of the local Langlands and Jacquet–Langlands correspondence. We recall the details that are relevant for our proof, mostly for the purpose of establishing notation.

Let $\underline{\mathrm{Nilp}}$ be the category of $\mathcal{O}_{\breve{F}}$-schemes on which ϖ is locally nilpotent. For an object S of $\underline{\mathrm{Nilp}}$, let $\phi_S : S \to \mathrm{Spec}\,\mathcal{O}_{\breve{F}}$ denote the structure morphism. Set $\overline{S} = S \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{F}}/\mathfrak{m}_{\breve{F}}$. Recall that a **formal $\mathcal{O}_F$-module** over S is a formal group X over S endowed with an $\mathcal{O}_F$-action $\iota : \mathcal{O}_F \to \mathrm{End}(X)$ such that the following two actions of $\mathcal{O}_F$ on the Lie algebra $\mathrm{Lie}(X)$ coincide:

- the action induced by ι, and
- the action induced by the $\mathcal{O}_S$-module structure of $\mathrm{Lie}(X)$ and the structure homomorphism $\mathcal{O}_F \to \mathcal{O}_{\breve{F}} \to \mathcal{O}_S$.

Fix a one-dimensional $\mathcal{O}_F$-module $\mathfrak{X}$ of $\mathcal{O}_F$-height n over $\overline{\mathbf{F}_q} = \mathcal{O}_{\breve{F}}/\mathfrak{m}_{\breve{F}}$. Such an $\mathfrak{X}$ is unique up to isomorphism. Set

$$D = \mathrm{End}_{\mathcal{O}_F}(\mathfrak{X}) \otimes_{\mathbf{Z}} \mathbf{Q},$$

which is known to be a central division algebra over F with invariant $1/n$.

Let $\mathcal{M} : \underline{\mathrm{Nilp}} \to \underline{\mathrm{Set}}$ be the functor that sends S to the set of isomorphism classes of pairs (X, ρ), where X is a formal $\mathcal{O}_F$-module over S and $\rho : \phi_{\overline{S}}^* \mathfrak{X} \to$

$X \times_S \overline{S}$ is a quasi-isogeny over $\mathcal{O}_F$. The functor $\mathcal{M}$ is represented by a formal scheme over $\mathcal{O}_{\breve{F}}$, which is non-canonically isomorphic to a disjoint union of countably many copies of $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}[\![T_1, \ldots, T_n]\!]$ (cf. [LT66], [Dri74], [RZ96]).

Given a one-dimensional formal $\mathcal{O}_F$-module $\mathfrak{X}$, the group of self-quasi-isogenies

$$\underline{\mathrm{QIsog}}_{\mathcal{O}_F}(\mathfrak{X}) = D^\times$$

acts naturally on the right of $\mathcal{M}$: namely, $h \in D^\times$ maps (X, ρ) to $(X, \rho \circ \phi_{\overline{S}}^* h)$.

The formal scheme $\mathcal{M}$ is also endowed with the structure of a Weil descent datum, that is, an isomorphism $\alpha : \mathcal{M} \to \mathcal{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\alpha} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, \mathcal{O}_{\breve{F}} & \xrightarrow{\sigma^*} & \mathrm{Spf}\, \mathcal{O}_{\breve{F}}. \end{array}$$

Here, $\sigma : \mathcal{O}_{\breve{F}} \to \mathcal{O}_{\breve{F}}$ is induced from the unique element of $\sigma \in \mathrm{Gal}(F^{ur}/F)$ lifting the arithmetic Frobenius automorphism $\overline{\sigma} : \overline{\mathbf{F}_q} \to \overline{\mathbf{F}_q}$. In order to describe this isomorphism, it suffices to construct a bijection $\alpha : \mathcal{M}(S) \to \mathcal{M}(S^\sigma)$ for each $S \in \underline{\mathrm{Nilp}}$ compatibly, where S^σ denotes the object $S \xrightarrow{\phi_S} \mathrm{Spec}\, \mathcal{O}_{\breve{F}} \xrightarrow{\sigma^*} \mathrm{Spec}\, \mathcal{O}_{\breve{F}}$ of $\underline{\mathrm{Nilp}}$. For $(X, \rho) \in \mathcal{M}(S)$, we define

$$\alpha(X, \rho) = (X, \rho \circ \phi_{\overline{S}}^* \mathrm{Frob}_{\mathfrak{X}}^{-1}),$$

where $\mathrm{Frob}_{\mathfrak{X}} : \mathfrak{X} \to (\overline{\sigma}^*)^* \mathfrak{X}$ denotes the q-th power Frobenius morphism, which is an $\mathcal{O}_F$-isogeny of $\mathcal{O}_F$-height 1. It is through the descent datum α that we obtain the necessary action of the Weil group W_F on cohomology.

We now consider the same moduli problem together with a choice of level structure. For an integer $m \geq 0$, let $\mathcal{M}_m : \underline{\mathrm{Nilp}} \to \underline{\mathrm{Set}}$ be the functor that sends S to the set of isomorphism classes of triples (X, ρ, η), where $(X, \rho) \in \mathcal{M}(S)$,and η is a Drinfeld level-m structure on X (cf. [Dri74, §4], [HT01, §II.2]). It is represented by a formal scheme that is finite and flat over $\mathcal{M}$, and $\{\mathcal{M}_m\}_{m \geq 0}$ forms a projective system called the **Lubin–Tate tower**. The action of $D^\times$ and the Weil descent datum on $\mathcal{M}$ naturally extend to $\mathcal{M}_m$, and they are compatible with the transition morphisms of the tower. Furthermore, the group $GL_n(F)$ acts on $\{\mathcal{M}_m\}_{m \geq 0}$ on the right as a pro-object (cf. [Str08, §2.2]). The principal congruence subgroup $K_m = \mathrm{Ker}(GL_n(\mathcal{O}_F) \to GL_n(\mathcal{O}_F/\mathfrak{m}_F^m))$ of $GL_n(F)$ acts trivially on $\mathcal{M}_m$.

By taking the rigid generic fiber M_m of the formal schemes representing $\mathcal{M}_m$, we obtain a projective system $\{M_m\}_{m \geq 0}$ of rigid analytic spaces, whose transition maps are finite and étale. Each M_m is an $(n-1)$-dimensional smooth rigid space over $\breve{F}$. For a compact open subgroup K of $GL_n(\mathcal{O}_F)$, we can define the rigid space M_K as the quotient

$$M_K := M_m/(K/K_m),$$

where $m \geq 0$ is an integer such that $K_m \subset K$. This is independent of the choice of m, and M_{K_m} coincides with M_m. These rigid spaces parametrized by compact open subgroups form a projective system $\{M_K\}_{K \subset GL_n(\mathcal{O}_F)}$ with finite étale transition maps. The actions of $D^\times$ and $GL_n(F)$, as well as the Weil descent datum, extend naturally to the system $\{M_K\}_{K \subset GL_n(\mathcal{O}_F)}$.

Now, let $\Gamma = \varpi^{d\mathbf{Z}}$ (where $d \in \mathbf{Z}_{\geq 1}$) be a discrete torsion-free cocompact subgroup of $F^\times$. Since $F^\times \subset D^\times$, we can consider Γ as a discrete subgroup of the division algebra $D^\times$. The actions of $GL_n(F)$ on these towers are trivial on $\Gamma \subset F^\times \subset GL_n(F)$ [RZ96, Lem. 5.36].

Let $\ell \neq p$ be a rational prime. We will denote the (compactly supported) ℓ-adic étale cohomology of the Lubin–Tate tower as follows:

$$H^i_{\Gamma,c} := \varinjlim_K H^i_c((M_K/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbf{Q}_\ell}),$$

where $\widehat{\overline{F}}$ is a completion of the algebraic closure $\overline{F}$. We also have the non-compactly supported analogue

$$H^i_\Gamma := \varinjlim_K H^i((M_K/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbf{Q}_\ell}).$$

The groups $GL_n(F)$ and $D^\times$ act on $H^i_{\Gamma,c}$. The action of $GL_n(F)$ on $H^i_{\Gamma,c}$ is smooth and admissible (cf. [Mie10, Thm. 3.5]). The action of $D^\times$ on $H^i_{\Gamma,c}$ is also smooth (cf. [Str08, Lem. 2.5.1]). By using the Weil descent datum α, we can define the actions of W_F on $H^i_{\Gamma,c}$ and H^i_Γ as follows. For $w \in W_F$, let $\nu(w)$ denote the integer satisfying $w|_{F^{ur}} = \sigma^{\nu(w)}$. By taking the fiber product of the diagrams

$$\begin{array}{ccc}
\mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}}) & \xrightarrow{w^*} & \mathrm{Spa}(\widehat{\overline{F}}\mathcal{O}_{\widehat{\overline{F}}}) \\
\downarrow & & \downarrow \\
\mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}) & \xrightarrow{(\sigma^*)^{\nu(w)}} & \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}),
\end{array}$$

and

$$\begin{array}{ccc}
M_K/\Gamma & \xrightarrow{\alpha^{\nu(w)}} & M_K/\Gamma \\
\downarrow & & \downarrow \\
\mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}) & \xrightarrow{(\sigma^*)^{\nu(w)}} & \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}})
\end{array}$$

we obtain an isomorphism $\alpha_w : (M_K/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}} \to (M_K/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}$ of adic spaces. The action of w is defined to be α_w^*. By these constructions, we obtain two representations $H^i_{\Gamma,c}$ and H^i_Γ of $W_F \times GL_n(F) \times D^\times$.

Recall that any irreducible representation V of $GL_n(F)/\Gamma$ admits a canonical decomposition

$$V = \left(\bigoplus_\pi V[\pi] \right) \oplus V_{non-cusp},$$

where π runs over the set of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ whose central characters are trivial on Γ, where the π-isotypic component $V[\pi]$ is a direct sum of finitely many copies of π, and the summand $V_{\mathrm{non-cusp}}$ contains no supercuspidal subquotient [Ber84, 1.11, Variantes (c)].

Let $(\frac{n-1}{2})$ denote the twist by the character $W_F \to \mathbf{C}^\times$ defined by $w \mapsto q^{\nu(w)\cdot(n-1)/2}$. In the following theorem, we summarize the main results of non-abelian Lubin–Tate theory that we require in our proof.

Theorem 2.2. ([Boy99], [HT01], [Mie10]) *Let π be an irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$ whose trivial character is trivial on Γ. As a $W_F \times \mathrm{GL}_n(F) \times D^\times$-module, we have*

$$H^{n-1}_{\Gamma,c}[\pi]\left(\tfrac{n-1}{2}\right) = \mathrm{rec}_F(\pi)^\vee \boxtimes \pi \boxtimes \mathrm{JL}(\pi)^\vee,$$

and the cohomology vanishes in all other degrees.

2.2. **Pairings on H^{n-1}.** In the proof of the main theorem, we need some precise results on pairings in the cohomology of the Lubin–Tate tower, and we collect them here.

The first result was obtained in [Mie10], in the course of the proof of the vanishing of the ℓ-adic cohomology of the Lubin–Tate tower outside of the middle degree.

Lemma 2.3. [Mie16, Thm. 3.2.] *For every integer i, the kernel and cokernel of the natural map*

$$H^i_{\Gamma,c} \to H^i_\Gamma,$$

when viewed as representations of $\mathrm{GL}_n(F)$, have no supercuspidal subquotient. In particular, if π is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ whose central character is trivial on Γ, the induced map on the π-isotypic components

$$H^i_{\Gamma,c}[\pi] \to H^i_\Gamma[\pi] \tag{2.4}$$

is an isomorphism.

For a compact open subgroup K of $\mathrm{GL}_n(\mathcal{O}_F)$, set

$$\mathrm{Tr}_K := (\mathrm{GL}_n(\mathcal{O}_F) : K)^{-1}\,\mathrm{Tr}_{M_K},$$

where Tr_{M_K} denotes the trace map

$$\mathrm{Tr}_{M_K} : H^{2(n-1)}_c((M_K/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbf{Q}_\ell})(n-1) \to \overline{\mathbf{Q}_\ell}.$$

It is easy to see that the maps Tr_K are compatible with the transition maps of the tower as K varies. Let Tr denote the homomorphism

$$\mathrm{Tr} : H^{2(n-1)}_{\mathrm{LT}/\Gamma,c}(n-1) \to \overline{\mathbf{Q}_\ell}$$

induced from the projective system of maps $\{\mathrm{Tr}_K\}_K$.

The following result is crucial; it is the primary technical tool for comparing the parity of $\mathrm{rec}_F(\pi)$ and $\mathrm{JL}(\pi)$ associated with a self-dual supercuspidal π of $\mathrm{GL}_n(F)$.

Lemma 2.5. *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ whose central character is trivial on Γ. The cup product pairing*

$$\mathrm{Tr}(-\cup-) : H^{n-1}_{\Gamma,c}(\tfrac{n-1}{2}) \times H^{n-1}_{\Gamma,c}(\tfrac{n-1}{2}) \to \overline{\mathbf{Q}_\ell},$$

induces a $W_F \times D^\times$-invariant pairing

$$(\,,\,) : H^{n-1}_{\Gamma,c}[\pi^\vee](\tfrac{n-1}{2}) \times H^{n-1}_{\Gamma,c}[\pi](\tfrac{n-1}{2}) \to \overline{\mathbf{Q}_\ell}$$

that satisfies the following property: for every compact open subgroup K of $\mathrm{GL}_n(F)$, the restriction of $(\,,\,)$ to the space of K-invariants

$$(\,,\,) : (H^{n-1}_{\Gamma,c}[\pi^\vee])^K(\tfrac{n-1}{2}) \times (H^{n-1}_{\Gamma,c}[\pi])^K(\tfrac{n-1}{2}) \to \overline{\mathbf{Q}_\ell}$$

is a perfect pairing.

Proof. By Poincaré duality for M_K/Γ, we know that the cup product pairing

$$(H^{n-1}_{\Gamma,c})^K(\tfrac{n-1}{2}) \times (H^{n-1}_{\Gamma})^K(\tfrac{n-1}{2}) \to \overline{\mathbf{Q}_\ell}$$

is perfect for every compact open subgroup K of $\mathrm{GL}_n(\mathcal{O}_F)$. Thus, the induced map

$$H^{n-1}_{\Gamma}(\tfrac{n-1}{2}) \to (H^{n-1}_{\Gamma,c}(\tfrac{n-1}{2}))^\vee$$

is an isomorphism. By taking π-isotypic parts and composing with the natural isomorphism (2.4), we obtain an isomorphism

$$H^{n-1}_{\Gamma,c}[\pi](\tfrac{n-1}{2}) \xrightarrow{\sim} H^{n-1}_{\Gamma}[\pi](\tfrac{n-1}{2}) \xrightarrow{\sim} (H^{n-1}_{\Gamma,c}[\pi^\vee](\tfrac{n-1}{2}))^\vee.$$

Therefore, for every compact open subgroup K of $\mathrm{GL}_n(F)$, we have an isomorphism

$$(H^{n-1}_{\Gamma,c}[\pi])^K(\tfrac{n-1}{2}) \xrightarrow{\sim} ((H^{n-1}_{\Gamma,c}[\pi^\vee])^K(\tfrac{n-1}{2}))^\vee.$$

It is easy to see that this isomorphism is induced from the restriction of the cup product pairing to $(H^{n-1}_{\Gamma,c}[\pi^\vee])^K(\frac{n-1}{2}) \times (H^{n-1}_{\Gamma,c}[\pi])^K(\frac{n-1}{2})$, giving us our desired result. □

3. Self-dual Representations

We recall some simple results on self-dual representations that we use in our proof.

Lemma 3.1. *A self-dual irreducible representation (π, V) of a group G is orthogonal if there exists a subgroup $H \subset G$ such that $\pi|_H$ is completely reducible and contains the trivial representation 1_H of H with multiplicity one.*

Proof. The unique non-degenerate bilinear form on (π, V) must be nonzero on the 1-dimensional subspace corresponding to 1_H and so must be symmetric. □

This has the following natural corollary, which applies to admissible representations of $\mathrm{GL}_n(F)$ that are *generic*, in the sense that they admit a Whittaker model (cf. [JPSS81b, §1.2]).

Corollary 3.2. *Every irreducible admissible self-dual* generic *representation* (π, V) *of* $\mathrm{GL}_n(F)$ *is orthogonal.*

Proof. By the theory of new vectors for generic representations of $\mathrm{GL}_n(F)$ (cf. [JPSS81b], [JPSS81a], [Mat13]), there exists an open compact subgroup $K \subset \mathrm{GL}_n(F)$ such that $\dim V^K = 1$. By admissibility, the restriction $\pi|_K$ is completely reducible. □

Since every discrete series representation of $\mathrm{GL}_n(F)$ is generic [Zel80, Thm 9.7], we obtain the following conclusion.

Corollary 3.3. *Every irreducible admissible self-dual discrete series representation of* $\mathrm{GL}_n(F)$ *is orthogonal.*

4. Proof of Theorem

Let π be a self-dual supercuspidal representation of $\mathrm{GL}_n(F)$, where n is even. We want to prove that

$$c(\mathrm{JL}(\pi)) = -c(\mathrm{rec}_F(\pi)).$$

The representation $\mathrm{rec}_F(\pi)^\vee \boxtimes \pi \boxtimes \mathrm{JL}(\pi)^\vee$ of the triple product group $W_F \times \mathrm{GL}_n(F) \times D^\times$ occurs in H^{n-1} by Theorem 2.2. Since each of the component representations is self-dual, we write this representation as $\mathrm{rec}_F(\pi) \boxtimes \pi \boxtimes \mathrm{JL}(\pi)$, noting that taking duals of self-dual representations does not change the representation and thus does not affect the parity.

By Poincaré duality, we have the following non-degenerate pairing given by the cup product:

$$\langle , \rangle : H^{n-1} \times H_c^{n-1} \to H_c^{2(n-1)},$$

which is equivariant under the action of $W_F \times \mathrm{GL}_n(F) \times D^\times$. Since (i) the representations $\mathrm{rec}_F(\pi)$, π, and $\mathrm{JL}(\pi)$ are all self-dual; (ii) H^{n-1} is multiplicity-free; and (iii) supercuspidals do not intertwine with other representations, the pairing $\langle , \rangle$ induces a non-degenerate bilinear form on the π-isotypic components $H_{\Gamma,c}^{n-1}[\pi](\frac{n-1}{2})$:

$$B : (\mathrm{rec}_F(\pi) \boxtimes \pi \boxtimes \mathrm{JL}(\pi)) \times (\mathrm{rec}_F(\pi) \boxtimes \pi \boxtimes \mathrm{JL}(\pi)) \to H^{2(n-1)},$$

which takes values in a one-dimensional subspace of $H^{2(n-1)}$ on which all three groups act by the trivial representation, since $\det(\mathrm{rec}_F(\pi))$ is trivial. The bilinear form B is skew-symmetric, since it is induced from the intersection pairing on H^{n-1} and $n-1$ is odd. Hence, $\mathrm{rec}_F(\pi) \boxtimes \pi \boxtimes \mathrm{JL}(\pi)$ is a symplectic representation of $W_F \times \mathrm{GL}_n(F) \times D^\times$.

Since every self-dual supercuspidal representation of $\mathrm{GL}_n(F)$ is generic, π is orthogonal (Cor. 3.3), that is, $c(\pi) = +1$. Since the bilinear form B on $\mathrm{rec}_F(\pi) \boxtimes \pi \boxtimes \mathrm{JL}(\pi)$ is skew-symmetric, this implies that

$$c(\mathrm{rec}_F(\pi))c(\mathrm{JL}(\pi)) = -1.$$

In other words, $\mathrm{rec}_F(\pi)$ is orthogonal (resp. symplectic) if and only if $\mathrm{JL}(\pi)$ is symplectic (resp. orthogonal). This concludes our proof.

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