Well-covered graphs and extendability

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Received 19 December 1990
Revised 24 March 1992

Abstract
A graph is \(k\)-extendable if every independent set of size \(k\) is contained in a maximum independent set. This generalizes the concept of a B-graph (i.e. 1-extendable graph) introduced by Berge and the concept of a well-covered graph (i.e. \(k\)-extendable for every integer \(k\)) introduced by Plummer. For various graph families we present some characterizations of well-covered and \(k\)-extendable graphs. We show that in order to determine whether a graph is well-covered it is sometimes sufficient to verify that it is \(k\)-extendable for small values of \(k\). For many classes of graphs, this leads to efficient algorithms for recognizing well-covered graphs.

1. Introduction

A maximum independent set of vertices in a graph is a set of pairwise nonadjacent vertices of largest cardinality. The difficulty of finding a maximum independent set has motivated research in a variety of areas. Plummer called a graph well-covered if every independent set is contained in a maximum independent set [15]. A maximum independent set can be found for a well-covered graph by using the most naive greedy algorithm. Several researchers have shown that checking whether a graph is not well-covered is an NP-hard problem ([3, 20, 21]). Thus, one would not expect that a simple characterization of well-covered graphs exists. However, we give a characterization (Theorem 2.1) of well-covered graphs that implies the existence of a polynomial-time algorithm to test whether a graph is well-covered for various classes...
of graphs including perfect graphs of bounded clique size. Our characterization generalizes results of Berge [1], Favaron [7], Ravindra [18], and Staples [22].

We call a graph $k$-extendable if every independent set of size $k$ is contained in a maximum independent set. This generalizes the concept of a $B$-graph (i.e. 1-extendable graph) introduced by Berge and the concept of a well-covered graph. A fair amount of study has been devoted to $B$-graphs, for example, see [1, 10, 19]. In general, there is no connection between $k$-extendability and $j$-extendability for $k \neq j$, and there are graphs which are not well-covered and which are $k$-extendable for any given values of $k$. The examples in Fig. 1 show that $k$-extendability does not imply $(k-1)$-extendability or vice versa. We show in Section 4 that 2-extendable graphs are either 1-extendable or constructed in a simple way from a complete graph and a 1-extendable graph. Further, we show that for various classes of graphs that if a graph is $k$-extendable for small values of $k$, then it is $k$-extendable for all values of $k$, i.e., the graph is well-covered (see Theorems 2.1, 5.3–5.5, and Corollary 3.4).

We show that certain restrictions allow us to give nice characterizations of well-covered graphs and imply the existence of polynomial-time checks for being well-covered. These restrictions involve the cycles of a graph $G$ and its independence number $\alpha(G)$, the size of a maximum independent set. We consider the cases of: trees (no cycles), bipartite graphs (no odd length cycles), triangle-free graphs, $C_4$-free graphs (no induced four cycle), and graphs with girth at least five. Some examples of this are given in Theorems 4.2, 4.3, 4.5, 5.5, and Corollary 4.4.

The set of vertices of a graph $G$ is denoted by $V(G)$, and the number of vertices is called the order of the graph. A graph is called very well-covered if it is well-covered and $\alpha(G) = \frac{1}{2}|V(G)|$. For well-covered graphs with no isolated vertices (vertices with no neighbors), the size of a maximum independent set is at most half the number of vertices. Favaron and Staples have characterized very well-covered graphs with no isolated vertices. In Section 3 we show how their result follows easily from our characterization.

![Even cycles are 1-extendable, but not 2-extendable.](https://example.com/fig1a.png)

![Stars are 2-extendable, but not 1-extendable.](https://example.com/fig1b.png)

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**Fig. 1.** The set of circled vertices does not extend to a maximum independent set.
An independent set is maximal if no vertex can be added to the set without destroying independence. A maximum independent set is a maximal independent set of largest cardinality. A graph is well-covered if every maximal independent set is a maximum independent set. This means that well-covered graphs are those that are $k$-extendable for all $k$.

We say an independent set $S$ of vertices in a graph $G$ satisfies Hall's condition if $|S| \leq |N(S)|$, where the neighbor set $N(S)$ of $S$ is

$$N(S) - \{v \in (V(G) - S) : v \text{ is adjacent to } u \text{ for some } u \in S\}.$$

We say a graph satisfies Hall's condition if every independent set of the graph satisfies Hall's condition.

A graph is bipartite if its vertex set can be partitioned into two sets $X$ and $Y$, called bipartite classes, such that every edge of the graph joins a vertex of $X$ with a vertex of $Y$. A matching of the graph is a subset of edges that are pairwise-nonadjacent. A perfect matching is a matching that contains every vertex of the graph.

**Hall's theorem.** Let $G$ be a bipartite graph with bipartite classes $A$ and $B$. If every set of vertices in $A$ satisfies Hall's condition, then there is a matching from $A$ to $B$.

A pendant edge is an edge with an endpoint of degree one. The join of two graphs $G$ and $H$ is formed by adding an edge $(u, v)$ between every pair of vertices $u \in V(G)$, $v \in V(H)$. The induced subgraph $G[U]$ on a set of vertices $U \subseteq V(G)$ of a graph $G$ is the graph with vertex set $U$ and edge set given by $\{(u, v) \in E(G) : u \in U \text{ and } v \in U\}$. The induced subgraph $G[F]$ on a set of edges $F \subseteq E(G)$ of a graph $G$ is the graph with edge set $F$ and vertex set given by $\{v \in V(G) : v \text{ lies in an edge of } F\}$. We will use the notation $G[E(V_1, V_2)]$ to denote the induced subgraph on the set of edges that have one endpoint in vertex set $V_1$ and the other endpoint in vertex set $V_2$. We call a graph triangle-free if there is no set of three vertices which induce a triangle. We say a set of vertices is a clique or a complete graph if every pair of vertices in the set is adjacent. A clique cover is a set of cliques whose union contains all the vertices of the graph. A minimum clique cover is a cover with the least number of cliques possible. The clique cover number $\theta$ is the number of cliques in a minimum clique cover.

2. **Main theorem and complexity**

In this section we give general conditions for a graph to be well-covered which depend on the relationship between clique covers and independent sets. One obvious but useful observation is the following: an independent set can contain at most one vertex from each clique in a clique cover. We use this in the main theorem and several places throughout this paper. In Section 4 there are several results where the clique cover number is equal to the independence number. In this case an independent set is of maximum size if and only if it contains a vertex from each clique of a minimum cover.
In this section we consider the general case when the clique cover number is not necessarily equal to the independence number and the clique cover under consideration is not necessarily minimum. Our main theorem, Theorem 2.1, gives a characterization of well-covered graphs which implies that it suffices to check $k$-extendability for small values of $k$. Theorem 2.1 also gives a characterization of well-covered graphs which depends on a property of any clique cover of the graph. In Corollaries 2.2 and 2.3, we consider further restrictions which allow us to show the existence of polynomial-time algorithms to check whether a graph is well-covered. In particular, we are able to show that for perfect graphs of bounded clique size there is a polynomial-time algorithm to check whether a graph is well-covered.

**Theorem 2.1.** Let $C$ be a clique cover consisting of $t$ cliques of a graph $G$ with independence number $\alpha(G) = t - d$, for some nonnegative integer $d$. Then the following are equivalent:

1. $G$ is well-covered.
2. $G$ is $k$-extendable for all $k \in \{1, 2, \ldots, \min(h, \alpha)\}$, where $h$ is the sum of the orders of the $d+1$ largest cliques in $C$.
3. For every $d+1$ cliques $C_1, C_2, \ldots, C_{d+1}$ of the clique cover $C$ with vertex set $W = \bigcup_{i=1}^{d+1} V(C_i)$, there is no nonmaximum independent set $S$ of $G - W$ such that $|W| \geq |S|$ and $W \subseteq N(S)$.

**Proof.** (1)⇒(2): By definition.

(2)⇒(3): Assume that there is a nonmaximum independent set $S$ of $G - W$ such that $|W| \geq |S|$ and $W \subseteq N(S)$. Since $W = \bigcup_{i=1}^{d+1} V(C_i)$ and $|W| \geq |S|$, we have that $|S| \leq h$. We know that any maximum independent set of $G$ must contain a vertex from $\alpha(t - d)$ of the cliques in $C$. However, any independent set containing $S$ cannot contain any of the vertices in $W$ since $W \subseteq N(S)$. Hence, $S$ cannot be extended to a maximum independent set, contradicting (2).

(3)⇒(1): If $G$ is not well-covered then there exists a maximal independent set $M$ which is not maximum. From the clique cover $C$, create a clique cover $C' = \{C_1', C_2', \ldots, C_{d+1}'\}$ that gives a partition of the vertex set (e.g. $C_i' = C_i - \bigcup_{k=1}^{i-1} C_k$). In $C'$, there are $d+1$ members, say $C_1', C_2', \ldots, C_{d+1}'$, that contain no vertex of $M$. Since $M$ is maximal every vertex $w_i$ in $W = \bigcup_{i=1}^{d+1} V(C'_i)$ has a neighbor $v_i$ in $M$. Let $S = \{v_i\}$. Such an $S$ contradicts the conditions of (3). \hfill $\square$

One important consequence of Theorem 2.1 is that to determine whether a graph is well-covered, it is sufficient to find a single clique cover and check whether either condition (2) or condition (3) holds.

Also note that if the clique cover $C$ in Theorem 2.1 is replaced by any partition of the vertices such that every maximum independent set can contain at most one vertex from each $C_i$ then the theorem still holds. Furthermore, it is clear from the proof that to prove a graph is well-covered, it is sufficient to find any partition $C$ of the vertices satisfying condition (3).
Condition (3) of Theorem 2.1 gives an algorithm which allows a fast check on whether a graph is well-covered for many classes of graphs. For example, this is the case in graphs with a perfect matching where \( x \) is half the number of vertices.

*Is Theorem 2.1 best possible?*

Condition (2) of Theorem 2.1 is the best possible statement in the sense that given \( t, d, \) and \( x \), we can construct a non-well-covered graph which is \( k \)-extendable for all values except one value \( m \leq h \). We will describe how to construct such a graph \( G_{t, p, m} \). These graphs are illustrated in Fig. 2. Naturally, there are some constraints on the values of \( t, d, \) and \( x \). It is clear that \( t \geq x, t \leq |V| - x, \) and \( 0 \leq d \leq |V| - x \). Let \( G_{t, p, m} \) for \( m \leq l \) be the graph constructed by taking a complete graph \( H \) on \( l \) vertices and attaching paths of length 2 to \( m - 1 \) vertices of \( H \) and connecting the remaining

\[
G_{2,p,2}
\]

The set \( \{a_2, b_2\} \) does not extend to a maximum independent set.

\[
G_{3,p,2}
\]

The set \( \{a_2, b_2\} \) does not extend to a maximum independent set.

\[
G_{8,p,1}
\]

is formed by replacing each vertex of the \( K_8 \) by an independent set \( K_p \). The set \( \{a_2, b_2, c_2, d_2, e_2\} \) is not extendable to a maximum independent set.

Fig. 2. Examples showing Theorem 2.1(2) is best possible. The graph \( G_{t, p, m} \) is \( k \)-extendable for all \( k \neq m \).
vertices of $H$ to one end of a $K_2$. Now replace each vertex of $H$ by an independent set $K_p$ on $p$ vertices. In this replacement, an edge $(u,v)$ of $H$ will become a complete bipartite joining of the respective copies of $K_p$ and an edge between $u \in H$ and $w \notin H$ becomes a complete bipartite joining between the corresponding copy of $K_p$ and the vertex $w$. The graph $G_{t,p,m}$ has $|V| = lp + 2m$ vertices and independence number $x = m + p$. The graph $G_{t,p,m}$ is $k$-extendable for all $k$ except $k = m$. There exists a clique cover of size $t$ for $G_{t,p,m}$ for all values of $t$ in the range $x \leq t \leq |V| - x$. Hence, condition (2) of Theorem 2.1 is the least requirement on $k$-extendability possible to insure that a graph be well-covered.

**Complexity**

One question that arises is what is the computational complexity of determining whether a graph is well-covered. For the class of all graphs it is NP-hard to determine whether a graph is not well-covered [3, 20, 21]. For some classes of graphs the computational complexity of determining whether a graph is well-covered is known to be polynomial. Some examples are: graphs with no 4 or 5 cycle [8], graphs with girth $\geq 5$ [9], bipartite graphs [18], line graphs [13], and very well-covered graphs with no isolated vertices [7, 22].

We observe that a graph is well-covered if the size of a maximum independent set (the independence number) is equal to the size of the smallest maximal independent set (the independent domination number). If both of these numbers can be calculated in polynomial time, then this gives a polynomial-time check for whether or not a graph is well-covered. For general graphs the problems of determining the independence number and the independent domination number are both NP-complete. There are several families of graphs for which it is known that both the independence number and the independent domination number can be calculated in polynomial time. Some of these families are: graphs of bounded tree width (e.g. series-parallel graphs), chordal graphs (e.g. interval graphs and split graphs), and permutation graphs. It is polynomial to find the independence number and the independent domination number for graphs of bounded tree width (P.D. Seymour and R. Thomas, personal communication). Although Grötschel et al. [11] showed that the independence number can be found in polynomial time for perfect graphs using the ellipsoid method, Corneil and Perl [4] showed that it is NP-complete to find the independent domination number for bipartite graphs. Since bipartite graphs are perfect, the method of calculating independence number and independent domination number cannot yield a fast algorithm for deciding whether a perfect graph is well-covered, unless P = NP. However, there are a number of classes of perfect graphs for which the independent domination number can be found in polynomial time. Farber [5] showed that it is linear to find the independent domination number for chordal graphs. Farber and Keil [6] show that it is polynomial to find the independent domination number for permutation graphs.
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One motivation for the following corollary is to show that there is a polynomial-time algorithm for determining whether a perfect graph of bounded clique size is well-covered (see Corollary 2.3).

**Corollary 2.2.** Let $F$ be a family of graphs closed under the operation of taking an induced subgraph, such that there exist polynomial-time algorithms for finding the independence number $\alpha$ and the clique cover number $\theta$. If $\theta - \alpha$ and the size of the largest clique are bounded, then there is a polynomial-time algorithm for checking whether the graph is well-covered.

**Proof.** Say the clique size is bounded by $b$. By condition (2) of Theorem 2.1, we need only check to see if the graph is $k$-extendable for $k \in \{1, 2, \ldots, \min(h, \alpha)\}$, where $h = b(d + 1)$ for $d = \theta - \alpha$. There are only a polynomial number of independent sets $S$ of size $k$ for $k \in \{1, 2, \ldots, \min(h, \alpha)\}$. For each such independent set $S$ we delete $S$ and its neighbors from $G$ to form an induced subgraph $H$ of $G$. It is fast to find the independence number of $H$ which tells us whether $S$ was extendable to a maximum independent set. □

**Corollary 2.3.** Let $G$ be a perfect graph with clique size bounded by an integer $b$. There is a polynomial-time algorithm for checking whether $G$ is well-covered.

**Proof.** The class of perfect graphs is closed under the operation of taking an induced subgraph. The independence number is equal to the clique cover number. Grötschel et al. [11] proved that there is a polynomial-time algorithm for finding the independence number of perfect graphs. □

3. One-extendable and two-extendable graphs

In this section we investigate properties of 1- and 2-extendable graphs that will be used later in this paper. We also establish a connection between 1-extendability and Hall's condition and this leads to results concerning perfect matchings. Finally, we show how a characterization of very well-covered graphs with no isolated vertices due to Favaron and Staples follows immediately from our main theorem.

The following theorem shows that 2-extendable graphs are either 1-extendable or formed in a simple way from 1-extendable graphs.

**Theorem 3.1.** If a graph $G$ is 2-extendable, then either $G$ is both 1-extendable and 2-extendable or $G$ is the join of a complete graph and a graph that is both 1-extendable and 2-extendable.

**Proof.** Let $G$ be 2-extendable and not 1-extendable. There exists a vertex $v$ of $G$ that is in no maximum independent set. Since $G$ is 2-extendable, the vertex $v$ must be
adjacent to every other vertex in $G$. Let $H$ be the subgraph of $G$ induced by the set of vertices that are not in any maximum independent set. The graph $H$ is a complete graph and is joined to the remainder of $G$. We only need to show that $G - H$ is a 1-extendable graph. From the definition of $H$ it follows that each vertex of $G - H$ is contained in a maximum independent set of $G$. It also follows from the definition of $H$ that every maximum independent set of $G$ is contained in $G - H$, and hence is a maximum independent set of $G - H$. So $G - H$ is 1-extendable.

We need the following two lemmas to prove Corollary 3.4. However, these lemmas are of independent interest because they establish a connection between 1-extendability and Hall's condition and hence perfect matchings. The following lemma is implied by [1, Theorems 2, 6 and Proposition 7]. For the convenience of the reader we give a concise direct proof.

**Lemma 3.2.** If a graph has no isolated vertices and is 1-extendable, then it satisfies Hall's condition.

**Proof.** Let $G$ be a 1-extendable graph without isolated vertices. Assume there is an independent set $S$ that does not satisfy Hall's condition, i.e. $|S| > |N(S)|$. Let $S$ be of minimum order. Consider any independent set $I$ that does not contain all the vertices of $S$. We will show that $I$ is not a maximum independent set by showing that the set $I' = I \cup S - N(S)$ is a larger independent set.

$$|I'| = |I| + |S| - |I \cap S| - |I \cap N(S)|.$$ 

By the minimality of $|S|$, we have $|I \cap S| \leq |N(I \cap S)|$. Since there are no edges between $I \cap S$ and $I \cap N(S)$, we have $N(I \cap S) \subseteq N(S) - (I \cap N(S))$. Thus, $|I \cap S| \leq |N(S)| - |I \cap N(S)|$. Also since $|S| > |N(S)|$ we have $|I'| > |I|$. This implies $S$ is in all maximum independent sets. Thus, any vertex $v \in N(S)$ is not contained in any maximum independent set. Since $G$ has no isolated vertices, $N(S)$ is not empty, contradicting the 1-extendability of $G$.  

Lemma 3.2 and Hall's theorem imply the following.

**Lemma 3.3.** A 1-extendable graph with independence number $\alpha \geq \frac{1}{2} |V|$ has a perfect matching if and only if it has no isolated vertices.

**Note:** If a 1-extendable graph has no isolated vertices, then by the above lemma there is a perfect matching; hence, $\alpha \leq \frac{1}{2} |V|$. Thus, $\alpha = \frac{1}{2} |V|$ is the extremal case for 1-extendable graphs with no isolated vertices. This motivates the idea of very well-covered graphs.

The equivalence of parts (1) and (2) in the following corollary of Theorem 2.1 shows that for a large class of graphs, which includes bipartite graphs, 1-extendable and 2-extendable are enough to ensure wellcoveredness. The equivalence of parts (1) and
Corollary 3.4. If a graph $G$ has no isolated vertices and $\chi(G) = \frac{1}{2}|V|$, then the following are equivalent:

1. $G$ is well-covered.
2. $G$ is both 1-extendable and 2-extendable.
3. $G$ has a perfect matching and for every edge $(u, v)$ of the perfect matching, $G[E(N(u), N(v))]$ is a complete bipartite graph.
4. $G$ has a perfect matching and for every edge $(u, v)$ of the perfect matching and independent set $S$ of $G - u - v$, at least one of the vertices $u, v$ has no neighbor in $S$.

Proof. Both (1) and (2) imply that $G$ has a perfect matching by Lemma 3.3. Now apply Theorem 2.1 with the perfect matching as the clique cover where $t = \chi$ and $d = 0$. It is immediate that (1) and (2) above are equivalent to (1) and (2) of Theorem 2.1. With slightly more work, we see that both (3) and (4) above are equivalent to (3) of Theorem 2.1. □

4. Cycle restrictions

In this section we consider various cycle restrictions. We consider the following: triangle-free graphs, graphs with no induced cycle of length four and graphs with girth at least five. In the next section we look at bipartite graphs.

Applying Theorem 3.1 to triangle-free graphs we get the following corollary.

Corollary 4.1. The only triangle-free graphs that are 2-extendable but not 1-extendable are stars $(K_{1,n})$.

When various cycle restrictions are coupled with the restriction that the independence number be equal to the clique cover number, we are able to give characterizations of various classes of well-covered graphs that would give polynomial-time checks.

The following result was proved jointly with Edward Scheinerman. A simplicial vertex is a vertex that is in only one maximal clique.

Theorem 4.2. Let $G$ be a graph with no induced four cycle and independence number $\chi$ equal to its clique cover number. Then $G$ is well-covered if and only if every minimum clique cover is a partition of the vertices and every clique of the cover contains a simplicial vertex.

Proof. ($\Rightarrow$): Any independent set can be extended to a maximum independent set by the addition of simplicial vertices.
Theorem 4.3. Let $G$ be a triangle-free graph with independence number $\alpha$ equal to its clique cover number, and no isolated vertices. Then $G$ is well-covered if and only if there exists a perfect matching $M$ such that for every edge $(u, v)$ of $M$ the graph $G[E(N(u), N(v))]$ is a complete bipartite graph.

Proof. Let $C$ be a minimum clique cover of $G$. Since $G$ is triangle-free, $C$ must consist of a maximum matching $M$ and an independent set $S$.

Let $G$ be well-covered. There can be no edge of $G$ joining a vertex $v$ of a matching edge $(u, v)$ of the matching $M$ and a vertex $s$ of the set $S$ for the following reason. The vertex $v$ must be contained in a maximum independent set since $G$ is well-covered. Each maximum independent set has size $|M| + |S|$. However, if $v$ is in an independent set, then that set cannot contain $s$ or $u$ and thus can contain at most $|v| + (|M| - 1) + (|S| - 1)$ vertices, giving us a contradiction. This means that the set $S$ consists of isolated vertices; thus, $S$ is empty. Hence, $G$ has a perfect matching and $\alpha = \frac{1}{2}|V|$. Thus, by parts (1) and (3) of Corollary 3.4, we have that the graph $G[E(N(u), N(v))]$ is a complete bipartite graph.

To prove the other direction, the existence of a perfect matching implies that $\alpha = \frac{1}{2}|V|$, so from parts (1) and (3) of Corollary 3.4 it follows that $G$ is well-covered.

Corollary 4.4. Let $G$ be a graph with girth at least five, with no isolated vertices, and with its independence number equal to its clique cover number. Then $G$ is well-covered if and only if $G$ has a perfect matching consisting of pendant edges.

Proof. If $G$ has a perfect matching consisting of pendant edges then by Theorem 4.3 it is well-covered. Conversely, if $G$ is well-covered then by Theorem 4.3 there exists a perfect matching $M$ such that for every edge $(u, v)$ of $M$ the graph $H - G[E(N(u), N(v))]$ is a complete bipartite graph. One of $u$ and $v$ must have degree one because, otherwise, $H$ would contain a cycle of length four.

We note that 1-extendability and 2-extendability do not imply well-coveredness if either of the hypotheses of the above theorem are removed. Independence number
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Fig. 3. This graph is 1-extendable, 2-extendable, and has no isolated vertices, but is not well-covered.

$\chi=\frac{1}{2}|V(G)|$ along with 1-extendability and 2-extendability is not enough to imply well-coveredness if we allow the graph to have isolated vertices (e.g. the graph consisting of an isolated vertex and a cycle of nine vertices). In Fig. 3 we give an 1-extendable and 2-extendable graph with no isolated vertices (and a perfect matching) that is not well-covered.

Our approach yields an easy proof (using Corollaries 3.4 and 4.4) of the following theorem. This theorem is also implicitly contained in Finbow et al. [9].

**Theorem 4.5.** Let $G$ be a graph with girth at least five and no isolated vertices. Then $G$ is very well-covered if and only if it has a perfect matching consisting of pendant edges.

5. Bipartite graphs

Now we can apply Theorem 2.1 to bipartite graphs. We note that finding a maximum independent set for bipartite graphs is as fast as finding a maximum matching, and hence there are efficient and practical algorithms for determining if a bipartite graph is $k$-extendable for any fixed $k$.

**Theorem 5.1.** For $G$ a bipartite graph with no isolated vertices, the following are equivalent:

1. $G$ is 1-extendable,
2. each bipartite class of $G$ is a maximum independent set,
3. $\chi(G)=\frac{1}{2}|V(G)|$,
4. $G$ has a perfect matching, and
5. $G$ satisfies Hall's condition.

**Proof.** Straightforward arguments will give the implications (2) $\Rightarrow$ (1), (2) $\Leftrightarrow$ (3), and (4) $\Leftrightarrow$ (5). To show (1) $\Leftrightarrow$ (4), one can either use Lemma 3.2, or refer to Berge's proof [1, Proposition 9]. 

The following corollary is obtained by applying Theorem 5.1 to obtain $\chi=\frac{1}{2}|V|$ and then using Corollary 3.4. The equivalence of (1) and (3) below was first proved by Staples [22].
Corollary 5.2. If \( G \) is a bipartite graph with no isolated vertices, then the following are equivalent:

1. \( G \) is well-covered,
2. \( G \) is both 1-extendable and 2-extendable,
3. \( G \) has a perfect matching \( M \) and, for every edge \((u, v)\) of the matching \( M\), \( G[N(u) \cup N(v)] \) is a complete bipartite graph.

Condition (3) of Corollary 5.2 yields an explicit, practical method for deciding whether a bipartite graph is well-covered.

We will see that the condition of being 1-extendable in condition (2) of Corollary 5.2 is not really necessary. We show for bipartite graphs that the class of 2-extendable graphs is basically the same as the class of well-covered graphs.

Theorem 5.3. Every bipartite graph which is not a star is well-covered if and only if it is 2-extendable.

Proof. \((\Leftarrow):\) Let \( G \) be a 2-extendable bipartite graph which is not a star. It suffices to show that every connected component \( H \) of \( G \) is well-covered. Since \( G \) is 2-extendable, \( H \) is not a star. Further, Corollary 4.1 implies that \( H \) is 1-extendable; thus, by Corollary 5.2, \( H \) is well-covered. \(\square\)

Theorem 5.4. For any tree \( T \) the following are equivalent:

1. \( T \) is well-covered,
2. \( T \) is both 1-extendable and \( k \)-extendable for some \( k \) between 2 and \( \alpha - 1 \).
3. \( T \) has a perfect matching consisting of pendant edges or it consists of a single vertex.

Proof. \((1) \Leftrightarrow (3):\) The equivalence of parts (1) and (3) above follow easily from parts (1) and (3) of Corollary 5.2 as shown by Ravindra [18].

We note that if a tree has a perfect matching comprised of pendant edges, then any maximum independent set consists of a set of vertices covering the pendant edges. Thus, one may extend any independent set \( S \) to a maximum independent set \( M \) by setting

\[
M = S \cup \{v: \text{degree}(v) = 1, \{v, N(v)\} \cap S = \emptyset\}.
\]

\((1) \Leftrightarrow (2):\) Consider a tree \( T \) that is 1-extendable and \( k \)-extendable for some \( k \) between 2 and \( \alpha - 1 \). By Theorem 5.1 the tree \( T \) has a perfect matching \( M = \{e_1, e_2, \ldots, e_{\lceil |V|/2 \rceil}\} \). Now by the equivalence of (1) and (3) above, we only need to show that the matching \( M \) consists of pendant edges. Assume that there is a matching edge \( e = (a, b) \) that is not pendant. We will show that this assumption contradicts \( T \) being \( k \)-extendable. Since there is a perfect matching, the independence number is at most \( \frac{1}{2} |V| \). To see that \( \alpha = \frac{1}{2} |V| \), we create a maximum independent set \( S \) by rooting the tree at an endpoint \( v \), and then for each matching edge \( e_i \) adding the vertex of \( e_i \) which
is farthest from $v$ to $S$. Now we create an independent set $S_k$ of size $k$ that is not contained in any maximum independent set. Since the matching edge $e=(a, b)$ is not pendant there must be a vertex $a'$ in some matching edge $e_i \neq e$ such that $a'$ is adjacent to $a$, and a vertex $b'$ in some matching edge $e_j \neq e$ such that $b'$ is adjacent to $b$. Put the vertices $a'$ and $b'$ in the set $S_k$. Then add to $S_k$ any $k-2$ vertices $u$ such that $u$ is the farthest vertex from edge $e$ in a matching edge $e_h$ where $e_h$ is not $e$, $e_i$, or $e_j$. The set $S_k$ is independent and of size $k$. Any independent set $S'$ containing $S_k$ cannot be of maximum size since $S'$ cannot contain a vertex from edge $e$. □

This corollary cannot be extended to bipartite graphs. Fig. 4 shows that a bipartite graph $G$ with no isolated vertices which is 1-extendable and 3-extendable is not necessarily well-covered.

The following theorem is an easy consequence of Corollary 3.4 (or equivalently [7] or [22]).

**Theorem 5.5.** The only connected, $r$-regular very well-covered graph is the complete bipartite graph $K_{r,r}$.

### 6. Further directions

We have shown that for two large classes of perfect graphs (those with bounded clique size and those with no induced four cycle), it is polynomial to check whether a graph is well-covered. It is unknown whether there is a polynomial-time algorithm to check whether a perfect graph is well-covered.

Is it polynomial to check whether a planar graph is well-covered?

A number of theorems concerning well-covered graphs do not actually require well-coveredness in their proofs. They only need $k$-extendability for some range of $k$. For example, Campbell and Plummer [2] show that there are only four well-covered graphs that are cubic planar and 3-connected. Their proof does not require well-coveredness, it only uses $k$-extendability for $k=1, 2, ..., 5$. It is not known if there are still only four such graphs if we only have $k$-extendability for $k=1, 2, 3, 4$ or whether there are infinitely many such graphs. The prisms on odd cycles give an infinite class of cubic, planar, 3-connected graphs that are $k$-extendable for $k=1, 2$.

A graph is *matching perfect* or *equimatchable* if every matching is contained in a maximum matching or, equivalently, all maximal matching have the same size. This
idea was first suggested by Grünbaum [12]. Lewin [14] characterizes well-covered line graphs, but does not give an efficient algorithm for showing that a line graph is well-covered. Lesk et al. [13] give a characterization which does yield a polynomial-time algorithm. Since the class of line graphs is properly contained in the class of claw-free graphs, it would be of interest to characterize well-covered claw-free graphs. Another area of interest to explore would be to characterize \( k \)-extendable line graphs or, equivalently, to characterize graphs where every matching of size \( k \) is extendable to a maximum matching. Plummer has considered matchings of size \( k \) which extend to perfect matchings [16, 17].

References


