Analog Loop Shifting in $H^2$ Optimization of Input-Delay Sampled-Data Systems

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Abstract—Available solutions to sampled-data $H^2$ optimal control problems for systems with input delays begin with a conversion to a discrete-time problem with input delays, followed by a matching optimization method geared to handle delays. We propose to reverse that order and address the delay first, by an analog loop transformation. Hence, discretization and optimization involve no delays and are amenable to any standard method. The proposed, streamlined, approach fits also previously unsolved problems where sampling and/or hold functions are design parameters.

Index Terms—Input-delay systems, sampled-data systems, $H^2$ optimization

I. INTRODUCTION

The analysis of the effect of fractional analog loop delays on sampled-data systems can be traced back to the study of the modified $z$-transform [1]. A standard control design procedure typically begins with a discretization step [2, Sec. 2.3], [3, §4.3.4], whereby the analog plant

$$\dot{x}(t) = Ax(t) + Bu(t - \tau)$$

(1)

(with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$) is converted to a discrete time system of the following form (or to a variant thereof):

$$\dot{x}_{k+1} = \tilde{A}x_k + \tilde{B}_0u_{k-1} + \tilde{B}_1u_{k-1}.$$  

(2)

Here, $x_k := x(kh)$, where $h > 0$ is the sampling period, $u(t) = \hat{u}_k$ for $kh \leq t < (k + 1)h$, $l := \lceil \tau/h \rceil$ is the round-up of the ratio, and the coefficients of (2) are appropriately defined matrices; e.g., $A := e^{Ah}$. In contrast to the intrinsically infinite-dimensional (1), the dynamics of (2) are of dimension $n + lm < \infty$, with $lm$ modes at the origin associated with the input delays. One way to address design problems is thus in the $(n + lm)$-order augmented state-space formulation of (2), like that in [2, Eq. 2.12]. An alternative is to eliminate the delays in (2) applying straightforward, albeit bulky, discrete-time counterparts of some continuous-time finite spectrum assignment (FSA) or dead-time compensation methods as in [4]–[7]. Such methods are particularly appealing when $\tau \gg h$, hence $l \gg 1$, due to the dramatic dimensionality reduction, from $n + lm$ to $n$.

Dead-time sampled-data systems with an analog $H^2$/LQG performance criterion have been solved by the augmented state-space approach (or its polynomial counterpart) in [8]–[10] and references therein, and by an appropriately modified FSA-like approaches, in [11], [12]. A particularly burdensome aspect of these methods is the dependency of both the discretization and the delay-elimination techniques on the specifics of the cost functional: an analog $H^2$ cost functional associated with (1) requires a nontrivial reformulation in order to be rewritten in terms of (2). All three solution components—discretization, delay elimination, and the optimal design—are then nontrivially coupled. The elevated complexity that results from that coupling seems to render different approaches applicable only to subsets of sampled-data analysis techniques. For example, [10, p. 49] argues that applying the lifting approach to study sampled-data delay systems is “problematic.”

This note puts forward an alternative approach, taking care of the delay, rather, as a first step, before discretization. We do so, using the loop-shifting approach of [13], which is a generalization of ideas of [14]. Following that step, the entire controller becomes a standard sampled data cascade of a discrete controller, sandwiched between a sample and a hold operators, without an added delay. Controller design is thus amenable to one’s favorite, delay-free, sampled-data technique. As a demonstration of the advantages of the proposed approach, we show that the optimal sampling and hold devices for the delay-free sampled-data $H^2$ problem, which are derived in [15], remain optimal for the delayed problem. It is worth emphasizing that although the application of analog loop-shifting arguments to sampled-data problems occupies a one-line derivation, the proof requires nontrivial modifications of the technical arguments used in [13].

Notation: The transpose of a matrix $M$ is denoted as $M^T$. By $L^2(\mathbb{R})$ we denote the space of square integrable signals on the real axis and by $L^2(\mathbb{R}^+)$—the space of $L^2(\mathbb{R})$ signals that are zero in $t < 0$. The Hardy space $H^\infty$ is the subspace of $L^\infty(\mathbb{R})$ (bounded functions on the imaginary axis), which consists of analytic and bounded functions on the open right half plane. A transfer function $G(s)$ is said to be inner if $G \in H^\infty$ and $G^*G = I$, where $G^*(s) := [G(-s)]^*$ is the conjugate transfer function. With some abuse of notation, we denote by $H^2$ the Hilbert space of $T$-periodic causal operators $G$ such that $\|G\|_2^2 := \langle G, G \rangle_2 < \infty$, where the inner product

$$\langle G_1, G_2 \rangle_2 := \frac{1}{T} \int_0^T \int_0^T \text{tr}[g_2^*(t, \theta)g_1(t, \theta)] \, dt \, d\theta$$

(3)

and $g_i(t, \theta)$ stands for the response of $G_i$ to the Dirac impulse $\delta(t - \theta)$ at the time instance $t$, see [17]. For time-
invariant systems \( T = 0 \) this reduces to the standard definition of \( H^2 \).

Finally, introduce the following abbreviated notations for the truncation and completion FIR (finite impulse response) systems

\[
\tau_r \left\{ \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right\} := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} e^{At} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-sr}
\]

and

\[
\pi_r \left\{ \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} e^{-sr} \right\} := \begin{bmatrix} A & B \\ C e^{-At} & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} e^{-sr},
\]

respectively, whose responses are supported over \([0, r)\).

**II. PROBLEM SETUP**

Consider the standard sampled-data problem presented by the block-diagram in Fig. 1. Here

\[
G(s) = \begin{bmatrix} G_{zw}(s) & G_{zv}(s) \\ G_{yw}(s) & G_{vy}(s) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix}
\]

is a given generalized plant, \( S \) is an A/D converted (sampler), \( H \) is a D/A converted (hold), and \( K_d(z) \) is the digital part of the controller. The element \( e^{-sr} \) represents an analog input delay that may result from communication, computation, etc. Since constant scalar delays commute with LTI systems, the method developed here, for systems with an input delay, is readily applicable to systems with an output delay.

The problem considered in this paper is as follows:

**OP_7:** Design a sampled-data controller \( H K_d S \) internally stabilizing the system in Fig. 1 and minimizing the \( H^2 \) norm of the closed-loop system \( T_{zw} \) from \( w \) to \( z \). For example, the delay-free sampled-data \( H^2 \) problem, extensively studied in the literature [15], [18]–[21], is problem **OP_0**.

Problem **OP_7** comprises four sub-categories, depending on whether the design of either the sample or hold devices is part of the optimization task, or whether these choices are made in advance [15], [22]. The variant studied in [8]–[12] is that where \( S \) and \( H \) are fixed to be the standard ideal sampler and zero-order hold, respectively, (the setup in [10] allows also a more general hold device) and the only designed part is \( K_d \). To the best of our knowledge, a solution of **OP_7**, including either the design of \( H \) or of \( S \), or of both these elements, has not been published to date. An advantage of our approach is that the treatment of the delay depends neither on the configuration nor on the specific choice of A/D and D/A converters. This fact greatly facilitates the solution of the combined problem, that includes all three compensator components.

**III. LOOP SHIFTING WITH ANALOG CONTROLLERS**

As a precursor for the solution of **OP_7**, we first consider the purely continuous-time problem of minimizing the \( H^2 \) norm of the closed-loop system in Fig. 2(a) by an analog controller \( K \). This simplified variant is equivalent to an LQG problem in a system with an input delay, and was first solved in [23]. An alternative solution, which plays an important role in this study, was proposed in [13]. This section reviews the solution in [13] and then extends its key steps to the case of a time-varying \( K \).

**A. Linear time-invariant \( K \)**

The idea is to apply stability-preserving loop transformations to convert the original problem in Fig. 2(a) to an equivalent problem, structured as in Fig. 2(b). Here \( \Pi(s) = \pi_r \{ G_{yw}(s) e^{-sr} \} \) and \( \Delta(s) = \tau_r \{ G_{zw}(s) \} \) are FIR systems and

\[
\begin{bmatrix} G_{zw}(s) & G_{zv}(s) \\ G_{yw}(s) & G_{vy}(s) \end{bmatrix} := \begin{bmatrix} A & B_w \\ C_z e^{-At} & D_{zw} \end{bmatrix} - \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} e^{-sr}.
\]

An important property of the transformed system is that the closed-loop mapping from \( w \) to \( z \),

\[
T_{zw} = \Delta + e^{-sr} T_{z\w}.
\]

is such that \( (\Delta, e^{-sr} T_{z\w}) \|_2 = 0 \) for all mappings from \( w \) to \( z \) such that \( T_{z\w} \in H^2 \). This, together with the fact that \( e^{-sr} \) is inner, implies that

\[
\| T_{z\w} \|_2^2 = \| \Delta \|_2^2 + \| T_{z\w} \|_2^2.
\]

Because \( \Delta \) does not depend on the controller, the minimization of \( \| T_{z\w} \|_2 \) is equivalent to the minimization of \( \| T_{z\w} \|_2 \), which is a standard delay-free \( H^2 \) problem for the plant in (5). The optimal \( K \) for the original problem is derived from the optimal \( \hat{K} \) for the latter, delay-free problem, by the transformation (cf. the dotted block in Fig. 2(b))

\[
K = \hat{K}(I - \Pi \hat{K})^{-1}.
\]

As a solution of a delay-free \( H^2 \) problem \( \hat{K}(s) \) is proper, the transformation (8) is guaranteed to be well defined. A point worth observing is that the resulting \( K \) is written in a Modified Smith Predictor form of [7].

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B. Linear time-varying $K$

Our next move is to show that the loop-shifting procedure above applies also when $K$ may be time varying.

The first step to this end is to extend the internal stability result of [13] to the case of a possibly time-varying $K$ for the system in Fig. 2(a). By the internal stability of the system in Fig. 2(a) we understand the boundedness of all nine systems from $(w, v_1, v_2)$ to $(z, y, u)$, as operators $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+),$

in the system in Fig. 3(a). This system is an augmentation of Fig. 2(a) with fictitious signals $v_1$ and $v_2$, introduced to rule out unstable cancellations in the loop.

In the LTI causal case, stability is equivalent to the condition [24, Thm. A.6.26] that the transfer functions of these nine systems belong to $H^\infty$. The analysis of [13] exploited the boundedness and analyticity properties of these transfer functions in the region $\text{Re } s > \alpha$ for some sufficiently large $\alpha > 0$. But these arguments are not readily applicable in the time-varying setting. We therefore prove the time-varying counterpart of the first two statements of [13, Lemma 1].

Lemma 1: A linear causal $K$ internally stabilizes the system in Fig. 2(a) iff it can be presented in the form (8) for some linear causal $\tilde{K}$, internally stabilizing the (delay-free) system in Fig. 3(b), where the generalized plant is given by (5).

Proof: Omitted due to copyright limitations. Available in [16].

To simplify the exposition, in what follows we assume that the compensator $K$ is $T$-periodic, with some $T > 0$. This is sufficiently general for the sampled-data analysis in the next section. Under this assumption we now consider the $H^2$ norm of the decomposition in Fig. 2(b). It is straightforward to see that the key property of the LTI decomposition in (6)—the orthogonality of the terms in its right-hand side—remains true in the periodic case as well. The following result can be seen as the counterpart of the third statement of [13, Lemma 1]:

Lemma 2: For every $T$-periodic $\tilde{K}$, the closed-loop system from $w$ to $z$ in Fig. 2(b) is described by (6) and

$$\langle \Delta, e^{-sT}T_{zw} \rangle_2 = 0$$

whenever $\tilde{K}$ is such that $T_{zw} \in H^2$.

Proof: Omitted due to copyright limitations. Available in [16].

The results of Lemmas 1 and 2 imply that the LTI results of [13] apply to a general periodic case. In particular, there exists only a design-independent constant difference between the $H^2$ performance delivered by any periodic $K$ in the delay problem in Fig. 3(a) and the $H^2$ performance delivered by the corresponding controller $\hat{K}$ in the delay-free standard $H^2$ problem in Fig. 3(b) i.e., with the generalized plant (5) as given by (7).

IV. SAMPLED-DATA CONTROLLERS

Return now to our sampled-data problem $\text{OP}_r$. The set of sampled-data controllers of the form $K = \mathcal{H}(K_dS)$, with the sampling period $h$, is a subset of the set of $h$-periodic controllers. From the loop-shifting viewpoint, it is then important to characterize the constraints that the sampled-data requirement to $K$ imposes on its primary part $\tilde{K}$ in (8).

This characterization turns out to be simple: the transformation $K \mapsto \tilde{K}$ in (8) preserves any sampled-data structure of $K$. Namely, let $\tilde{K} = \mathcal{H}(K_dS)$ for some causal $\mathcal{H}$ and $S$ and proper $\tilde{K}_d(\tilde{s})$. Then (8) can be transformed as

$$K = \mathcal{H}(\tilde{K}_dS)(I - \Pi\mathcal{H}(\tilde{K}_dS))^{-1} = \mathcal{H}(\tilde{K}_d(I - \Pi_d\tilde{K}_d)^{-1}S,$$

where $\Pi_d := S\Pi\mathcal{H}$ is the discretized version of $\Pi$ (the second equality follows by $X(I - XY)^{-1} = (I - XY)^{-1}X$, which is true whenever either $I - XY$ or $I - YX$ is nonsingular). Thus, if $\tilde{K}$ is designed as a sampled-data controller, with some sampling and hold functions, the resulting $K$ is a sampled-data controller too, with the very same sampling and hold functions and the discrete part

$$K_d = \tilde{K}_d(\Pi_d\tilde{K}_d)^{-1},$$

which is a discrete dead-time compensator. Moreover, since $\Pi_d(\tilde{s})$ is always strictly proper (this is shown in [16], by construction) and $\tilde{K}_d(\tilde{s})$ is proper, transformation (9) is always well posed. Thus, the following result can be formulated:

Theorem 1: The solution of $\text{OP}_r$ is given by

![Diagram](image)

where $\mathcal{H}(\tilde{K}_dS)$ solves the standard sampled-data $H^2$ problem for the generalized plant (5) and $\Pi_d = S\Pi\mathcal{H}(\{P_y\mathcal{H}e^{-sT}\})\mathcal{H}$.\end{proof}

With the help of Theorem 1, $\text{OP}_r$ can now be solved by any available approach to the delay-free sampled-data $H^2$ optimization, see [15], [18]–[21] and the references therein.

In addition, the fact that the (1, 2) and (2, 1) parts of the generalized plant in (5) are the same as those in (4) can be exploited to simplify the delay-free optimization. For example, by arguments, similar to those in [13, §3.1], it can be shown that the Riccati equations required in the state-space design of $K$ are independent of $r$. Likewise, it can be readily shown that the stabilizability of (5) by a sampled-data controller is equivalent to that of (4) under any choice of $S$ and $\mathcal{H}$.
(2, 1) subblocks). This implies that the $H^2$-optimal sampler and hold for $\text{OP}_r$ (i.e., $S$ and $H$ obtained as a part of the optimization procedure) have the same structure as those for a delay-free $H^2$ problem. Moreover, the optimal $S$ and $H$ in [15] are independent of each other (i.e., they are optimal irrespective of the choice of each other) and are functions of the state-space realizations of only the subsystems $G_{zw}$ and $G_{zw}$ in (5), respectively. Hence, the optimal $H$ for $\text{OP}_r$ coincides with that of $\text{OP}_p$, whereas the optimal $S$ is only altered by its post-multiplication by the matrix $e^{At}$ (see [13, §3.1] for details). Since this matrix can always be absorbed in the discrete part $K_d$, we may conclude that the optimal $S$ and $H$ for $\text{OP}_r$ are exactly the same as those for its delay-free version $\text{OP}_p$.

V. ILLUSTRATIVE EXAMPLE

Consider the sampled-data tracking problem described by the block-diagram in Fig. 4, where the plant $e^{-At}/s$ is controlled by a sampled-data controller with the ideal sampler $S$ and the antialiasing filter $a/(s + a)$, $a > 0$, in the IDOF configuration. The reference signal $r$ is modeled via the exosystem $1/s$, which is the model for the step signal. The purpose of this academic, stripped-down, example is to show how our approach leads to a simple characterization of the effect of the plant delay $\tau$ on the achievable performance for different choices of the hold function $H$.

Let the regulated signal be

$$z = \begin{bmatrix} e^{\rho u} \end{bmatrix},$$

for some $\rho \geq 0$. In other words, we penalize both the analog tracking error $e = r - y_p$ and the control effort. This tracking problem can be cast as the standard problem in Fig. 1 with the generalized plant

$$G(s) = \begin{bmatrix} \begin{smallmatrix} \frac{1}{s} & -\frac{1}{s} \\ 0 & \sqrt{2} \\ \frac{1}{s + a} & \frac{1}{s + a} \end{smallmatrix} \end{bmatrix} = \begin{bmatrix} -a & a & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(its construction involves canceling the mode of the exosystem, which can be done because these are fictitious uncontrollable dynamics).

Applying the loop-shifting arguments, the problem reduces to the delay-free sampled-data problem for the generalized plant

$$\tilde{G}(s) = \begin{bmatrix} -a & a & 1 - e^{-\alpha t} & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ e^{\alpha t} & 1 - e^{\alpha t} & 0 & 0 \end{bmatrix}.$$

Because

$$\Delta(s) = \begin{bmatrix} \tau \{1/s\} \\ 0 \end{bmatrix},$$

its $H^2$ norm is merely the $L^2[0, \tau]$-norm of the impulse response of the integrator. Hence, (7) reads

$$\|T_{zw}\|^2 = \tau + \|T_{zw}\|^2;$$

where the second term is the optimal achievable $H^2$ performance for the generalized plant $\tilde{G}$ and no loop delay. Calculating this quantity is a standard problem. We use the formula of [15, Lemma 5.5], which are applicable to the case of both the zero-order hold $H_{ZOH}$ and the optimal hold $H_{opt}$ designed there. Lengthly albeit straightforward calculations yield that $\|T_{zw}\|^2$ depends on $\tau$ in either case and

$$\|T_{zw}\|^2 = \left\{ \begin{array}{ll} \tau + h^2/2 + \sqrt{\rho + h^2/12} \cdot J_a & \text{if } H = H_{ZOH} \\ \tau + h^2/2 + \sqrt{\rho} + J_a & \text{if } H = H_{opt} \end{array} \right.$$

where, defining $\eta := ah(1 + e^{-ah}) - 2(1 - e^{-ah}) > 0$,

$$J_a := \frac{ah(1 + e^{-ah}) \eta - \eta}{2a(1 - e^{-ah})} \in [0, \frac{h}{2\sqrt{3}}]$$

accounts for the effect of the antialiasing filter. Thus, the effect of the loop delay on the achievable performance in this case amounts to the addition of the quantity $\tau \cdot \rho$ to the cost function.

Remark 5.1: The optimal hold for this problem does not depend on the antialiasing filter and acts as

$$u(kh + t) = \frac{e^{-t/\sqrt{\rho}}}{\sqrt{\rho}} \hat{u}_k, \quad t \in [0, h].$$

If the penalty on the control effort vanishes, $\rho \rightarrow 0$, this hold function effectively approaches the impulse hold acting as $u(kh + t) = \delta(t)\hat{u}_k$, which is the hold used in [25].

VI. CONCLUDING REMARKS

The paper has studied the standard sampled-data $H^2$ problem for analog systems with a single input delay. Our message is that by inverting the conventional course of action, i.e., by first handling the delay in an analog setting and only then imposing a sampled-data structure upon the controller, the treatment can be substantially simplified and extended to more general formulations. In particular, we have shown that the delay problem reduces to a delay-free problem by the use of simple loop shifting arguments and that the resulting dead-time compensation scheme can be digitalized in a straightforward manner. The proposed approach seamlessly extends to output delay systems.

REFERENCES


