Local Properties of Triangular Graphs

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Abstract. In the paper triangular graphs are discussed. The class of triangular graphs is of special interest as unifying basic features of complete graphs with trees and being used on many various occasions. They model processor networks in which local computations can be easily implemented. The main issue addressed in the paper is to characterize class of triangular graphs (defined globally) by local means. Namely, it is proved that any triangular graph can be constructed from a singleton by successive extensions with nodes having complete neighborhoods. Next, the proved theoretical properties are applied for designing some local algorithms: for elections a leader and for constructing spanning trees of triangular graphs. The fairness of these algorithms is proved, which means that any node can be elected and any spanning tree can be obtained due to application of these algorithms.

Keywords: triangular graphs, local computations, leader elections, spanning trees.

1 Introduction.

Graphs serve as a natural theoretical model of computational networks. Some features of networks can be formulated and discussed via graphs of special class, like trees, circuits, complete graphs, and others. Essential feature of computational networks is a possibility of performing distributed algorithms, enabling processors to perform independent computations and to communicate with others via links joining network nodes. Among different classes of graphs triangular graphs are of special interest, combining basic features of trees and complete graphs. Triangular graphs are defined as graphs in which all cycles with more than 3 nodes have diagonals. Triangular graphs can model networks more reliable than trees but less expensive than complete graphs; it justifies their practical value. Viewing graphs as models of computational networks, the issue of local extension or local reduction of graphs is of particular interest; in case of widespread networks the only changes that can be made are local. On the other hand, triangular graphs offer some theoretical means for more theoretical reasons, as structures of dependency relation discussed in [6].

Notions concerning graphs and their properties have local or global character (see [1]). The neighborhood relation and the neighborhood set is local; to
check whether two nodes are neighbors or not the whole graph need not to be known. However, the notion of path, cycle, connectivity, and all derived from them are global; one cannot decide whether two nodes are connected or not without knowing the whole graph. In general, a property of a graph is \( k \)-local, if it can be decided by inspecting subgraphs of \( G \) with diameter at most \( k \). Similarly, an operation on a graph is \( k \)-local, if it is defined for graphs with diameter (the maximum distance between nodes) at most \( k \) and then extended to arbitrary graphs. An algorithm transforming graphs (or labels assigned to their elements) is local, if any of its action is local. In what follows we shall be concentrated on 1-local properties and operations, calling them simply \( \text{local} \). More facts about local computations and the locality notions the reader can find in [5].

The class of triangular graphs, defined globally, is a broad class of graphs representing reliable processing network. For such networks a possibility of local extensions (or reduction) play an essential part, provided such operations preserve global properties of transformed graphs. In our setup the nature of networks nodes is irrelevant; the only local property of the network nodes (represented here by nodes of a graph) is their neighborhoods structure; other properties are either global or irrelevant in our abstract approach. The main issue addressed in the present paper is to construct triangular graphs by a sequence of local extensions. Namely, we shall prove the theorem to the extent that the class of triangular graphs is the least class of graphs containing all singletons and closed with respect to extensions by nodes meeting some local properties.

The standard mathematical notation is used through the paper. The union, intersection, and difference of sets \( X, Y \) are denoted by \( X \cup Y, X \cap Y, \) and \( X/Y \), respectively. If it causes no ambiguity, we shall frequently omit braces around singletons, writing e.g. \( X \cup x, X \cap x, X/x \) rather than \( X \cup \{x\}, X \cap \{x\}, X/\{x\} \). The cardinality of set \( X \) is denoted by \( |X| \).

2 Graphs.

Basic notions from graph theory can be found e.g. in [2]. Here, by a graph we understand a finite set (of its \textit{nodes}) together with a family of two element sets of nodes, called its \textit{edges}. If \( G \) is a graph, \( V_G \) will denote the set of its nodes and \( E_G \) the set of its edges. We shall write \( x \in G, X \subseteq G \) rather than \( x \in V_G, X \subseteq V_G \). Here, we say that graph \((V', E')\) is a \textit{subgraph} of graph \((V'', E'')\) and write \((V', E') \subseteq (V'', E'')\), if \( V' \subseteq V'' \) and \( E' = E' \cap 2^{V''} \). It means that a subgraph of a graph is uniquely determined by its set of nodes. Results of set-theoretical operations on subgraphs are then induced by corresponding operations on their sets of nodes. Graph \( G \) is a \textit{singleton}, if it has precisely one node. The number of nodes of a graph is its \textit{size}. Graphs of size greater than 1 are \textit{non-trivial}. Nodes \( x, y \) are \textit{incident} to edge \( e \) (and edge \( e \) is incident to nodes \( x, y \)), if \( e = \{x, y\} \). Say that \( x \) and \( y \) are \textit{neighbors} (in \( G \)), if \( \{x, y\} \in E_G \), (then each of them is said to be a \textit{neighbor} of the other); otherwise they are said to be \textit{separated}. For any
node $x \in G$ the subgraph induced by $\{y \mid \{x, y\} \in E_G\}$ is denoted by $\gamma(G, x)$ and called the neighborhood of $x$. Two sets $X, Y \subseteq G$ are separated, if each $x \in X$ and each $y \in Y$ are separated. A graph is complete (or, equivalently, a clique), if any two different nodes of it are neighbors; otherwise it is incomplete. A node of a graph is perfect, if its neighborhood is a clique. A clique is nontrivial, if it contains at least 3 nodes. The diameter of a set of nodes is the maximum length of all shortest paths joining nodes of this set.

Set $P$ of nodes is a path connecting $x$ with $y$ (or between $x$ and $y$), if $P = \{x_1, x_2, \ldots, x_n\}$ with $n > 0$, where $x_1 = x, x_n = y$ and $x_i$ is a neighbor of $x_{i+1}$ for all $1 \leq i < n$. Number $n$ is the length of $P$. Notice that there can be a number of different paths connecting the same pair of nodes. A set $X$ of nodes is connected, if for any two nodes of $X$ there is a path connecting them. A set which is not connected is said to be disconnected. A graph is connected, if the set of all its nodes is connected, otherwise it is disconnected. For any $X \subseteq G$, any maximum connected subgraph of $X$ is its (connected) component.

Set $C$ of nodes is a cycle, if $C = \{x_1, x_2, \ldots, x_n\}$ with $n > 2$, $x_i$ is a neighbor of $x_{i+1}$ for all $1 \leq i < n$, and $x_1$ is a neighbor of $x_n$. Number $n$ is the length of $C$. If, moreover, $x_1$ is a neighbor of $x_j$ for $2 < j < k$, pair $\{x_1, x_j\}$ is a diagonal of $C$. A cycle is simple, if it does not contain any other cycles. A cycle of length 3 is a triangle. A cycle of length 4 is a square. A cycle of length greater than 3 is non-trivial.

3 Triangular graphs.

Graph is triangular if it is connected and each of its non-trivial cycle has a diagonal. In particular, any tree is triangular (as containing no cycles at all) as well as any complete graph (as containing all possible diagonals of any cycle). Triangular graphs, generalizing trees and complete graphs, are worth to be discussed for theoretical as well as practical reasons. Observe that the definition of triangularity is global: to check whether a graph meets the above definition or not, the whole graph should be known. Proposition 1 gives another, alternative definition of triangular graphs.

**Proposition 1.** Connected graph $G$ is triangular if and only if each cycle in $G$ contains a triangle.

**Proof.** If cycle $C$ with $|C| > 3$ contains a triangle, it has obviously a diagonal. Assume cycle $C$ has a diagonal. If length of $C$ is $n = 3$, it clearly contains a triangle, namely itself. Let $n > 3$. Assume that each cycle in $G$ of length less than $n$ contains a triangle and let $C = \{x_1, x_2, \ldots, x_n\}$ be a cycle such that pair $\{x_1, x_k\}$ with $2 < k < n$ is a diagonal of $C$. Then $C' = \{x_1, x_2, \ldots, x_k\}$ is also a cycle in $G$ and the length of $C'$ is less than $n$. By assumption, $C'$ contains a triangle, hence all the more $C$ contains a triangle. By induction, any cycle with a diagonal contains a triangle. From this it follows that $G$ is triangular if and only if each cycle in $G$ contains a triangle. \qed
Proposition 2. Connected subgraphs of triangular graphs are triangular.

Proof. If connected subgraph of triangular graph were not triangular, this subgraph would contain a non-trivial cycle without a diagonal; then the whole graph would contain the same cycle without a diagonal. □

Let $G$ be a non-empty graph. Set $X \subseteq G$ is separating $G$, if $G/X$ is non-empty and disconnected; any minimum set separating $G$ is a cut of $G$. Clearly, a graph can have a number of different cuts. If $X$ is separating $G$, we also say that $G$ is separated by $X$.

Proposition 3. Any incomplete and non-empty graph has a cut.

Proof. Let $G$ be a graph. If $G$ is non-empty and disconnected, the empty set is a cut. If $G$ is non-empty and incomplete, it contains two separated nodes, say $x$ and $y$. Set $X = G/\{x, y\}$; then $G/X$ is non-empty, as containing nodes $x$, and $y$, and disconnected, since $x$ is not connected with $y$ in $G/X$. Thus, the family of all sets $X$ separating $G$ is non-empty; any minimum member of this family is a cut of $G$. □

Fig. 1 Example of a triangular graph. Sets $\{4\}$, $\{5\}$, $\{6, 7\}$ are cuts; set $\{1, 4\}$ is separating, but not a cut (as containing smaller separating set, namely $\{4\}$).

Lemma 4. Let $X$ be a cut of $G$ and $H$ be a connected component of $G/X$. Then each $x \in X$ has a neighbor in $H$.

Proof. Let $X$ be a cut of $G$ and $H$ be a connected component of $G/X$. By way of contradiction, suppose $x$ is a node of $X$ without any neighbor in $H$. In such a case set $Y = X/x$ would be also separating $G$. Indeed, any node of $H$ would not be connected with any node of $G/Y = (G/X) \cup x$. It would mean that $Y$ is also separating $G$. This contradiction to the assumed minimality of $X$ proves the lemma. □

Proposition 5. Any cut of a triangular graph is a clique.
Proof. Let $G$ be a triangular graph and $X$ be a cut of $G$ such that $x', x'' \in X$. Let $Y', Y''$ be different connected components of $G/X$. By Lemma 4 there are neighbors $y'_1, y''_1$ of $x'$ such that $y'_1 \in Y', y''_1 \in Y''$ and neighbors $y'_2, y''_2$ of $x''$ such that $y'_2 \in Y', y''_2 \in Y''$. Since $Y', Y''$ are connected components, there are paths between $y'_1$ and $y'_2$ in $Y'$ and a path between $y''_1$ and $y''_2$ in $Y''$. It means that there is a cycle $C$ in $G$ containing nodes $x', x'', y'_1, y''_1, y'_2, y''_2$. We prove that there are neighbors $y', y''$ of $x'$ as well as of $x''$ such that $y' \in Y', y'' \in Y''$ and that $x', x''$ are neighbors. Suppose the contrary, i.e. that either there is no such neighbors $y', y''$, or $x', x''$ are not neighbors. In the first case there would be a cycle contained in $C$ with not less than 4 nodes and no diagonal, regardless $x', x''$ are neighbors or not:

![Diagram](image)

In the second case when there are neighbors $y', y''$ of $x'$ and of $x''$ such that $y' \in Y', y'' \in Y''$ but $x', x''$ are not neighbors, the square $x', y', x'', y''$ would have no diagonals, since $y', y''$ are not neighbors as members of disjoint connected components, and $x', x''$ are not neighbors by assumption:

![Diagram](image)

Thus, in any case, $G$ would not be triangular, a contradiction. This contradiction proves $x', x''$ to be neighbors and, consequently, that $X$ is a clique. 

Corollary 6. Any triangular graph is either complete, or separated by a clique.

Theorem 7. Each non-trivial triangular graph contains at least two different perfect nodes.

Proof. Let $G$ be a triangular graph. The proof will be carried out by induction on the size of $G$. If $G$ is a singleton, it is complete. Assume, as induction hypothesis, that the theorem holds for all triangular graphs of size less than $n$ and let the size of $G$ be $n > 1$. If $G$ is complete, the proof is over, since any complete graph with more than 1 node contains 2 different perfect nodes. Otherwise, by Corollary 6, there a complete graph $X$ which is a cut of $G$, with connected components $G', G''$ of $G/X$. Consider graphs $G_1 = G/G'', G_2 = G/G'$. Both of them are triangular, of size less than $n$, and their intersection is the clique $X$. By induction hypothesis graph $G_1$ contains different perfect nodes $x_1, y_1$ and graph $G_2$ contains different perfect nodes $x_2, y_2$. One of $x_1, y_1$ (say $x_1$) is not in $X$, since $X$ is a clique, and one of $x_2, y_2$ (say $x_2$) is not in $X$ for the same reasons. It means that nodes
$x_1, x_2$ are perfect in $G$ and different of each other, since they belong to different connected components of $G/X$. By induction, the proof is completed.

\[ \square \]

**Corollary 8.** Graphs with all nodes having at least two separated neighbors are not triangular.

Say that graph $G$ is an extension of graph $H$ by a node with property $P$, if there exists node $x \in P$ such that $H = G/x$. In particular, $G$ is an extension of $H$ by a perfect node, if there is $x \in G$ with complete neighborhood such that $H = G/x$.

**Theorem 9.** Class of triangular graphs is the least class of graphs containing all singletons and closed w.r. to the extensions with perfect nodes.

*Proof.* If $G$ is triangular, then either it is a singleton, or it has a perfect node $x$ such that $G/x$ is triangular. It proves that each triangular graph is generated from singletons by extensions with perfect nodes. Conversely, if a graph is a singleton, it is triangular; if a graph arises from another triangular graph by extending it with a perfect node, it remains triangular, since adding a node with a complete neighborhood does not introduce any cycle without a diagonal: all diagonals are supplied by the complete neighborhood. Thus, any graph generated from singletons by extension with perfect nodes is triangular. It completes the proof.

\[ \square \]

A question arises whether the requirement concerning introducing nodes by above extensions are too strong or not. In other words, can the limitation to nodes with complete neighborhoods be relaxed, e.g. to nodes with triangular neighborhoods or not. In Fig.2 a non-triangular graph which is an extension a triangular graph by a node with triangular (but not perfect) neighborhood is presented.

![Graph](image)

Example of a graph which is not triangular; it can arise from triangular graph $\{1, 2, 3, 5\}$ by introducing node 4 with neighborhood $\{1, 3, 5\}$ which is triangular, but not complete.
Definition 10. Node $x$ of a triangular graph $G$ external, if $\gamma(G, x)$ is connected.
Edge $\{x, y\}$ of triangular graph $G$ is external, if $\gamma(G, x) \cap \gamma(G, y)$ is a clique. □

From property formulated in Proposition 2 we get immediately the following:

Proposition 11. A graph resulting from a triangular graph by removing its external node is triangular. □

For any edge $e$ of $G$ denote by $G/e$ graph $(V_G, E_G/e)$.

Lemma 12. Let $G$ be a triangular graph and $e$ be an external edge of $G$. Then $G/e$ is triangular.

Proof. Let $G$ be a triangular graph, $e = \{x, y\}$ be an external edge in $G$. Suppose, by way of contradiction, that $G/e$ is not triangular. It means that $G/e$ is either disconnected, or contains a nontrivial cycle without diagonal. If $G/e$ were disconnected in effect of removing edge $e$, it would mean that $\gamma(G, x) \cap \gamma(G, y) = \emptyset$, contradicting externality of $e$. If $G/e$ would contain a nontrivial cycle without diagonal, this cycle should be a square, since otherwise $G$ would contain a nontrivial cycle without diagonal, against the assumed triangularity of $G$. If this cycle in $G/e$ is a square, say $\{x, z', y, z''\}$, not containing a diagonal, then nodes $z', z''$ would be separated; since both of them are in $\gamma(G, x) \cap \gamma(G, y)$, the neighborhood of $e$ would be not complete and, consequently, the edge $e$ would be not external. This contradiction proves the Lemma. □

Lemma 13. Let $G$ be triangular and $e$ be an edge of $G$ such that $G/e$ is triangular. Then $e$ is external in $G$.

Proof. Let $G$ be triangular, $e = \{x, y\}$ be an edge of $G$ such that $G/e$ is triangular. Consider subgraph $\gamma(G, x) \cup \gamma(G, y)$ of $G$. As a connected subgraph of $G$, it is triangular. Since $G/e$ is triangular, subgraph $H = (\gamma(G, x) \cup \gamma(G, y))/e = (\gamma(G, x)/y) \cup (\gamma(G, y)/x)$ is triangular either. Since $x$ and $y$ are nodes of $H$ and there is no edge $\{x, y\}$ in $H$, set $H_0 = \gamma(G, x) \cap \gamma(G, y)$ is separating $x$ from $y$; moreover, $H_0$ is the minimum set separating $x$ from $y$, since by removing from $H_0$ any node $H_0$ ceases its separation property. Thus, by Proposition 5, $H_0$ is a clique, which means that $e$ is external. It completes the proof. □

Theorem 14. Let $G$ be a triangular graph, $e$ be an edge of $G$. Then $G/e$ is triangular if and only if $e$ is an external edge of $G$.

Proof. It follows directly from Lemma 12 and Lemma 13. □
**Theorem 15.** Triangular graphs without external edges are trees.

*Proof.* Let $G$ be a triangular graph without external edges. If $G$ is a singleton, $G$ is a tree and the theorem is proved. If $G$ is not a tree, then it has a perfect node, say $x$. If $\gamma(G, x)$ is a singleton, $G/x$ is triangular and by induction hypothesis, a tree, therefore $G$ is a tree as well. If $\gamma(G, x)$ is not a singleton, any edge incident with $x$ is external, contradicting the assumptions about $G$. It completes the proof. □

**Proposition 16.** Any node of any triangular graph is incident to none or more than one external edge.

*Proof.* Let $G$ be a triangular graph, $x$ an arbitrary node of $G$. Consider subgraph $\gamma(G, x)$. If all connected components of $\gamma(G, x)$ are singletons, $x$ is not incident to any external edge. Let some of connected components be not singletons and let $X$ be one of them. Then $X$, as connected subgraph of a triangular graph, is triangular and by Theorem 7, $X$ contains two perfect nodes $x', x''$ of $X$. Since $X \subseteq \gamma(G, x)$, each of edges $\{x, x'\}, \{x, x''\}$ is external. Thus, $x$ is incident to at least two external edges. It completes the proof. □

Say that a set $Y \subseteq G$ is *cyclic*, if any node of $Y$ has at least two neighbors in $Y$. It is clear that any cyclic set contains a cycle. By definition, any tree does not contain any cyclic subgraph. From here it follows the following proposition.

**Proposition 17.** Let $G$ be a triangular graph, $T$ be a tree contained in $G$. Then there exists an external edge of $G$ not contained in $E_T$. □

To close this section, observe that the number of external edges in triangular graph is either 0 or not less than 3. It follows from the derivation property of triangular graphs formulated in Theorem 9, since the least triangular graph with external edges is a triangle.
4 Local algorithms in triangular graphs.

Algorithm is local, if execution of its step concerning a node (an edge) of the processed graph is depending solely on the neighborhood of this node (this edge) and the effect of such a step concerns this node exclusively. In algorithms described below only one type of actions is taken into account, namely removing of a node or an edge from the graph. Two basic local algorithms will be designed and discussed in this section election a leader and determining a spanning tree of triangular graphs. Consider an algorithm for election a leader from triangular graphs first, and then an algorithm for construction of a spanning tree for such graphs. Recall that external nodes are nodes with connected neighborhoods and external edges are edges with complete neighborhoods.

Algorithm L (for leader election) Data of the algorithm: any triangular graph $G$. Step of algorithm: remove an arbitrary perfect node from $G$. The algorithm terminates, if there is no external nodes in the graph.

Proposition 18. Algorithm $L$ applied to any triangular graph $G$ terminates with a singleton graph contained in $G$; moreover, any singleton graph contained in $G$ can be the result of a run of the algorithm.

Proof. Since removing from a triangular graph a node with the connected neighborhood results in a triangular graph, steps of algorithm preserve triangularity of the processed graph. Since after execution of a step the size of graph is reduced by one, any run terminates. Since any non-trivial triangular graph has a node with the connected neighborhood, the algorithm terminates with a singleton. Let now $x_0$ be an arbitrary node of the processed graph. Execute any step of the algorithm in such a way that $x_0$ is not removed; it is always possible, since any triangular graph contains at least two different perfect nodes, hence two different nodes with connected neighborhoods. Such an execution preserves $x_0$ contained in the graph. Thus, the algorithm terminates with the singleton $x_0$. 

In the definition of the step of algorithm $L$ the word "perfect" can be replaced with "external" (which means "having connected neighborhood"). Proposition 2 claims that removing a node with the connected neighborhood from a triangular graph results in a triangular graph, so the correctness of the algorithm is guaranteed with this substitution. Nevertheless, using perfect nodes in the above algorithm preserves its 1-locality.

Algorithm T (for spanning tree construction) Data of the algorithm: any triangular graph $G$. Step of algorithm: remove any external edge from $G$. The algorithm terminates, if there is no external edges in the graph.

Proposition 19. Algorithm $T$ applied to any triangular graph $G$ terminates with a spanning tree of $G$; moreover, any spanning tree of $G$ can be the result of a run of algorithm $T$. 

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Proof. If there is an external edge in the processed graph, by Theorem 14 the graph obtained by removing from it an external node is triangular. If no external nodes are in the processed graph, by Theorem 15 the graph itself is (the unique) its spanning tree. Since removing an edge of $G$ by one, the algorithm terminates with a tree $T$. Since any node of $G$ is a node of $T$ and any edge of $T$ is an edge of $G$, $T$ is a spanning tree of $G$. Let $T_0$ be an arbitrary spanning tree of $G$. Then proceed with the execution of $T$: if there are some external edges of $G$, then, by Proposition 17, there exists an external edge of $G$ which does not belong to tree $T_0$. Remove any such edge; in effect we get a triangular graph again, with all edges belonging to $T_0$. Thus, the final result of $T$ be a tree with all edges belonging to $T_0$, that is, the required tree $T_0$.

5 Final remarks.

In the paper triangular graphs, i.e. graphs in which any cycle with more than 3 nodes has a diagonal, or equivalently, graphs in which all cycles contain triangles, have been discussed. The main issue of the paper was to show that although the triangular graphs definition is global, a local definition is possible. Namely, one can obtain each of such graphs by successive local extensions starting from singletons and introducing new perfect points. It has been also shown that triangular graphs can be obtained from trees by successive introducing new external edges. Locality is here limited to subgraphs with diameter 1. It makes possible to define local algorithms, with the same degree of locality, one for election a leader out of graph nodes, the other for constructing a spanning tree for any triangular graph.

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