Disjoint Cycles in Eulerian Digraphs and the Diameter of Interchange Graphs

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Let $R=(r_1, \ldots, r_m)$ and $S=(s_1, \ldots, s_n)$ be nonnegative integral vectors with $\sum r_i = \sum s_j$. Let $\mathcal{A}(R, S)$ denote the set of all $m \times n \{0, 1\}$-matrices with row sum vector $R$ and column sum vector $S$. Suppose $\mathcal{A}(R, S) \neq \emptyset$. The interchange graph $G(R, S)$ of $\mathcal{A}(R, S)$ was defined by Brualdi in 1980. It is the graph with all matrices in $\mathcal{A}(R, S)$ as its vertices and two matrices are adjacent provided they differ by an interchange matrix. Brualdi conjectured that the diameter of $G(R, S)$ cannot exceed $mn/4$. A digraph $G=(V, E)$ is called Eulerian if, for each vertex $u \in V$, the out-degree and indegree of $u$ are equal. We first prove that any bipartite Eulerian digraph with vertex partition sizes $m$, $n$, and with more than $(\sqrt{17} - 1) mn/4 (\approx 0.78mn)$ arcs contains a cycle of length at most 4. As an application of this, we show that the diameter of $G(R, S)$ cannot exceed $(3+\sqrt{17}) mn/16 (\approx 0.445mn)$. The latter result improves a recent upper bound on the diameter of $G(R, S)$ by Qian. Finally, we present some open problems concerning the girth and the maximum number of arc-disjoint cycles in an Eulerian digraph.

Key Words: interchange graph; Eulerian digraph; diameter; girth; $\{0, 1\}$-matrix row (column) sum vector; arc-disjoint cycles.

1. INTRODUCTION

Let $m$ and $n$ be positive integers, and let $R=(r_1, \ldots, r_m)$ and $S=(s_1, \ldots, s_n)$ be nonnegative integral vectors with $\sum r_i = \sum s_j$. Denote by $\mathcal{A}(R, S)$ the set of all $m \times n \{0, 1\}$-matrices $A=(a_{ij})$ satisfying

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Thus a \( \{0, 1\} \)-matrix belongs to \( \mathcal{A}(R, S) \) provided its row sum vector is \( R \) and its column sum vector is \( S \). The set \( \mathcal{A}(R, S) \) was the subject of intensive study during the late 1950s and early 1960s by many researchers. (See [3] for a survey paper.)

In 1957, Ryser [8] defined an interchange to be a transformation which replaces the \( 2 \times 2 \) submatrix

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

of a matrix \( A \) of 0’s and 1’s with the \( 2 \times 2 \) submatrix

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\]

or vice versa. Clearly an interchange (and hence any sequence of interchanges) does not alter the row and column sum vectors of a matrix and therefore transforms a matrix in \( \mathcal{A}(R, S) \) into another matrix in \( \mathcal{A}(R, S) \). Ryser [8] proved the converse of the result by inductively showing that given \( A, B \in \mathcal{A}(R, S) \) there is a sequence of interchanges which transforms \( A \) into \( B \). Therefore interchanges play an important role when one studies the matrices in \( \mathcal{A}(R, S) \), and Ryser [9, p. 68] raised the question of how many interchanges are needed to achieve such a transformation: “We remark that the minimal number of interchanges required to transform \( A \) into \( A’ \) is apparently a hopelessly complicated function of \( A \) and \( A’ \).”


Suppose \( \mathcal{A}(R, S) \neq \emptyset \). In 1980, Brualdi [3] defined the interchange graph \( G(R, S) \) of \( \mathcal{A}(R, S) \). It is the graph with all matrices in \( \mathcal{A}(R, S) \) as its vertices and two vertices (representing two such matrices) \( A \) and \( B \) are adjacent if \( A \) can be obtained from \( B \) by a single interchange. Much research has been done in investigating various properties of \( G(R, S) \). For example, Shao [10] showed that interchange graphs are 3-connected with a few 2-connected exceptions. Chen et al. [5] proved that the edge connectivity of an interchange graph equals its minimum degree. The Hamiltonicity property of interchange graphs were also studied by some other researchers [6, 12, 13].

Recall that a digraph \( G = (V, E) \) is called Eulerian if, for each vertex \( u \in V \), the outdegree \( \deg^+(u) \) of \( u \) equals the indegree \( \deg^-(u) \) of \( u \). Let \( \mathcal{C}_{m,n} \) denote the set of all \( m \times n \) \( \{-1, 0, 1\} \)-matrices with each row and column
sum equal to 0. For any \( m \times n \{−1, 0, 1\}\-matrix \( C = (c_{ij}) \), one can define a bipartite digraph \( G(C) = (V, E) \) with vertex set partition \( V = \{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_n\} \) such that \((u_i, v_j) \in E\) if \(c_{ij} = 1\) and \((v_j, u_i) \in E\) if \(c_{ij} = -1\). It can be seen that, for a \( \{−1, 0, 1\}\-matrix \( C, C \in \mathbb{F}_{m,n} \) if and only if \( G(C) \) is a bipartite Eulerian digraph with vertex set bipartitioned into sets of sizes \( m \) and \( n \), respectively. It is known that any Eulerian digraph can be decomposed into arc-disjoint cycles.

We assume throughout that no graph (resp. digraph) contains multiple edges (resp. arcs). The distance between two vertices in a graph is the length of a shortest path linking the two vertices in the graph. Thus, for any \( A, B \in \mathcal{A}(R, S) \), the distance between \( A \) and \( B \) in \( G(R, S) \), denoted by \( i(A, B) \), is the minimum number of interchanges which transform \( A \) into \( B \). Let \( d(A, B) \) denote the number of nonzero entries in \( A - B \), and let \( q(A, B) \) denote the maximum number of arc-disjoint cycles in a cycle decomposition of the bipartite Eulerian digraph corresponding to \( A - B \in \mathbb{F}_{m,n} \). The following result was obtained by Walkup [11] in 1965 (see also [3, p. 172]).

**Lemma 1.** Let \( A, B \in \mathcal{A}(R, S) \). Then

\[
i(A, B) = \frac{d(A, B)}{2} - q(A, B).
\]

The diameter of a graph is the greatest distance between a pair of vertices in the graph. Lemma 1 implies that the diameter of \( G(R, S) \), denoted by \( D(G(R, S)) \), cannot exceed \( mn/2 - 1 \), and Brualdi [3] made the following conjecture in 1980.

**Conjecture 1.** For any nonnegative integral \( m \)-vector \( R \) and \( n \)-vector \( S \) such that \( \mathcal{A}(R, S) \neq \emptyset \),

\[
D(G(R, S)) \leq \frac{mn}{4}.
\]

Recently, Qian [7] made some progress on the conjecture by proving that

\[
D(G(R, S)) \leq \frac{mn}{2} - \frac{m}{2} \ln \frac{n + 2}{4}.
\]

In this paper, we show that \( D(G(R, S)) \leq (3 + \sqrt{17}) \frac{mn}{16} \approx 0.445mn \). In the course of proving this, we obtain a result concerning the length of a shortest cycle in a bipartite Eulerian digraph.
2. 4-CYCLES IN BIPARTITE EULERIAN DIGRAPHS

Let \( G = (V, E) \) denote a digraph. For any two subsets \( S_1 \) and \( S_2 \) of \( V \), we use \( E(S_1, S_2) \) to denote the set of all arcs from \( S_1 \) to \( S_2 \) in \( G \). If \( G \) is Eulerian, then it is easy to show that \( |E(S, V \setminus S)| = |E(V \setminus S, S)| \) for any subset \( S \) of \( V \). We will repeatedly use this fact in the following theorem.

**Theorem 1.** Any bipartite Eulerian digraph with vertex partition sizes \( m \), \( n \), and with more than \( (\sqrt{17} - 1) mn/4 \approx 0.78mn \) arcs contains a cycle of length at most 4.

**Proof.** Suppose, contrary to the theorem, that \( G \) contains no digons or 4-cycles. Let \( G \) have bipartition \((V_1, V_2)\), where \( |V_1| = m \) and \( V_2 = n \). Without loss of generality, it may be supposed that \( G \) is strongly connected. We shall prove that

\[
m^2n - m|E| \geq 4 \sum_{v \in V_2} (\deg^+(v))^2 \geq \frac{|E|^2}{n}.
\]

The desired inequality then follows by dividing by \( m^2n \) and applying the quadratic formula. The inequality on the right follows from the Cauchy–Schwarz inequality. We now prove the inequality on the left by a counting argument.

For each \( u \in V_1 \) and each positive integer \( i \), let \( \Gamma^+_i(u) \) and \( \Gamma^-_i(u) \) denote the set of all distance \( i \) out-neighbors of \( u \) and the set of all distance \( i \) in-neighbors of \( u \), respectively. Since \( G \) is bipartite without digons, \( \Gamma^+_1(u) \) and \( \Gamma^-_1(u) \) are disjoint subsets of \( V_2 \). Since \( G \) is bipartite without 4-cycles, \( \Gamma^+_2(u) \) and \( \Gamma^-_2(u) \) are disjoint subsets of \( V_1 \) and \( E(\Gamma^+_2(u), \Gamma^-_1(u)) = E(\Gamma^-_2(u), \Gamma^+_1(u)) = \emptyset \). Thus

\[
|E(\Gamma^+_2(u), V_2 \setminus \Gamma^-_1(u))| = |E(\Gamma^-_2(u), V_2 \setminus \Gamma^+_1(u))| + |E(\Gamma^+_2(u) \setminus \Gamma^+_1(u), \Gamma^-_1(u))|
\]

(2)

Since \( G \) contains no digons,

\[
|\Gamma^+_2(u)| \cdot (n - |\Gamma^-_1(u)|) \geq |E(\Gamma^+_2(u), V_2 \setminus \Gamma^-_1(u))| + |E(\Gamma^-_2(u), \Gamma^+_1(u))|.
\]

(3)

Since \( \Gamma^+_1(u) \) and \( \Gamma^-_1(u) \) are disjoint subsets of \( V_2 \), we have \( V_2 \setminus \Gamma^-_1(u) \supseteq \Gamma^+_1(u) \) and so

\[
|E(V_2 \setminus \Gamma^-_1(u), \Gamma^+_2(u))| \geq |E(\Gamma^+_1(u), \Gamma^+_2(u))|.
\]

(4)
Since $G$ is Eulerian and $V_2 \cong \Gamma^+_1(u)$, we have
\[ |E(\Gamma^+_2(u), V_2)| = |E(V_2, \Gamma^+_2(u))| \geq |E(\Gamma^+_1(u), \Gamma^+_2(u))|. \tag{5} \]

Inequalities (2) through (5) imply
\[ |\Gamma^+_2(u)\cdot(n-|\Gamma^-_1(u)|)| \geq 2|E(\Gamma^+_1(u), \Gamma^+_2(u))|. \tag{6} \]

Dually,
\[ |\Gamma^-_2(u)\cdot(n-|\Gamma^+_1(u)|)| \geq 2|E(\Gamma^-_1(u), \Gamma^-_2(u))|. \tag{7} \]

Since $G$ is Eulerian, we have
\[ |\Gamma^-_1(u)| = |\Gamma^+_1(u)|. \tag{8} \]

Since $\Gamma^+_2(u)$ and $\Gamma^-_2(u)$ are disjoint subsets of $V_1$, we have
\[ m \geq |\Gamma^+_2(u)| + |\Gamma^-_2(u)|. \tag{9} \]

By (8) and (9), adding inequalities (6) and (7) implies
\[ m(n-|\Gamma^+_1(u)|) \geq 2|E(\Gamma^+_1(u), \Gamma^+_2(u))| + 2|E(\Gamma^-_1(u), \Gamma^-_2(u))|. \tag{10} \]

Next we sum inequality (10) over all $u \in V_1$. The sum of the left side of the inequality is
\[ \sum_{u \in V_1} m(n-|\Gamma^+_1(u)|) = m^2n - m \sum_{u \in V_1} \deg^+(u) = m^2n - m|E|/2. \tag{11} \]

It is an easy observation that both $\sum_{u \in V_1} |E(\Gamma^+_1(u), \Gamma^+_2(u))|$ and $\sum_{u \in V_1} |E(\Gamma^-_1(u), \Gamma^-_2(u))|$ equal the number of 2-paths in $G$ with initial vertex in $V_1$. On the other hand, these 2-paths can also be counted by considering their mid-vertices, which are in $V_2$. Thus
\[ \sum_{u \in V_1} |E(\Gamma^+_1(u), \Gamma^+_2(u))| = \sum_{u \in V_1} |E(\Gamma^-_1(u), \Gamma^-_2(u))| = \sum_{v \in V_2} \deg^+(v) \deg^-(v). \tag{12} \]

Since $\deg^+(v) = \deg^-(v)$ for all $v$ in $G$, the left inequality of (1) follows from (10), (11) and (12). This completes the proof of the theorem. \]

**Remark 1.** The bound $(\sqrt{17} - 1) mn/4$ in Theorem 1 is almost certainly not sharp. In fact, the following family of examples suggests that $2mn/3$ could be the best possible bound in Theorem 1.
Example 1. Suppose \( m \) and \( n \) are multiples of 3. Let \( V(G) = \bigcup_{i=1}^{6} V_i \) such that \( |V_i| = m/3 \) for all odd \( i \) and \( |V_i| = n/3 \) for all even \( i \). Let each vertex in \( V_i, 1 \leq i \leq 6 \), adjacent to each vertex in \( V_{i+1} \) (\( V_7 = V_1 \)). Then \( G \) is an Eulerian digraph with \( 2mn/3 \) arcs, and \( G \) has no 4-cycles.

3. MAIN RESULT

As an application of Theorem 1, we are ready to prove our main result.

**Theorem 2.** For any nonnegative integral \( m \)-vector \( R \) and \( n \)-vector \( S \) such that \( \mathcal{A}(R, S) \neq \emptyset \),

\[
D(G(R, S)) \leq \frac{3 + \sqrt{17}}{16} mn \approx 0.445mn.
\]

**Proof.** We show, equivalently, that for any \( A, B \in \mathcal{A}(R, S), \)

\[
i(A, B) \leq \frac{3 + \sqrt{17}}{16} mn.
\]

This follows immediately from Lemma 1 if \( d(A, B) \leq \frac{(\sqrt{17} - 1) mn}{4} \) (\( \approx 0.78mn \)) since \( (\sqrt{17} - 1)/8 < (3 + \sqrt{17})/16 \). Thus it may be supposed that \( d(A, B) > (\sqrt{17} - 1) mn/4 \). Let \( G_1 \) be the bipartite Eulerian digraph corresponding to \( A - B \in \mathcal{C}_{m,n} \). Then \( G_1 \) has \( d(A, B) > (\sqrt{17} - 1) mn/4 \) arcs. By Theorem 1, \( G_1 \) contains a 4-cycle \( C_1 \). Let \( G_2 \) be the bipartite Eulerian digraph obtained from \( G_1 \) by removing all arcs in \( C_1 \). In case \( G_2 \) contains more than \( (\sqrt{17} - 1) mn/4 \) arcs, by applying Theorem 1 again, \( G_2 \) contains a 4-cycle. The cycles \( C_1 \) and \( C_2 \) are arc-disjoint in \( G_1 \). Therefore, by recursively applying Theorem 1, one can obtain \( t = \left\lfloor \frac{d(A, B) - (\sqrt{17} - 1) mn/4}{4} \right\rfloor = \left\lfloor \frac{4d(A, B) - (\sqrt{17} - 1) mn}{16} \right\rfloor \)

arc-disjoint 4-cycles from \( G_1 \). Thus \( q(A, B) \geq t \). By Lemma 1,

\[
i(A, B) \leq \frac{d(A, B)}{2} - \frac{4d(A, B) - (\sqrt{17} - 1) mn}{16} \leq \frac{3 + \sqrt{17}}{16} mn,
\]

where the last inequality holds since \( d(A, B) \leq mn \). This completes the proof of the theorem.
4. FURTHER RESEARCH

We may generalize the problem in Theorem 1 as follows: For each integer \( k \geq 2 \), how many arcs must a bipartite Eulerian digraph have in order to guarantee the existence of a cycle of length at most \( 2k \)? We have the following conjecture.

**Conjecture 2.** Suppose \( k \geq 2 \) is an integer. Then any bipartite Eulerian digraph with vertex partition sizes \( m, n \), and with more than \( 2mn/(k+1) \) arcs contains a cycle of length at most \( 2k \).

The following family of examples shows that the bound \( 2mn/(k+1) \) in Conjecture 2 cannot be decreased. Suppose \( m \) and \( n \) are multiples of \( k+1 \). Let \( V(G) = \bigcup_{i=1}^{2k+2} V_i \) such that \( |V_i| = m/(k+1) \) for all odd \( i \) and \( |V_i| = n/(k+1) \) for all even \( i \). Let each vertex in \( V_i \), \( 1 \leq i \leq 2k+2 \), adjacent to each vertex in \( V_{i+1} \) \( (V_{2k+3} = V_1) \). Then \( G \) is an Eulerian digraph with \( 2mn/(k+1) \) arcs and girth \( 2k+2 \).

Now we turn to another problem of maximizing the number of arc-disjoint cycles in a digraph. By Lemma 1, a strong lower bound on \( q(A, B) \) for all \( A, B \in \mathcal{A}(R, S) \) should be obtained in order to settle Conjecture 1. Recall that \( q(A, B) \) equals the maximum number of arc-disjoint cycles in the bipartite Eulerian digraph corresponding to \( A \rightleftharpoons B \in \mathcal{G}_{m,n} \). The problem of bounding from below the maximum number of arc-disjoint cycles in a digraph has been studied by Alon et al. [2], who made the following conjecture in 1996.

**Conjecture 3.** Any \( r \)-regular digraph contains a collection of at least \( r(r+1)/2 \) arc-disjoint cycles.

Here, \( r \)-regular means that \( \deg^+(u) = \deg^-(u) = r \) for all vertices \( u \). Thus an \( r \)-regular digraph is Eulerian. For a more general conjecture on the maximum number of arc-disjoint cycles in a digraph with given minimum outdegree, we refer the reader to [1]. To conclude the paper, we present a similar conjecture, which might be a step in the right direction for attacking Conjecture 1.

**Conjecture 4.** Any bipartite Eulerian digraph \( G = (V, E) \) with vertex partition of sizes \( m \) and \( n \) contains a collection of at least \( |E|^2/(4mn) \) arc-disjoint cycles.

Conjecture 4, if true, will imply Conjecture 1 since \( |E|/2 - |E|^2/(4mn) \leq mn/4 \). The family of examples following Conjecture 2 also shows that Conjecture 4 is sharp. As suggested by one of the referees, the special case of Conjecture 4 in which the digraph is an Eulerian bipartite tournament...
seems particularly interesting, the conjecture asserting that any such bipar-
tite tournament admits a decomposition into 4-cycles. We do not have
a proof or a disproof even in this special case. The same referee also
reminded us the following analogous question for a regular tournament
which was originally raised by Brualdi and Li [4]: Does any regular tourna-
ment admit a decomposition into \((\frac{n-1}{2})^2\) 4-cycles and \((n-1)/2\) 3-cycles?

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REFERENCES

2. N. Alon, C. McDiarmid, and M. Molloy, Edge-disjoint cycles in regular directed graphs,
   *J. Graph Theory* 22 (1996), 231–237.
3. R. A. Brualdi, Matrices of zeros and ones with fixed row and column sum vectors, *Linear
4. R. A. Brualdi and Q. Li, The interchange graph of tournaments with the same score
   vector, in “Progress in Graph Theory,” pp. 129–151, Academic Press, Toronto, Ontario,
   1984.
5. R. Chen, X. Guo, and F. Zhang, The edge connectivity of interchange graphs of classes of
7. J. Qian, On the upper bound of the diameter of interchange graphs, *Discrete Math.* 195
10. J. Shao, The connectivity of interchange graph of class \(\mathcal{A}(R, S)\) of \((0, 1)\)-matrices, *Acta
11. D. W. Walkup, Minimal interchanges of \((0, 1)\)-matrices and disjoint circuits in a graph,
13. H. Zhang, Hamiltonicity of interchange graphs of a type of matrices of zeros and ones,
    *J. Xinjiang Univ.* 9 (1992), 1–6.