Persistence Analysis of Interconnected Positive Systems with Communication Delays

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Abstract—This is a continuation of our preceding studies on the analysis of interconnected positive systems. Under mild conditions on positive subsystems and a nonnegative interconnection matrix, we showed that the state of the interconnected positive system converges to a positive scalar multiple of a prescribed positive vector. As a byproduct of this property, called persistence, it turned out that the output converges to the positive right eigenvector of the interconnection matrix. This result is effectively used in the formation control of multi-agent positive systems. The goal of this paper is to prove that the essential property of persistence is still preserved under arbitrary (time-invariant) communication delays. In the context of formation control, this preservation indicates that the desired formation is achieved robustly against communication delays, even though the resulting formation is scaled depending upon initial conditions for the state. From a mathematical point of view, the key issue is to prove that the delay interconnected positive system has stable poles only except for a pole of degree one at the origin, even though it has infinitely many poles in general. To this end, we develop frequency-domain (s-domain) analysis for delay interconnected positive systems.

Keywords: positive system, communication delay, persistence, formation control.

I. INTRODUCTION

The analysis and synthesis of positive systems (PSs) receive great attention recently. A linear time-invariant (LTI) system is said to be positive if its state and output are both nonnegative for any nonnegative initial state and nonnegative input [8], [11]. This property can be seen naturally in biology, network communications, economics and probabilistic systems. Moreover, simple dynamical systems such as integrator and first-order lag and their series/parallel connections are all positive. Even though their dynamics are pretty simple, large-scale systems constructed from those subsystems exhibit complicated behavior and deserve investigation in the area of multi-agent systems [13], [17], [18]. The study on PSs is also attractive because it has close relationship with convex optimization and especially, linear programming. Nowadays, it is known that PSs admit extremely particular (strong) results for analysis and synthesis that are not valid for general systems; see, e.g., [14], [15], [16], [3]. This particularity still holds for LTI time-delay positive systems (TDPSs), and plenty of fruitful results have already been reported, see, e.g., [4], [1], [12], [7]. Among them, it is shown in [4], [1] that a TDPS is asymptotically stable if and only if its corresponding delay-free system is asymptotically stable.

Even though most of existing studies for positive systems focus on stability and stabilization, it is important to bring the system of interest to a stability boundary in the (consensus-based) formation control of multi-agent systems [13], [17], [18]. This is the motivation of our preceding studies [5], [6] on interconnected positive systems (see Fig. 1). Under mild conditions on positive subsystems and a nonnegative interconnection matrix, we showed that the interconnected positive system is on a stability boundary and its state converges to a positive scalar multiple of a prescribed positive vector. As a byproduct of this property, called persistence, it turned out that the output converges to the positive right eigenvector of the interconnection matrix. As expected, this result is effectively used in the formation control of multi-agent positive systems [5], [6].

The goal of this paper is to prove that the essential property of persistence is still preserved for delay interconnected positive systems (see Fig. 2). In the context of formation control of multi-agent positive systems, this preservation indicates that the desired formation is achieved robustly against arbitrary (time-invariant) communication delays, even though the resulting formation is scaled depending upon initial conditions for the state. From a mathematical point of view, the key issue is to prove that the delay interconnected positive system has stable poles only except for a pole of degree one at the origin, even though it has infinitely many poles in general. To this end, we develop frequency-domain (s-domain) analysis for delay interconnected positive systems.

We use the following notations. We denote by $\mathbb{R}$ and $\mathbb{C}$ the set of real and complex numbers, respectively. We also use $\mathbb{R}_+ := \mathbb{R}_+ \cup \{0\}$ for the set nonnegative real numbers, and complex numbers with nonpositive (negative) real parts. The set of positive integers up to $N$ is denoted by $\mathbb{Z}_N$, i.e., $\mathbb{Z}_N := \{1, \ldots, N\}$. For given two matrices $A$ and $B$ of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all $(i,j)$, where $A_{ij}$ stands for the $(i,j)$-entry of $A$. In relation to this notation, we also define $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x > 0\}$ and $\mathbb{R}_+^{n \times m}$ with obvious modifications. For $A \in \mathbb{R}^{n \times n}$, we denote by $\sigma(A)$ and $\rho(A)$ the set of the eigenvalues of $A$ and the spectral radius of $A$, respectively. For $A \in \mathbb{R}_+^{n \times n}$, Theorem 8.3.1 in [10] states that there is an eigenvalue equal to $\rho(A)$. This eigenvalue is related...
to the Perron-Frobenius theorem and denoted by $\lambda_F(A)$ in this paper. We finally define the set of $n$-vector-valued continuous functions over $[a, b]$ by $C([a, b], \mathbb{R}^n)$, and the set of nonnegative $n$-vector-valued continuous functions over $[a, b]$ by $C([a, b], \mathbb{R}^+_n)$.

We finally define the set of nonnegative for any nonnegative initial state and nonnegative input.

**Definition 1 (Metzler Matrix):**[8] A matrix $A \in \mathbb{R}^{n \times n}$ is said to be Metzler if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ ($i \neq j$).

In the sequel, we denote by $\mathbb{M}^+ \times \mathbb{M}^+$ (the set of Metzler (Hurwitz stable) matrices of size $n$).

\[ \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & G_N \end{bmatrix} \]

**Proposition 1:**[8] The system (1) is positive if and only if $A \in \mathbb{M}^+, B \in \mathbb{R}^{n \times w+n}$, $C \in \mathbb{R}^{n \times n}$, and $D \in \mathbb{R}^{n \times w}$.

**III. Persistence of Interconnected Positive Systems**

In this section we quickly review our preceding results on the persistence of interconnected positive systems [5].

Consider the stable, SISO, strictly proper, and positive subsystem $G_i$ ($i \in \mathbb{Z}_N$) represented by

\[ G_i: \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i w_i(t), \\ z_i(t) = C_i x_i(t), \end{cases} \]

$A_i \in \mathbb{M}^+ \cap \mathbb{M}^{n \times n}$, $B_i \in \mathbb{R}_{+}^{n \times 1}$, $C_i \in \mathbb{R}_{+}^{1 \times n_i}$.

With these positive subsystems, we define a positive and stable system $G$ by $G := \text{diag}(G_1, \cdots, G_N)$. The state space realization of $G$ is given by

\[ \dot{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_x}, \quad \text{n}_2 := \sum_{i=1}^{N} n_i, \]

\[ \hat{z} := \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{R}^{N}, \quad \hat{w} := \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \in \mathbb{R}^{N}. \]

For given interconnection matrix $\Omega \in \mathbb{R}_{+}^{N \times N}$, we studied in [5], [6] the interconnected system $G \ast \Omega$ defined by

\[ \hat{w}(t) = \Omega \hat{z}(t). \]

The block-diagram of the interconnected system $G \ast \Omega$ is shown in Fig. 1. Note that $G \ast \Omega$ is positive since its state-space realization is given by

\[ \hat{x}(t) = A_3 \hat{x}(t), \quad A_3 := A + B \Omega C \in \mathbb{M}^{n_x} \]

We next recall the definition of persistence of $G \ast \Omega$.

**Definition 3:**[5] For given positive subsystems $G_i$ ($i \in \mathbb{Z}_N$) represented by (2) and interconnection matrix $\Omega \in \mathbb{R}_{+}^{N \times N}$, consider the interconnected positive system $G \ast \Omega$. Then, $G \ast \Omega$ is said to have the property of persistence if there exist $\xi_0, \xi_\infty \in \mathbb{R}_{+}^{n_x}$ such that

\[ \lim_{t \to \infty} \hat{x}(t) = (\xi_0^T \hat{w}(0)) \xi_\infty \]

for any initial state $\hat{x}(0) \in \mathbb{R}^{n_x}$.

This definition requires that the state $\hat{x}$ of $G \ast \Omega$ converges to a positive scalar multiple of a positive vector (i.e., all the states $\hat{x}_i$ ($i = 1, \cdots, n_2$) become positive and hence “excited” eventually) as long as $\hat{x}(0) \in \mathbb{R}_{+}^{n_x} \setminus \{0\}$. This is the reason why we call the property persistence. On the persistence of $G \ast \Omega$, the next result has been shown in [5].

**Theorem 1:**[5] For given positive subsystems $G_i$ ($i \in \mathbb{Z}_N$) represented by (2) and interconnection matrix $\Omega \in \mathbb{R}_{+}^{N \times N}$, suppose the following conditions are satisfied:

The diagram of the interconnected system $G \ast \Omega$ is given in Fig. 1. Note that $G \ast \Omega$ is positive since its state-space realization is given by

\[ \hat{x}(t) = A_3 \hat{x}(t), \quad A_3 := A + B \Omega C \in \mathbb{M}^{n_x}. \]
(i) \((A_i, B_i)\) is controllable and \((A_i, C_i)\) is observable for all \(i \in \mathbb{Z}_N\).
(ii) For \(G_i(s) := C_i(sI - A_i)^{-1}B_i\) \((i \in \mathbb{Z}_N)\), condition \(G_1(0) = \cdots = G_N(0) =: \gamma(> 0)\) holds.
(iii) The interconnection matrix \(\Omega \in \mathbb{R}^{N \times N}\) is irreducible (i.e., the directed graph \(\Gamma(\Omega)\) is strongly connected).
(iv) \(\lambda_p(\Omega) = 1/\gamma\) holds.

Then, for the interconnected positive system \(G \star \Omega\), the next result holds.

(I) The matrix \(A_{cl}\) given by (6) has an eigenvalue zero that is algebraically (and hence geometrically) simple. Moreover, \(A_{cl}\) satisfies \(\Re(\lambda) < 0 \forall \lambda \in \sigma(A_{cl}) \setminus \{0\}\).

(II) If we denote the right and left eigenvectors of \(\Omega\) associated with the eigenvalue \(\lambda_p(\Omega)\) by \(v_R \in \mathbb{R}_+^N\) and \(v_L \in \mathbb{R}_+^N\), respectively, we have \(A_{cl}v_R = 0\) and \(\xi^T_L A_{cl} = 0\) where
\[
\xi_R = -A^{-1}B v_R \in \mathbb{R}_+^N,
\]
\[
\xi_L = -A^{-T}C^T v_L \in \mathbb{R}_+^N,
\]
where \(\xi^T_L \xi_R = 1\). Here the eigenvectors \(v_R, v_L \in \mathbb{R}_+^N\) are appropriately scaled so that \(\xi^T_L \xi_R = 1\) is satisfied.

(III) For any initial state \(\hat{x}(0) \in \mathbb{R}_+^N\), the state \(\hat{x}\) of \(G \star \Omega\) satisfies
\[
\lim_{t \to \infty} \hat{x}(t) = (\xi^T_L \hat{x}(0))\xi_R \in \mathbb{R}_+^N.
\]
Namely, the interconnected system \(G \star \Omega\) has the property of persistence.

(IV) The output \(\hat{z}\) of \(G \star \Omega\) satisfies
\[
\lim_{t \to \infty} \hat{z}(t) = \gamma(\xi^T_L \hat{x}(0)) v_R \in \mathbb{R}_+^N.
\]

(V) If we define a linear function \(V : \mathbb{R}_+^N \to \mathbb{R}\) by

\[
V(\hat{x}(t)) := \xi^T_L \hat{x}(t),
\]
we have
\[
V(\hat{x}(t)) = V(\hat{x}(0)) (\forall t \in \mathbb{R}_+).
\]
Namely, the quantity \(V\) serves as the first integral (conserved quantity) of the system \(G \star \Omega\).

As shown in (9) of (III), the interconnected system \(G \star \Omega\) satisfying the conditions (i)-(iv) has the property of persistence. Beyond that, the result (10) clearly shows that the output \(\hat{x}(t) = [z_1(t) \cdots \ z_N(t)]^T\) of the interconnected system \(G \star \Omega\) converges to \(\gamma(\xi^T_L \hat{x}(0)) v_R \in \mathbb{R}_+^N\). From the viewpoint of formation control, this result implies that, for given \(v_{obj} \in \mathbb{R}_+^N\) that represents the “shape” of the desired formation, we can achieve \(\lim_{t \to \infty} \hat{z}(t) = \gamma(\xi^T_L \hat{x}(0)) v_{obj} \in \mathbb{R}^N\) by designing an interconnection matrix \(\Omega \in \mathbb{R}^{N \times N}\) satisfying (iii) and \(\Omega v_{obj} = (1/\gamma) v_{obj}\) \(^{11}\). Based on these ideas, a formation control of multi-agent positive systems is achieved in a sound way in [5], [6].

IV. COMMUNICATION DELAY CASES: THE COUNTERPART RESULTS

Let us consider the case where the interconnected system \(G \star \Omega\) operates with communication delays. More precisely, we consider the case where (5) is replaced by

\[
\tilde{w}(t) = \sum_{i,j=1}^{N} \Omega_{ij} \tilde{z}(t - h_{ij}).
\]

Here, \(h_{ij} \geq 0\) stands for the delay over the communication from subsystem \(G_i\) to \(G_i\). Note that the entries of \(\Omega_{ij} \in \mathbb{R}_+^{N \times N}\) are all zero except for the \((i,j)\)-entry and \(\sum_{i,j=1}^{N} \Omega_{ij} = \Omega\). We denote by \(G \star \Omega_h\) the interconnected system constructed by \(G\) and (12). The block-diagram of \(G \star \Omega_h\) is shown in Fig. 2. The state-space realization of \(G \star \Omega_h\) is given by

\[
\hat{x}(t) = A\hat{x}(t) + \sum_{i,j=1}^{N} B\Omega_{ij}C\hat{z}(t - h_{ij}),
\]

(13)

Here, \(h := \max_{i,j} h_{ij}\) and \(\phi \in C([-h, 0], \mathbb{R}_+)\) is the initial state. We see from [4] that \(G \star \Omega_h\) is positive, in the sense that \(\hat{x}(t) \geq 0 \forall t \in \mathbb{R}_+\) holds for any initial state \(\phi \in C([-h, 0], \mathbb{R}_+)\). The positivity of \(G \star \Omega_h\) is ensured by the positivity of subsystems \(G_i (i \in \mathbb{Z}_N)\) and \(\Omega \in \mathbb{R}^{N \times N}_+\).

The characteristic function of (13) is given by

\[
\Delta(s) = \det(F(s)),
\]

\[
F(s) := I - A - \sum_{i,j=1}^{N} B\Omega_{ij}Ce^{-sh_{ij}}.
\]

(14)

A zero of \(\Delta(s)\) is referred to as a pole of \(G \star \Omega_h\). If we denote the set of poles of \(G \star \Omega_h\) by \(\Lambda\), which is an infinite set in general, it is known that \(G \star \Omega_h\) is stable if and only if \(\Lambda \subset \mathbb{C}_-\) [9].

Our goal in this paper is to prove that the essential property of persistence is still preserved against arbitrary (time-invariant) communication delays. For rigorous treatment, we first make the definition of persistence for delay interconnected positive systems in a conformable fashion to the delay free case, i.e., Definition 3.

Definition 4: For given positive subsystems \(G_i (i \in \mathbb{Z}_N)\) represented by (2), interconnection matrix \(\Omega \in \mathbb{R}^{N \times N}_+\), and communication delays \(h_{ij} \geq 0 \forall (i,j) \in \mathbb{Z}_N\), consider the delay interconnected positive system \(G \star \Omega_h\). Then, \(G \star \Omega_h\) is said to have the property of persistence if there exist a linear functional \(\alpha : \mathbb{C}([-h, 0], \mathbb{R}_+) \to \mathbb{R}\) and \(\xi_\infty \in \mathbb{R}_+^N\) such that the following conditions hold:

(i) \(\lim_{t \to \infty} \hat{x}(t) = \alpha(\phi) \xi_\infty\) \(\forall \phi \in \mathbb{C}([-h, 0], \mathbb{R}_+)\) \(\Rightarrow \) \(\alpha(\phi) > 0\).

(ii) The linear functional \(\alpha\) satisfies \(\alpha(\phi) > 0\) for all \(\phi \in \mathbb{C}([-h, 0], \mathbb{R}_+)\) and \(\alpha(0) = 0\).

In this definition, the requirement (ii) corresponds to \(\xi_0 \in \mathbb{R}_+^N\) in Definition 3. Indeed, if we let \(h = 0\) in Definition 4, then the linear functional \(\alpha\) satisfying (ii) reduces to

\[
\alpha(\phi) = \alpha(\phi(0)) = \xi_0^T \phi(0) =: \xi_0^T \phi(0)\]

for some \(\xi_0 \in \mathbb{R}_+^N\). It follows that Definition 4 is a natural extension of Definition 3 to communication-delay-cases.

We are now in the position to state the main result of this paper on the persistence of \(G \star \Omega_h\). For concise statements of the main theorem, we make the next definition beforehand.

\(^{11}\)If \(\Omega \in \mathbb{R}^{N \times N}_+\) is irreducible, only the Frobenius eigenvalue has associated eigenvector that is strictly positive [10]. Therefore \(\Omega v_{obj} = (1/\gamma) v_{obj}\) for \(v_{obj} \in \mathbb{R}_+^N\) ensures \(\lambda_p(\Omega) = 1/\gamma\).
Definition 5: We define the linear functional \( \alpha_0 : C([-h, 0], \mathbb{R}^{n_x}) \rightarrow \mathbb{R} \) by
\[
\alpha_0(\phi) = \begin{cases} 
1 + \xi_T L \left( \sum_{i,j=1}^{N} h_{ij} B \Omega_{ij} C \right) \xi_R & \\
\times \xi_T L \phi(0) + \sum_{i,j=1}^{N} B \Omega_{ij} C \int_{-h_{ij}}^{0} \phi(\tau) d\tau .
\end{cases}
\]
(16)

Here, \( \xi_L, \xi_R \in \mathbb{R}^{n_z} \) are given by (8).

In this definition, note that the linear functional \( \alpha_0 \) satisfies the condition (ii) in Definition 4 since
\[
\alpha_0(\phi) \geq \begin{cases} 
1 + \xi_T L \left( \sum_{i,j=1}^{N} h_{ij} B \Omega_{ij} C \right) \xi_R & \\
\times \xi_T L \phi(0) > 0
\end{cases}
\]
holds for all \( \phi \in C([-h, 0], \mathbb{R}^{n_z}) \) and \( \phi(0) \neq 0 \). Under these preparations, we can state the next result.

Theorem 2: For given positive subsystems \( G_i (i \in \mathbb{Z}_N) \), interconnection matrix \( \Omega \in \mathbb{R}^{N \times N}_+ \), and communication delays \( h_{ij} \geq 0 \) \( (i, j \in \mathbb{Z}_N) \), suppose the conditions (i)-(iv) in Theorem 1 are satisfied. Then, for the delay interconnected positive system \( \mathcal{G} \ast \Omega_h \), the next results hold.

(\text{I}') The delay interconnected positive system \( \mathcal{G} \ast \Omega_h \) has stable poles only except for the pole of degree one at the origin.

(\text{III}') For any initial state \( \phi \in C([-h, 0], \mathbb{R}^{n_x}) \), the state \( \tilde{x} \) of \( \mathcal{G} \ast \Omega_h \) satisfies
\[
\lim_{t \to \infty} \tilde{x}(t) = \alpha_0(\phi) \xi_R \in \mathbb{R}^{n_z}
\]
where the linear functional \( \alpha_0 : C([-h, 0], \mathbb{R}^{n_z}) \rightarrow \mathbb{R} \) is given by (16) and \( \xi_R \in \mathbb{R}^{n_z} \). Namely, the delay interconnected positive system \( \mathcal{G} \ast \Omega_h \) has the property of persistence.

(\text{IV}') The output \( \tilde{z} \) of \( \mathcal{G} \ast \Omega_h \) satisfies
\[
\lim_{t \to \infty} \tilde{z}(t) = \gamma \alpha(\phi) v_R \in \mathbb{R}^{N}
\]
where \( v_R \in \mathbb{R}^N_+ \) is the right eigenvector of \( \Omega \in \mathbb{R}^{N \times N}_+ \) associated with the eigenvalue \( \lambda_\gamma(\Omega) \).

(V') Let us define a linear functional \( V_d : C([-h, t], \mathbb{R}^{n_z}) \rightarrow \mathbb{R} \) by
\[
V_d(\tilde{x}_d) := \xi_L \tilde{z}(t) + \sum_{i,j=1}^{N} B \Omega_{ij} C \int_{-h_{ij}}^{0} \tilde{z}(t + \tau) d\tau .
\]
(19)

where \( \tilde{x}_d(\tau) = \tilde{z}(t + \tau) (-h \leq \tau \leq 0) \). Then, we have
\[
V_d(\tilde{x}_d) = V_d(\tilde{x}_d) \quad (\forall t \in \mathbb{R}_+).
\]

Namely, the quantity \( V_d \) serves as the first integral (conserved quantity) of the system \( \mathcal{G} \ast \Omega_h \).

Even though the proof of this theorem is the theoretical core of this paper, we only provide a brief sketch for it in the appendix section due to limited space. The next important remarks follow on the results (I')-(V').

Remark 1:

- The result (I') implies that the delay-free interconnected positive system \( \mathcal{G} \ast \Omega \) that is on a stability boundary still remains to be on a stability boundary against arbitrary (time-invariant) communication delays. This is a novel result for the stability analysis of time-delay positive systems (TDPS), where an available result is that if a delay-free positive system is stable, then this system remains to be stable against arbitrary (time-invariant) delays [4]. For the proof of (I'), the standard time-domain arguments using Lyapunov-Krasovskii functionals do not work fine since we have to prove rigorously the “degree one property” of the pole at the origin. For this reason, we developed explicit proof of (I') by the arguments in frequency-domain (s-domain). See the appendix section for a brief sketch.

- If we let \( h = 0 \) in Theorem 2, the linear functional \( \alpha_0 \) given by (16) reduces to \( \alpha_0(\phi) = \xi_T L \phi(0) = \xi_T L \tilde{x}(0) \). It follows that the results (17) and (18) reduce respectively to (9) and (10). In this sense, the results (III') and (IV') are natural extension of (III) and (IV) in Theorem 1 to communication-delay-cases. Exactly the same comment applies also to the result (V') on the first integral of \( \mathcal{G} \ast \Omega_h \).

- The result (IV') clearly shows that the output \( \tilde{z} \) converges to a scalar multiple of \( v_R \), that is equal to the delay-free case. In the context of formation control of multi-agent positive systems [5], [6] that is briefly reviewed in the preceding section, this result ensures that the desired formation for an appropriately constructed interconnected positive system \( \mathcal{G} \ast \Omega \) is achieved robustly against arbitrary (time-invariant) communication delays, even though the resulting formation is scaled depending upon the initial state \( \phi \). Note that the dependence on the initial state, which is observed also in the delay-free case, is unavoidable in the current problem setting where we do not allow to equip external inputs for the interconnected positive systems.

To summarize, we have shown in Theorem 2 the counterpart results of Theorem 1 for delay interconnected positive systems. We derived explicit closed-form formulas (17), (18), and (19) by introducing the linear functional given by (16).

V. NUMERICAL EXAMPLES

Let us consider the formation control problem of a multi-agent system constructed from \( N \) agents [5]. The \( i \)-th agent \( \phi \in \mathbb{Z}_N \) can move over the \((x, y)\)-plane with independent (interference-free) dynamics \( Z_{i,x}(s) \) and \( Z_{i,y}(s) \) along \( x \) and \( y \) axes, respectively. Assume
\[
Z_{i,j}(s) = \frac{k_{i,j}}{s(s + a_{i,j})} U_{i,j}(s) \quad (i = 1, \ldots, N, \ j = x, y)
\]
where \( k_{i,j}, a_{i,j} > 0 \). Applying a local feedback
\[
U_{i,j}(s) = -f_{i,j}(Z_{i,j}(s) - W_{i,j}(s))
\]
with \( 0 < f_{i,j} < a_{i,j}^2 / 4k_{i,j} \), we have
\[
Z_{i,j}(s) = G_{i,j}(s) W_{i,j}(s)
\]
\[
G_{i,j}(s) = \begin{bmatrix} -p_{i,j} & 1 \\ 0 & -q_{i,j} \\ p_{i,j}q_{i,j} & 0 \end{bmatrix},
\]
\[
p_{i,j} + q_{i,j} = a_{i,j}, \quad p_{i,j}q_{i,j} = f_{i,j} k_{i,j}.
\]

It turns out that each subsystem \( G_{i,j}(i \in \mathbb{Z}_N, \ j = x, y) \) is stable, SISO, strictly proper, positive and satisfies \( G_{i,j}(0) = \)}
1. For simplicity, we consider the case \( k_{i,j} = k = 1 \), \( a_{i,j} = a = 10 \), and \( f_{i,j} = 0.8 \times a^2/4k = 20 \) (\( i \in \mathbb{Z}_N, \ j = x, \ y \)).

Assuming that the agent \( i \) can communicate with agent \( i - 1 \) and \( i + 1 \) (agent 0 and \( N + 1 \) should be regarded as agent \( N \) and 1, respectively), our goal here is to design a communication scheme over the agents with respect to each agent’s position so that prescribed formation can be achieved. To form a circle, we let \( [v_{obj,x} \ v_{obj,y}] = \left[ \frac{1}{2} + \cos(2\pi i/N) \ 2 + \sin(2\pi i/N) \right] \) and constructed two interconnection matrices \( \Omega_x, \ \Omega_y \) for independent communications along \( x- \) and \( y- \) axes, respectively, by following [5], [6].

Figs. 3 and 4 at the last page show the simulation results for delay-free-case, where we let the initial states of two interconnected systems along \( x- \) and \( y- \) axes as \( x_{i,j}(0) = [z_{i,j}(0) \ 0]^T (i \in \mathbb{Z}_N, \ j = x, y) \). In these figures, the blue dots show the terminal positions of agents computed beforehand from (10). We can confirm that the desired formation has been achieved around 5 [sec]. On the other hand, in Figs. 5-7, we show the simulation results under uniform communication delay 1 [sec]. We let the initial condition as \( \phi_{i,j}(t) = x_{i,j}(0) (-1 \leq t \leq 0, \ i \in \mathbb{Z}_N, \ j = x, y) \). The blue dots in these figures show the terminal positions of agents computed from (18). Again we can confirm that the desired formation has been achieved eventually. However, the convergence is definitely slower than the delay-free case.

VI. CONCLUSION

In this paper, we analyzed the persistence of interconnected positive systems with communication delays. We clarified that the essential property of persistence is still preserved against arbitrary communication delays. In the context of formation control of multi-agent positive systems, this result indicates that the formation is achieved robustly against arbitrary communication delays, even though the resulting formation is scaled depending upon the initial states. We verified all of the theoretical results by numerical examples on the formation control of multi-agent positive systems.

REFERENCES


APPENDIX

BRIEF SKETCH FOR THE PROOF OF THEOREM 2

A. Proof of (I’)

The assertion (I’) can be proved by the next lemma, in conjunction with the positiveness of each subsystem. Details are omitted due to limited space.

**Lemma 1:** For given \( A_0 \in \mathbb{M}_n \) and \( A_i \in \mathbb{R}^{n \times n} (i = 1, \ldots, L) \), suppose the following conditions hold.

I) The matrix \( A := \sum_{i=0}^{L} A_i \) has an eigenvalue zero that is algebraically (and hence geometrically) simple. Moreover, \( A \) satisfies \( \text{Re}(\lambda) < 0 \) (\( \forall \lambda \in \sigma(A) \setminus \{0\} \)).

II) \( \exists \xi_R \in \mathbb{R}_+^n \) such that \( A \xi_R = 0 \), \( \xi_R A = 0 \), and \( \xi_R^T \xi_R = 1 \).

Then, for any \( h_i \in \mathbb{R}_+ (i = 1, \ldots, L) \), the entire function \( \Delta_0(s) := \det \left( sI - A_0 - \sum_{i=1}^{L} A_ie^{-sh_i} \right) \)

has a zero of degree one at the origin. Moreover, the residue of \( \left( sI - A_0 - \sum_{i=1}^{L} A_ie^{-sh_i} \right)^{-1} \) at the origin is given by

\[
1 + \xi_R^T \left( \sum_{i=1}^{L} h_i A_i \right) \xi_R \xi_R^T.
\]  

(20)

B. Proof of (III’), (IV’), and (V’)

Once (I’) is proved, the proof of (III’), (IV’), and (V’) can be done straightforwardly by following the standard routine for dealing with time-delay systems [9], [2]. By applying the Laplace transform to (13) and denote by \( \tilde{X}(s) \) the Laplace transform of \( \tilde{z}(t) \), we have

\[
\tilde{X}(s) = F(s)^{-1} p(s)
\]  

(21)

where \( F(s) \) is given by (14) and \( p(s) \) is given by

\[
p(s) = \phi(0) + \sum_{i,j=1}^{N} \mathcal{B}_{ij} \mathcal{C} e^{-sh_{ij}} \int_{h_{ij}}^{0} \phi(\tau)e^{-\sigma \tau} d\tau.
\]  

(22)

By applying the inverse Laplace transform to (21), we can obtain the next lemma.
Lemma 2: [9] For sufficiently large $c$, we have
\[
\hat{x}(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{st} F(s)^{-1} p(s) ds \quad (t > 0)
\] (23)
where $F(s)$ and $p(s)$ are given by (14) and (22), respectively.

We next expand the solution in (23) as an infinite series. Intuitively, such expansion can be done by replacing the integration to a counter integration that includes the poles of $e^{ts}F(s)^{-1}p(s)$ (i.e., the zeros of $\Delta(s)$) and applying the residue theorem. Such intuition can indeed be justified and we can obtain the next lemma.

Lemma 3 ([2], p. 109): Let $\{s_r\}$ be the sequence of zeros of $\Delta(s)$ given by (14) arranged in order of decreasing real parts. Then
\[
\hat{x}(t) = \sum_{r=1}^{\infty} e^{s_r t} p_r(t) \quad (t > 0)
\] (24)
where $e^{s_r t} p_r(t)$ is the residue of $e^{ts}F(s)^{-1}p(s)$ at $s_r$ and $p_r(t)$ is a polynomial of degree less than the degree of $s_r$.

In (24), we see from the results of (I)' that $s_1 = 0$ whose degree is one and $\Re(s_r) < 0 \quad (i = 2, 3, \ldots)$. Since $p_r(t)$ is a polynomial, $\lim_{t \to \infty} \sum_{r=2}^{\infty} e^{s_r t} p_r(t) = 0$. It follows that
\[
\lim_{t \to \infty} \hat{x}(t) = \lim_{t \to \infty} p_1(t) + \lim_{t \to \infty} \sum_{r=2}^{\infty} e^{s_r t} p_r(t) = p_1
\] (25)
where $p_1(t) = p_1 \in \mathbb{R}^n$ is a constant. For the computation of $p_1$ that is the residue of $e^{ts}F(s)^{-1}p(s)$ at $s = s_1 = 0$, we see from (20) that
\[
\lim_{s \to 0} sF(s)^{-1} = \left\{ 1 + \xi^T \left( \sum_{i,j=1}^{N} h_{ij} B \Omega_{ij} C \right) \xi \right\}^{-1} \xi \xi^T.
\]
Therefore we arrive at
\[
p_1 = \lim_{s \to 0} s e^{ts} F(s)^{-1} p(s)
= \left\{ 1 + \xi^T \left( \sum_{i,j=1}^{N} h_{ij} B \Omega_{ij} C \right) \xi \right\}^{-1} \xi \xi^T p(0)
= \alpha_0(\phi) \xi \xi^T.
\]
Here, $\alpha_0: C([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ is given by (16). This completes the proof of (III'). The validity of (IV') and (V') follows via elementary mathematics.

Fig. 3. Positions of agents at $t = 0$[sec] (delay-free case).

Fig. 4. Positions of agents at $t = 5$[sec] (delay-free case).

Fig. 5. Positions of agents at $t = 0$[sec] (under communication delays).

Fig. 6. Positions of agents at $t = 5$[sec] (under communication delays).

Fig. 7. Positions of agents at $t = 16$[sec] (under communication delays).