

# Multi-species extension of the solvable partially asymmetric reaction-diffusion processes

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## Abstract

By considering the master equation of the partially asymmetric diffusion process on a one-dimensional lattice, the most general boundary condition (i.e. interactions) for the multi-species reaction-diffusion processes is considered. Resulting system has various interactions including diffusion to left and right, two-particle interactions  $A_\alpha A_\beta \rightarrow A_\gamma A_\delta$  and the extended  $n$ -particle drop-push interactions to left and right. We obtain three distinct new models. The conditions on reaction rates to ensure the solvability of the resulting models are obtained. The two-particle conditional probabilities are calculated exactly.

## 1 Introduction

The understanding of non-equilibrium statistical physics is still much more incomplete than that of equilibrium theory, due to the absence of an analogue of the Boltzman-Gibbs approach and in spite of considerable recent progress [1]. Therefore non-equilibrium systems have to be specified by some defining dynamical rules which are then analyzed. The topic has received a lot of attention and many reviews exist, e.g. [2]-[7].

One of the interesting and important examples of the non-equilibrium systems is the one-dimensional reaction-diffusion processes, which have application in various fields of physics like study of the shocks [8], noisy Burgers equation [9], polymers in random media [10], traffic models [11], and biopolymerization [12]. As these systems are interacting systems with  $N$ -particle, even simple models may pose a formidable problem if one wants to approach them analytically. See [13]-[16] for more recent references.

The simplest reaction-diffusion process is the totally asymmetric simple exclusion process (TASEP). In this model, each lattice site is occupied by at most one particle and all particles can only hop with equal rate to their right-neighboring sites, if these sites are not occupied. TASEP has been studied in [17] by introducing a master equation which describes the evolution equation of the particles when they are not in neighboring sites, and a so-called boundary

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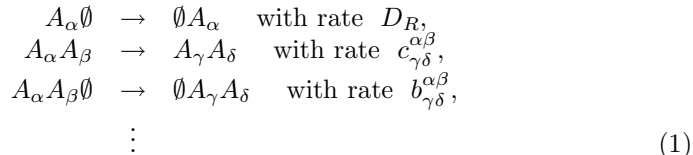
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condition, which specifies the situation in which the probabilities go outside the physical regions. This happens when some of the particles are in adjacent sites and the master equation can not be applied to them. It has been shown that the model is integrable in the sense that the  $N$ -particle  $S$ -matrix is factorized into a product of two-particle  $S$ -matrices. The coordinate Bethe ansatz has been used in this proof.

The interesting observation is that if one chooses other boundary conditions, with the same master equation, one can in principle introduce other interactions (besides diffusion to right-neighboring sites), which may be integrable in the abovementioned sense. This is what is first done in [18], in which the so-called drop-push model has been studied by this method. In this model the particle hops to the next right site, even it is occupied. It can hop by pushing all the neighboring particles to their next right sites, with a rate depending on the number of these particles. Some other generalization of TASEP can be found in [19]-[21].

The generalization of one-species reaction-diffusion processes to  $p$ -species is an important task. The main problem in this generalization, besides introducing a set of suitable boundary conditions to model an interacting system, arises from the above mentioned factorization of  $N$ -particle scattering matrix. It was shown in [22] that in order that a more-than-one species system be solvable, in the sense of the Bethe ansatz, certain relations should be satisfied between the rates. These relations can be written as some kind of a spectral Yang-Baxter (SYB) equation. By this method, all the solvable two-species reaction-diffusion models, without annihilation and creation reactions and with equal reaction rates, have been obtained in [22].

The multi-species generalization of the reactions considered in [22] has been studied in [23], and the drop-push reaction of [18] has been generalized to  $p$ -species in [24]. The most general totally asymmetric reaction-diffusion processes has been recently studied in [25]. These processes are

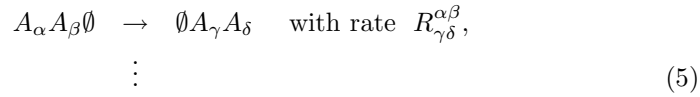
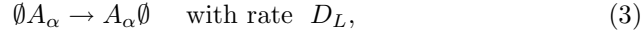
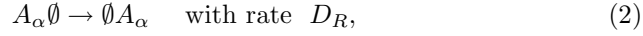


where the dots indicate the other drop-push reactions with  $n$ -adjacent particles, in which in the meantime the types of the particles can also be changed. These latter reactions are called the extended drop-push processes. It has been shown that the reaction rates of processes (1) must satisfy some specific constraints, in order that we have a set of consistent evolution equations. Also the corresponding two-particle  $S$ -matrices must satisfy the SYB equation. Some classes of the solutions of these equations have been discussed in [25].

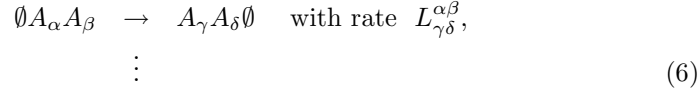
In all of the above studies, only the totally asymmetric exclusion processes have been considered, i.e. the particles can only diffuse to their next right neighboring sites. If one wants to consider the left and right diffusions simultaneously, one must consider a more general master equation with suitable boundary conditions and then seek the situations in which the model is integrable. In [17], one-species model with only simple diffusion to left and right (i.e. partially asymmetric) has been considered, and in [26], the one-species partially asymmetric drop-push model has been studied. Finally a two-species model in which

the particles, besides diffusion to left and right, have exchange-reaction has been studied in [27].

In this paper we want to study the most general  $p$ -species integrable models with partially asymmetric reaction-diffusion processes, which all the previous studied models are the special cases of them. These general models may have some or all of the following reactions:



and



In above equations  $\alpha, \beta, \dots = (1, \dots, p)$ ,  $\emptyset$  stands for vacancy, and dots in eqs.(5) and (6) indicate the drop-push of  $n$ -adjacent particles to right and left sites, respectively, in which in the meantime the types of the particles can also be changed. We call interactions (5) and (6) as right-drop-pushing and left-drop-pushing, respectively. We show that there are three distinct models which are integrable and each of these models contains reactions (2) and (3) and one or two of the reactions (4) to (6).

The sheme of the paper is as follows. There are two kinds of boundary conditions that can be generalized to  $p$ -species cases. In section 2, we generalize the first kind of boundary condition, which was introduced in [26], to the most general  $p$ -species case. Using the law of conservation of number of particles, it is shown that there exists five constraints that must be satisfied by reaction rates of eqs.(2)-(6), in order to have a set of consistent evolution equations to express the interactions (2)-(6). But it is seen that there is no solution for these constraints. The situation does not change even if we relax one of the constraints by including the annihilation processes. Therefore one can not explain all the reactions (2)-(6) by this method. But it will be shown that we can have two distinct models. In the first type model the reactions are eqs.(2), (3), (4) and (5) and in the second type the reactions are eqs.(2), (3), (4) and (6).

The second kind of boundary condition, which was used in [17] and [27], is generalized to the most general  $p$ -species case in section 3. We show that the resulting consistent boundary condition can explain the reactions (2), (3) and (4). This is the type 3 model. It must be mentioned that the type 3 model is not a subclass of types 1 and 2 and is a new distinct one. In section 4 we investigate the Bethe ansatz solution for these models and discuss the solutions of the corresponding SYB equations. We see that the  $S$ -matrix of type 3 model

is much more involved than two other ones and therefore only some special classes of solutions of its SYB equation can be obtained. Finally we study the conditional probabilities of these models and in special two-particle sector, we obtain the exact expressions.

## 2 First kind generalization

Consider a  $p$ -species system with particles  $A_1, A_2, \dots, A_p$ . The basic objects we are interested in are the probabilities  $P_{\alpha_1 \dots \alpha_N}(x_1, \dots, x_N; t)$  for finding at time  $t$  the particle of type  $\alpha_1$  at site  $x_1$ , particle of type  $\alpha_2$  at site  $x_2$ , etc.. We take the physical region of coordinates as  $x_1 < x_2 < \dots < x_N$ . The master equation for a partially asymmetric exclusion process is

$$\begin{aligned} \frac{\partial}{\partial t} P_{\alpha_1 \dots \alpha_N}(x_1, \dots, x_N; t) &= D_R \sum_{i=1}^N P_{\alpha_1 \dots \alpha_N}(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_N; t) \\ &+ D_L \sum_{i=1}^N P_{\alpha_1 \dots \alpha_N}(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_N; t) \\ &- N P_{\alpha_1 \dots \alpha_N}(x_1, \dots, x_N; t). \end{aligned} \quad (7)$$

This equation describes a collection of  $N$  particles, diffusing to the next-right sites by rate  $D_R$  and to the next-left sites by rate  $D_L$ . In eq.(7) we have used a time scale so that

$$D_R + D_L \equiv 1. \quad (8)$$

This master equation is only valid for  $x_i < x_{i+1} - 1$ . For  $x_i = x_{i+1} - 1$ , there will be some terms with  $x_i = x_{i+1}$  in the right hand side of eq.(7) which are out of the physical region. But one can assume that (7) is valid for all the physical region  $x_i < x_{i+1}$  by imposing certain boundary conditions for  $x_i = x_{i+1}$ . Different boundary condition introduces different interactions for particles. Following the argument which have been given in [25], it can be easily seen that the master equation (7) leads to following relation for two-particle probabilities:

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{x_2} \sum_{x_1 < x_2} P_{\alpha_1 \alpha_2}(x_1, x_2; t) &= \sum_x [D_R P_{\alpha_1 \alpha_2}(x, x; t) + D_L P_{\alpha_1 \alpha_2}(x + 1, x + 1; t)] - \sum_x P_{\alpha_1 \alpha_2}(x, x + 1; t) \\ &= \sum_x P_{\alpha_1 \alpha_2}(x, x; t) - \sum_x P_{\alpha_1 \alpha_2}(x, x + 1; t). \end{aligned} \quad (9)$$

This equation leads us to take  $P_{\alpha_1 \alpha_2}(x, x; t)$  as linear combination of  $P_{\beta_1 \beta_2}(x, x + 1; t)$  and  $P_{\beta_1 \beta_2}(x - 1, x; t)$ 's as the only choice for having a consistent set of evolution equations in more-than-two-particle sectors [25]. Therefore the most general boundary condition is

$$P_{\alpha_1 \alpha_2}(x, x) = \sum_{\beta} b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x - 1, x) + \sum_{\beta} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x, x + 1). \quad (10)$$

$\beta$  stands for  $(\beta_1 \beta_2)$  and  $b$  and  $c$  are  $p^2 \times p^2$  matrices determine the interactions. In the probabilities appear in eq.(10), we have suppressed all the other coordinates and the time  $t$  for simplicity. In fact  $P_{\alpha_1 \alpha_2}(x, x) := P_{\gamma_1 \dots \gamma_i \alpha_1 \alpha_2 \gamma_{i+3} \dots \gamma_N}(x_1, \dots, x_i, x, x, x_{i+3}, \dots, x_N)$ . In the first step, let us exclude the creation and annihilation processes (it can

be shown that one can not study the creation processes by this method, so in fact in this step, we exclude the annihilation processes). Since the number of particles is constant in time, summing over  $\alpha_1$  and  $\alpha_2$  makes the left-hand side of (9) zero and results:

$$-\sum_x \sum_\alpha P_{\alpha_1 \alpha_2}(x, x+1) + \sum_x \sum_\beta \left( \sum_\alpha (b+c)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \right) P_{\beta_1 \beta_2}(x, x+1) = 0, \quad (11)$$

in which eq.(10) has been used. Clearly eq.(11) gives:

$$\sum_\alpha (b+c)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = 1 \quad \text{constraint (I)}. \quad (12)$$

Note that in  $p = 1$ , the boundary condition (10) and constraint (12) reduce to those considered in [26]. Also in the case of totally asymmetric processes in which  $D_L = 0$ , our problem reduces to one considered in [25]. Following the same steps as [25], we first consider  $\dot{P}_{\alpha_1 \alpha_2}(x, x+1)$ . Using eqs.(7) and (10), it is found

$$\begin{aligned} \dot{P}_{\alpha_1 \alpha_2}(x, x+1) &= D_R P_{\alpha_1 \alpha_2}(x-1, x+1) + D_L P_{\alpha_1 \alpha_2}(x, x+2) \\ &\quad + D_R \sum_\beta b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x-1, x) + D_L \sum_\beta c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x+1, x+2) \\ &\quad + (D_R \sum_\beta c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} + D_L \sum_\beta b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}) P_{\beta_1 \beta_2}(x, x+1) - 2P_{\alpha_1 \alpha_2}(x, x+1) \\ &= D_R P_{\alpha_1 \alpha_2}(x-1, x+1) + D_L P_{\alpha_1 \alpha_2}(x, x+2) + D_R \sum_\beta b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x-1, x) \\ &\quad + D_L \sum_\beta c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x+1, x+2) + \sum_{\beta \neq \alpha} (D_R c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} + D_L b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}) P_{\beta_1 \beta_2}(x, x+1) \\ &\quad - [D_R + D_L + \sum_{\beta \neq \alpha} (D_R c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} + D_L b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}) + \sum_\beta (D_R b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} + D_L c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2})] P_{\alpha_1 \alpha_2}(x, x+1), \end{aligned} \quad (13)$$

in which we use eqs.(8) and (12). The latter can be written as:

$$c_{\alpha_1 \alpha_2}^{\alpha_1 \alpha_2} = 1 - \sum_\beta b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} - \sum_{\beta \neq \alpha} c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}, \quad (14)$$

or

$$b_{\alpha_1 \alpha_2}^{\alpha_1 \alpha_2} = 1 - \sum_{\beta \neq \alpha} b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} - \sum_\beta c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}. \quad (15)$$

It is seen that the evolution equation (13) describes the following two-particle interactions:

$$\begin{aligned} A_\alpha \emptyset &\rightarrow \emptyset A_\alpha && \text{with rate } D_R, \\ \emptyset A_\alpha &\rightarrow A_\alpha \emptyset && \text{with rate } D_L, \\ A_\alpha A_\beta &\rightarrow A_\gamma A_\delta && \text{with rate } D_R c_{\gamma\delta}^{\alpha\beta} + D_L b_{\gamma\delta}^{\alpha\beta}, \\ A_\alpha A_\beta \emptyset &\rightarrow \emptyset A_\gamma A_\delta && \text{with rate } D_R b_{\gamma\delta}^{\alpha\beta}, \\ \emptyset A_\alpha A_\beta &\rightarrow A_\gamma A_\delta \emptyset && \text{with rate } D_L c_{\gamma\delta}^{\alpha\beta}. \end{aligned} \quad (16)$$

To study the consistency of our formalism and also deriving the more-than-two particle interactions, we consider  $\dot{P}_{\alpha_1 \dots \alpha_n}(x, x+1, \dots, x+n-1)$ . In  $n=3$ , we encounter two boundary terms  $P_{\alpha_1 \alpha_2 \alpha_3}(x, x+1, x+1)$  and  $P_{\alpha_1 \alpha_2 \alpha_3}(x+1, x+1, x+2)$ . Using (10), the first one becomes:

$$P_{\alpha_1 \alpha_2 \alpha_3}(x, x+1, x+1) = \sum_{\beta \gamma} b_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} [b_{\alpha_1 \beta_2}^{\gamma_1 \gamma_2} P_{\gamma_1 \gamma_2 \beta_3}(x-1, x, x+1) + c_{\alpha_1 \beta_2}^{\gamma_1 \gamma_2} P_{\gamma_1 \gamma_2 \beta_3}(x, x+1, x+1)] + \sum_{\beta} c_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} P_{\alpha_1 \beta_2 \beta_3}(x, x+1, x+2) \quad (17)$$

which describes the boundary term  $P_{\alpha_1 \alpha_2 \alpha_3}(x, x+1, x+1)$  as a linear combination of other boundary terms, i.e.  $P_{\gamma_1 \gamma_2 \beta_3}(x, x+1, x+1)$ 's. As has been shown in [25], the only consistent solution to this problem is the vanishing of these terms in the right-hand side of eq.(17), which results:

$$\sum_{\beta_2} c_{\alpha_1 \beta_2}^{\gamma_1 \gamma_2} b_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} = 0 \quad \text{constraint (II)}, \quad (18)$$

or

$$(1 \otimes b)(c \otimes 1) = 0, \quad (19)$$

in which 1 stands for the  $p \times p$  identity matrix. The second boundary term is

$$P_{\alpha_1 \alpha_2 \alpha_3}(x+1, x+1, x+2) = \sum_{\beta} b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2 \alpha_3}(x, x+1, x+2) + \sum_{\beta \gamma} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} [b_{\beta_2 \alpha_3}^{\gamma_2 \gamma_3} P_{\beta_1 \gamma_2 \gamma_3}(x+1, x+1, x+2) + c_{\beta_2 \alpha_3}^{\gamma_2 \gamma_3} P_{\beta_1 \gamma_2 \gamma_3}(x+1, x+2, x+3)], \quad (20)$$

which again leads us to take

$$\sum_{\beta_2} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} b_{\beta_2 \alpha_3}^{\gamma_2 \gamma_3} = 0 \quad \text{constraint (III)}, \quad (21)$$

or

$$(c \otimes 1)(1 \otimes b) = 0. \quad (22)$$

Assuming constraints (18) and (21) and using eqs.(7) and (10),  $\dot{P}_{\bar{\alpha}}(x, x+1, x+2)$  is

$$\begin{aligned} \dot{P}_{\bar{\alpha}}(x, x+1, x+2) &= D_R P_{\bar{\alpha}}(x-1, x+1, x+2) + D_L P_{\bar{\alpha}}(x, x+1, x+3) \\ &+ D_R \sum_{\beta} b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2 \alpha_3}(x-1, x, x+2) + D_L \sum_{\beta} c_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} P_{\alpha_1 \beta_2 \beta_3}(x, x+2, x+3) \\ &+ \sum_{\beta \neq \alpha} (D_R c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} + D_L b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}) P_{\beta_1 \beta_2 \alpha_3}(x, x+1, x+2) \\ &+ \sum_{\beta \neq \alpha} (D_R c_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} + D_L b_{\alpha_2 \alpha_3}^{\beta_2 \beta_3}) P_{\alpha_1 \beta_2 \beta_3}(x, x+1, x+2) \\ &+ D_R \sum_{\gamma} b_{\bar{\alpha}}^{\gamma} P_{\bar{\gamma}}(x-1, x, x+1) + D_L \sum_{\gamma} c_{\bar{\alpha}}^{\gamma} P_{\bar{\gamma}}(x+1, x+2, x+3) \\ &- [D_R + D_L + D_R \sum_{\beta} b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} + D_L \sum_{\beta} c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} + D_R \sum_{\beta} b_{\beta_2 \beta_3}^{\alpha_2 \alpha_3} + D_L \sum_{\beta} c_{\beta_2 \beta_3}^{\alpha_2 \alpha_3} \\ &+ \sum_{\beta \neq \alpha} (D_R c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} + D_L b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}) + \sum_{\beta \neq \alpha} (D_R c_{\beta_2 \beta_3}^{\alpha_2 \alpha_3} + D_L b_{\beta_2 \beta_3}^{\alpha_2 \alpha_3})] P_{\bar{\alpha}}(x, x+1, x+2), \end{aligned}$$

(23)

in which we have used eqs.(14) and (15) for diagonal elements of matrix  $D_R c + D_L b$ .  $b_{\vec{\alpha}}^{\vec{\gamma}}$  and  $c_{\vec{\alpha}}^{\vec{\gamma}}$  are defined as following

$$b_{\vec{\alpha}}^{\vec{\gamma}} = \sum_{\gamma} b_{\alpha_1 \beta}^{\gamma_1 \gamma_2} b_{\alpha_2 \alpha_3}^{\beta \gamma_3}, \quad (24)$$

$$c_{\vec{\alpha}}^{\vec{\gamma}} = \sum_{\gamma} c_{\alpha_1 \alpha_2}^{\gamma_1 \beta} c_{\beta \alpha_3}^{\gamma_2 \gamma_3}. \quad (25)$$

Looking at source terms of eq.(23), it is obvious that they describe the reactions (16) and the following three-particle drop-push reactions:

$$A_{\gamma_1} A_{\gamma_2} A_{\gamma_3} \emptyset \rightarrow \emptyset A_{\alpha_1} A_{\alpha_2} A_{\alpha_3} \quad \text{with rate } D_R b_{\vec{\alpha}}^{\vec{\gamma}}, \quad (26)$$

and

$$\emptyset A_{\gamma_1} A_{\gamma_2} A_{\gamma_3} \rightarrow A_{\alpha_1} A_{\alpha_2} A_{\alpha_3} \emptyset \quad \text{with rate } D_L c_{\vec{\alpha}}^{\vec{\gamma}}. \quad (27)$$

The sink terms are consistent with this description, provided

$$\sum_{\beta} b_{\vec{\beta}}^{\vec{\alpha}} = \sum_{\beta \gamma} b_{\beta_1 \gamma}^{\alpha_1 \alpha_2} b_{\beta_2 \beta_3}^{\gamma \alpha_3} = \sum_{\beta} b_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \quad \text{constraint (IV)}, \quad (28)$$

and

$$\sum_{\beta} c_{\vec{\beta}}^{\vec{\alpha}} = \sum_{\beta \gamma} c_{\beta_1 \beta_2}^{\alpha_1 \gamma} c_{\gamma \beta_3}^{\alpha_2 \alpha_3} = \sum_{\beta} c_{\beta_2 \beta_3}^{\alpha_2 \alpha_3} \quad \text{constraint (V)}. \quad (29)$$

By calculating other  $\dot{P}_{\vec{\alpha}}(x, x+1, \dots, x+n-1)$ 's it can be shown that we need not any more constraints and therefore the master equation (7) with boundary condition (10) and five constraints (I)-(V) can consistently describe the following reactions:

$$A_{\alpha} \emptyset \rightarrow \emptyset A_{\alpha} \quad \text{with rate } D_R \quad (30)$$

$$\emptyset A_{\alpha} \rightarrow A_{\alpha} \emptyset \quad \text{with rate } D_L \quad (31)$$

$$A_{\alpha} A_{\beta} \rightarrow A_{\gamma} A_{\delta} \quad \text{with rate } D_R c_{\gamma \delta}^{\alpha \beta} + D_L b_{\gamma \delta}^{\alpha \beta} \quad (32)$$

$$A_{\alpha_0} \cdots A_{\alpha_n} \emptyset \rightarrow \emptyset A_{\gamma_0} \cdots A_{\gamma_n} \quad \text{with rate } D_R (b_{n-1, n} \cdots b_{0,1})_{\gamma_0 \cdots \gamma_n}^{\alpha_0 \cdots \alpha_n}, \quad (33)$$

$$\emptyset A_{\alpha_0} \cdots A_{\alpha_n} \rightarrow A_{\gamma_0} \cdots A_{\gamma_n} \emptyset \quad \text{with rate } D_L (c_{0,1} \cdots c_{n-1, n})_{\gamma_0 \cdots \gamma_n}^{\alpha_0 \cdots \alpha_n}. \quad (34)$$

In above equations we use the following definition for  $b_{k, k+1}$  and  $c_{k, k+1}$  :

$$a_{k, k+1} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{a}_{k, k+1} \otimes 1 \otimes \cdots \otimes 1. \quad (35)$$

Note that for  $D_L = 0$ , the five classes of the above reactions reduce to three ones discussed in [25]. In [25], the constraints between reaction rates are three relations (I), (II), and (IV). Note that at  $D_L = 0$ , the constraints (III) and (V) do not appear since the multiplication factors of their corresponding terms in evolution equation is  $D_L$ , which is zero.

To find the set of solutions of five constraints (I) to (V), one can consider the solutions of equations (I), (II) and (IV), that is the solutions derived in [25], and then considers the subset of them satisfy (III) and (V). We must note that in our models, the diagonal elements of matrix  $c$  are the reaction rates of the last line of eq.(16) and must be positive. This is in contrast to the case studied in [25] in which the diagonal elements of  $c$  can be negative.

We can also follow another approach. That is trying to find the solution of equations (II) to (IV) and then seek ones which satisfy relation (I). As these relations are rather complex, we can not completely solve them for arbitrary  $p$ , but we try them as much as possible.

As all the matrix elements of matrices  $b$  and  $c$  are reaction rates, they can not be negative, so the only solution of eq.(18) is:

$$c_{\alpha_1\beta_2}^{\gamma_1\gamma_2} b_{\alpha_2\alpha_3}^{\beta_2\beta_3} = 0 \quad (\text{without sum over } \beta_2). \quad (36)$$

This relation has two following solutions ( for each  $\beta_2$ )

$$c_{\alpha_1\beta_2}^{\gamma_1\gamma_2} = 0 \quad \text{and} \quad b_{\alpha_2\alpha_3}^{\beta_2\beta_3} = 0. \quad (37)$$

So for each  $\beta_2$  we have two solutions, and as  $\beta_2$  runs from 1 to  $p$ , we have  $2^p - 2$  set of solutions for constraint (II). We exclude two of the solutions in which all of the elements of  $c$  or  $b$  is zero, since we look for the situations in which  $b \neq 0$  and  $c \neq 0$ . We will later study the cases  $b = 0$  or  $c = 0$  in which the number of independent classes of reactions (30)-(34) reduces to four. By the same argument, the solutions of eq.(21) are (for each  $\beta_2$ )

$$c_{\alpha_1\alpha_2}^{\beta_1\beta_2} = 0 \quad \text{and} \quad b_{\beta_2\alpha_3}^{\gamma_2\gamma_3} = 0, \quad (38)$$

and therefore we again have  $2^p - 2$  set of solutions for constraint (III). Note that from  $2^p - 2$  solutions of constraints II ( and III ) only  $(p - 1)$  of them are independent, that is does not transform to each other under interchanging of the labels of the species of the particles. So the number of independent solutions of constraints (II) and (III) are  $(p - 1)(2^p - 2)$ . For example in  $p = 2$ , the independent solutions of (II) and (III) are

$$\begin{cases} c_{\alpha_1 1}^{\gamma_1 \gamma_2} = 0, b_{\alpha_2 \alpha_3}^{2 \beta_3} = 0, c_{\alpha_1 \alpha_2}^{\beta_1 1} = 0, b_{2 \alpha_3}^{\gamma_2 \gamma_3} = 0 \}, \\ \{ c_{\alpha_1 1}^{\gamma_1 \gamma_2} = 0, b_{\alpha_2 \alpha_3}^{2 \beta_3} = 0, c_{\alpha_1 \alpha_2}^{\beta_1 2} = 0, b_{1 \alpha_3}^{\gamma_2 \gamma_3} = 0 \}, \end{cases} \quad (39)$$

which can be written as

$$b = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & c_{24} \\ 0 & 0 & 0 & 0 \\ 0 & c_{42} & 0 & c_{44} \end{pmatrix}, \quad (40)$$

and

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 \\ b_{41} & b_{42} & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{21} & 0 & c_{23} & 0 \\ 0 & 0 & 0 & 0 \\ c_{41} & 0 & c_{43} & 0 \end{pmatrix}, \quad (41)$$



respectively. We label the states as  $|1 \rangle = (1, 1)$ ,  $|2 \rangle = (1, 2)$ ,  $|3 \rangle = (2, 1)$  and  $|4 \rangle = (2, 2)$ . Putting eqs.(40) and (41) into the constraint (IV) and (V) (eqs.(28) and (29)) results

$$b = \begin{pmatrix} 1 & 1 & 0 & 0 \\ b_{21} & b_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & c_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad (42)$$

and

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ b_{41} & b_{41} & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ c_{41} & 0 & c_{41} & 0 \end{pmatrix}, \quad (43)$$

respectively. Now eq.(12) (constraint I) says that the sum of the elements of each column of matrix  $(b + c)$  must be one, which unfortunately does not satisfy by (42) and (43). The sum of the elements of the second column of  $(b + c)$  of eq.(42) and the first column of eq.(43) are greater than (or equal to) 2. So reactions (30)-(34) have not any representation in  $p = 2$ , the situation which we expect to be true for other  $p$ 's. For example in  $p = 3$ , constraints (II) and (III) have 12 independent solutions, which are in two categories: the number of constraints on  $b_i^j$  and  $c_k^l$ 's are equal (three on  $b_i^j$  and three on  $c_k^l$ ), and one which these numbers differ, i.e. 4 and 2. As an example of the first category, we consider the case in which  $c_{\alpha 1}^j = 0, c_{\alpha 2}^j = 0, c_i^{\beta 1} = 0, b_k^{3\gamma} = 0, b_{1\alpha}^k = 0, b_{3\alpha}^k = 0$ . It means that in matrix  $c$ , the rows 1,2,4,5,7,8 and columns 2,5, and 8 are zero, so it has 18 non-zero elements, and  $b$  is a matrix in which the rows 1,2,3,7,8,9 and columns 7,8, and 9 are zero so it also has 18 non-zero elements. Putting these  $b$  and  $c$  matrices in constraint (IV) and (V) results two following solutions for each  $b$  and  $c$ :

$$\begin{aligned} c_1 &: \{c_{31} = c_{34} = c_{37} = 1, c_{33} \text{ and } c_{36} \text{ arbitrary}\}, \\ c_2 &: \{c_{93} = c_{96} = c_{99} = 1, c_{91} \text{ and } c_{94} \text{ arbitrary}\}, \\ b_1 &: \{b_{41} = b_{42} = b_{43} = 1, b_{44} \text{ and } b_{45} \text{ arbitrary}\}, \\ b_2 &: \{b_{54} = b_{55} = b_{56} = 1, b_{51} \text{ and } b_{52} \text{ arbitrary}\}, \end{aligned} \quad (44)$$

in which we only write down the non-zero elements. It can be easily seen that none of the combinations  $b_1 + c_1$ ,  $b_1 + c_2$ ,  $b_2 + c_1$ ,  $b_2 + c_2$  are acceptable in the sense of constraint (I), as at least the sum of the elements of one of the columns of these matrices are greater than (or equal to) 2. We have checked that the same situation arises in other 11 solutions. So again in  $p = 3$ , we have no representation. We can not generally prove this, but we believe that the set of constraints (I)-(V) have no solution for arbitrary  $p$ .

One may suppose that if we somehow change the constraint (I), then it may be possible to find some solution for our equations. So we add the annihilation processes to our previous interactions. Note that these interactions appear only in the sink terms of the evolution equation, as if we consider the initial state with  $n$  particles, no annihilation processes can lead to a  $n$ -particle state at any other time  $t$ . So if we change the constraint (I) to: (as we have not the conservation of particles)

$$\sum_{\alpha} (b + c)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = 1 - \lambda_{\beta_1 \beta_2}, \quad (45)$$

and using it in calculation of  $\dot{P}_{\alpha_1\alpha_2}(x, x+1)$ , we find the same equation as (13), except an extra term  $\lambda_{\alpha_1\alpha_2}P_{\alpha_1\alpha_2}(x, x+1)$  which is added to sink terms. So  $\lambda_{\alpha_1\alpha_2}$  is the sum of the rates of all annihilation processes with initial state  $(\alpha_1\alpha_2)$  and therefore is a positive quantity. Therefore adding the annihilation processes to interactions (30)-(34) means that the sum of the elements of each column of  $(b+c)$  can now be less than or equal to one. But as we have shown in eqs.(42), (43) and (44), the sum of the elements of some of the columns of  $(b+c)$  in these examples are at least 2, which differs from what is suggested by eq.(45). In brief, including the annihilation processes can not alter our result and the set of processes (30)-(34) have no representation, with or without adding the annihilation processes.

Now it is interesting to note that even if one of the matrices  $b$  or  $c$  be equal to zero, we have yet all four desired reactions: Diffusion to left *and* right, two-particle reactions  $A_\alpha A_\beta \rightarrow A_\gamma A_\delta$ , and the extended drop-push reactions, which the latter occur only in one side (left *or* right). These are almost the general reactions that one can study in this framework. Let us check the constraints in these cases.

## 2.1 Type 1 model

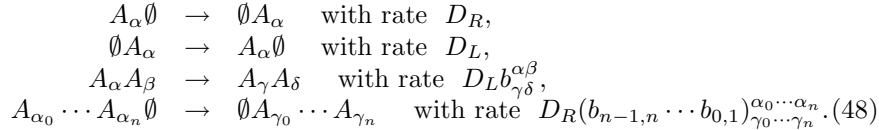
Take  $c = 0$ . Eq.(12) becomes

$$\sum_{\alpha} b_{\alpha_1\alpha_2}^{\beta_1\beta_2} = 1. \quad (46)$$

Constraints (II), (III) and (V) are satisfied trivially and constraint (IV) is also satisfied: using eq.(46), both sides of (IV) become one. Therefore master equation (7) with boundary condition

$$P_{\alpha_1\alpha_2}(x, x) = \sum_{\beta} b_{\alpha_1\alpha_2}^{\beta_1\beta_2} P_{\beta_1\beta_2}(x-1, x), \quad (47)$$

and constraint (46), describe consistently the following reactions:



## 2.2 Type 2 model

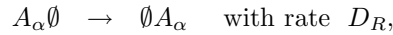
In the same way, for  $b = 0$  it can be seen that the master equation (7) with boundary condition

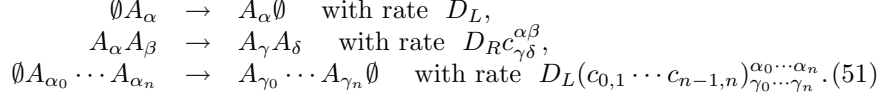
$$P_{\alpha_1\alpha_2}(x, x) = \sum_{\beta} c_{\alpha_1\alpha_2}^{\beta_1\beta_2} P_{\beta_1\beta_2}(x, x+1), \quad (49)$$

and constraint

$$\sum_{\alpha} c_{\alpha_1\alpha_2}^{\beta_1\beta_2} = 1, \quad (50)$$

describe successfully the reactions:





The condition of solvability of these models will be discussed in next sections.

### 3 Second kind generalization

By noting the first line of eq.(9), it is seen that eq.(10) is not the only possible  $p$ -species boundary condition. In fact, one can instead consider the following boundary condition :

$$D_R P_{\alpha_1 \alpha_2}(x, x) + D_L P_{\alpha_1 \alpha_2}(x+1, x+1) = \sum_{\beta} b_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x-1, x) + \sum_{\beta} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x, x+1). \quad (52)$$

This is the multi-species generalization of the boundary condition considered in [17] and [27].

To study the interactions introduced by (7) and (52), we must again consider  $\dot{P}_{\alpha_1 \cdots \alpha_n}(x, x+1, \cdots, x+n-1)$ . In  $n=3$ , we encounter the boundary term  $D_R P_{\alpha_1 \alpha_2 \alpha_3}(x, x+1, x+1) + D_L P_{\alpha_1 \alpha_2 \alpha_3}(x, x+2, x+2)$ , where using (52) results  $\sum_{\beta} b_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} P_{\alpha_1 \beta_2 \beta_3}(x, x, x+1) + \sum_{\beta} c_{\alpha_2 \alpha_3}^{\beta_2 \beta_3} P_{\alpha_1 \beta_2 \beta_3}(x, x+1, x+2)$ . But the first term  $P_{\alpha_1 \beta_2 \beta_3}(x, x, x+1)$  can not be written in terms of physical probabilities, since in this case only the linear combination  $D_R P_{\alpha_1 \alpha_2 \cdots}(x, x, \cdots) + D_L P_{\alpha_1 \alpha_2 \cdots}(x+1, x+1, \cdots)$  can be written in terms of physical function (eq.(52)). This is in contrast with the case studied in section 2. The only solution to this problem is taking

$$b = 0. \quad (53)$$

So our second kind  $p$ -species model is defined through the master equation (7) and the following boundary condition :

$$D_R P_{\alpha_1 \alpha_2}(x, x) + D_L P_{\alpha_1 \alpha_2}(x+1, x+1) = \sum_{\beta} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x, x+1). \quad (54)$$

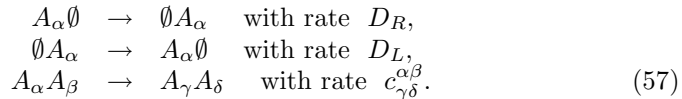
Conservation of number of particle gives

$$\sum_{\alpha} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = 1, \quad (55)$$

and calculating  $\dot{P}_{\alpha_1 \alpha_2}(x, x+1)$  results

$$\begin{aligned}
\dot{P}_{\alpha_1 \alpha_2}(x, x+1) &= D_R P_{\alpha_1 \alpha_2}(x-1, x+1) + D_L P_{\alpha_1 \alpha_2}(x, x+2) + \sum_{\beta \neq \alpha} c_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} P_{\beta_1 \beta_2}(x, x+1) \\
&\quad - (D_R + D_L + \sum_{\beta \neq \alpha} c_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}) P_{\alpha_1 \alpha_2}(x, x+1). \quad (56)
\end{aligned}$$

This equation describes the following reactions as source and sink terms



Calculating other  $\dot{P}_{\alpha_1 \cdots \alpha_n}(x, x+1, \cdots, x+n-1)$ 's confirms these reactions without any further constraint. So the **type 3 model** is defined by master equation (7), boundary condition (54), constraint (55), and reactions (57).

## 4 Bethe ansatz solution

Until now, we have constructed a consistent formalism to study some reaction-diffusion processes. Now we want to solve the resulting evolution equations and check the solvability of these models. To solve the master equation (7), we consider the following Bethe ansatz

$$P_{\alpha_1 \dots \alpha_N}(\mathbf{x}; t) = e^{-E_N t} \psi_{\alpha_1 \dots \alpha_N}(\mathbf{x}), \quad (58)$$

with

$$\Psi(\mathbf{x}) = \sum_{\sigma} \mathbf{A}_{\sigma} e^{i\sigma(\mathbf{p}) \cdot \mathbf{x}}. \quad (59)$$

$\Psi$  is a tensor of rank  $N$  with components  $\psi_{\alpha_1 \dots \alpha_N}(\mathbf{x})$  and the summation runs over the elements of the permutation group of  $N$  objects [28, 29]. Inserting (58) in (7), results

$$E_N = \sum_{k=1}^N (1 - D_R e^{-ip_k} - D_L e^{ip_k}). \quad (60)$$

Inserting (58) in boundary condition (47) gives

$$\Psi(\dots, x_k = x, x_{k+1} = x, \dots) = b_{k,k+1} \Psi(\dots, x_k = x-1, x_{k+1} = x, \dots), \quad (61)$$

which using (59) results

$$[1 - e^{-i\sigma(p_k)} b_{k,k+1}] \mathbf{A}_{\sigma} + [1 - e^{-i\sigma(p_{k+1})} b_{k,k+1}] \mathbf{A}_{\sigma \sigma_k} = 0. \quad (62)$$

$\sigma_k$  is an element of permutation group which only interchanges  $p_k$  and  $p_{k+1}$ :

$$\sigma_k : (p_1, \dots, p_k, p_{k+1}, \dots, p_N) \rightarrow (p_1, \dots, p_{k+1}, p_k, \dots, p_N). \quad (63)$$

Eq.(62) gives  $\mathbf{A}_{\sigma \sigma_k}$  in terms of  $\mathbf{A}_{\sigma}$  as following:

$$\mathbf{A}_{\sigma \sigma_k} = S_{k,k+1}^{(1)}(\sigma(p_k), \sigma(p_{k+1})) \mathbf{A}_{\sigma}, \quad (64)$$

where

$$S_{k,k+1}^{(1)}(z_1, z_2) = 1 \otimes \dots \otimes 1 \otimes \underbrace{S^{(1)}(z_1, z_2)}_{k,k+1} \otimes 1 \otimes \dots \otimes 1, \quad (65)$$

and  $S^{(1)}(z_1, z_2)$  is the following  $p^2 \times p^2$  matrix

$$S^{(1)}(z_1, z_2) = -(1 - z_2^{-1}b)^{-1}(1 - z_1^{-1}b), \quad (66)$$

in which  $z_k = e^{ip_k}$ . The same procedure for boundary conditions (49) and (54), i.e. the type 2 and type 3 models, results

$$S^{(2)}(z_1, z_2) = -(1 - z_1c)^{-1}(1 - z_2c), \quad (67)$$

and

$$S^{(3)}(z_1, z_2) = -(D_R + z_1 z_2 D_L - z_1 c)^{-1}(D_R + z_1 z_2 D_L - z_2 c), \quad (68)$$

respectively. Eq.(64) allows one to compute all  $\mathbf{A}_{\sigma}$ 's in terms of  $\mathbf{A}_1$  (which is set to unity).

As the generators of permutation group satisfy  $\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}$ , so one also needs

$$\mathbf{A}_{\sigma_k \sigma_{k+1} \sigma_k} = \mathbf{A}_{\sigma_{k+1} \sigma_k \sigma_{k+1}}. \quad (69)$$

This, in terms of  $S$ -matrices becomes

$$S_{12}(z_2, z_3) S_{23}(z_1, z_3) S_{12}(z_1, z_2) = S_{23}(z_1, z_2) S_{12}(z_1, z_3) S_{23}(z_2, z_3). \quad (70)$$

In the terms of  $R$ -matrix defined through

$$S_{k,k+1} =: \Pi_{k,k+1} R_{k,k+1}, \quad (71)$$

where  $\Pi$  is the permutation matrix, eq.(70) is transformed to

$$R_{23}(z_2, z_3) R_{13}(z_1, z_3) R_{12}(z_1, z_2) = R_{12}(z_1, z_2) R_{13}(z_1, z_3) R_{23}(z_2, z_3). \quad (72)$$

This is the spectral Yang-Baxter equation.

The Bethe ansatz solution exists, if the scattering matrix satisfies (70). In other words, the matrix  $b$  in (66) and  $c$  in (67) and (68) is acceptable only if the resulting  $S$ -matrices satisfy (70). This is a very restricted condition and needed for having the solvability.

The  $S$ -matrices (66) and (67) are exactly the ones considered in [24] and [23], respectively. Using the fact that  $S^{(1)}$  is a binomial of degree one with respect to  $z_1^{-1} = e^{-ip_1}$  and  $S^{(2)}$  is of degree one with respect to  $z_2$ , it can be shown that SYB equation (70) for  $S^{(1)}$  and  $S^{(2)}$  reduces to

$$b_{23}[b_{23}, b_{12}] = [b_{23}, b_{12}] b_{12}, \quad (73)$$

and

$$c_{12}[c_{12}, c_{23}] = [c_{12}, c_{23}] c_{23}, \quad (74)$$

respectively [23, 24]. Note that although the above equations are much simpler than eq.(70), but they are very complicated yet. In  $p$ -species, each one is an equality between two  $p^3 \times p^3$  matrices which results a system of  $p^6$  equations to be solved for  $p^4 - p^2$  elements of  $b$  (or  $c$ ), which may or may not have solution (eq.(46) and (50) reduce the number of independent elements of  $b$  and  $c$  to  $p^4 - p^2$ ). The general properties of the solutions of eq.(73) and eq.(74) have been discussed in [24] and [23], respectively, which can be directly used here. In other words, for every solution of eq.(74), there exists a corresponding solvable model which have been discussed in [23], i.e.

$$\begin{aligned} A_\alpha \emptyset &\rightarrow \emptyset A_\alpha && \text{with rate } 1, \\ A_\alpha A_\beta &\rightarrow A_\gamma A_\delta && \text{with rate } c_{\gamma\delta}^{\alpha\beta}. \end{aligned} \quad (75)$$

and a type 2 model with reactions written in eq.(51) (note that the reactions (75) are a subset of (51) with  $D_L = 0$ ). The same is true for solutions of (73). They can describe the following solvable model (discussed in [24]):

$$\begin{aligned} A_\alpha \emptyset &\rightarrow \emptyset A_\alpha && \text{with rate } 1, \\ A_{\alpha_0} \cdots A_{\alpha_n} \emptyset &\rightarrow \emptyset A_{\gamma_0} \cdots A_{\gamma_n} && \text{with rate } (b_{n-1,n} \cdots b_{0,1})_{\gamma_0 \cdots \gamma_n}^{\alpha_0 \cdots \alpha_n}, \end{aligned} \quad (76)$$

and a type 1 model with reaction (48) (again at  $D_L = 0$ , (48) reduces to (76)).

The reasoning which leads the SYB equations of  $S^{(1)}$  and  $S^{(2)}$  to (73) and (74) does not work for  $S^{(3)}$  since it is not a binomial of degree one with respect

to  $z_1$  or  $z_2$ , in fact it contains all powers of  $z_1$  and  $z_2$ . So obtaining the solutions of (70) for  $S^{(3)}$  is more difficult than for  $S^{(1)}$  and  $S^{(2)}$ , even in the simplest case  $p = 2$ . In  $p = 2$  case we encounter a system of 64 equations that must be solved for 12 non-diagonal elements of  $c$  (the diagonal elements are determined by eq.(55)). The solution must be momentum-independent (independent of  $z_1$ ,  $z_2$  and  $z_3$ ) and non-negative. We can not solve this equations generally ( taking all  $c_{ij} \neq 0$  ) by standard mathematical softwares and therefore restrict ourselves to some specific cases. For example taking

$$c = \begin{pmatrix} c_{11} & 0 & 0 & c_{14} \\ c_{21} & 1 & 0 & c_{24} \\ 1 - c_{11} - c_{21} & 0 & 1 & 1 - c_{14} - c_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (77)$$

or

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{21} & 1 & 0 & c_{24} \\ c_{31} & 0 & 1 & c_{34} \\ 1 - c_{21} - c_{31} & 0 & 0 & 1 - c_{24} - c_{34} \end{pmatrix}, \quad (78)$$

which are the four-parameters cases, one obtains two solutions

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ D_R & 1 & 0 & D_L \\ D_L & 0 & 1 & D_R \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (79)$$

and one with  $D_L \leftrightarrow D_R$ . Taking

$$c = \begin{pmatrix} 1 - c_{41} & 1 - c_{42} & 1 - c_{43} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_{41} & c_{42} & c_{43} & 1 \end{pmatrix}, \quad (80)$$

or

$$c = \begin{pmatrix} 1 & 1 - c_{42} & 1 - c_{43} & 1 - c_{44} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c_{42} & c_{43} & c_{44} \end{pmatrix}, \quad (81)$$

as some three-parameters cases, we find four solutions

$$c = \begin{pmatrix} 1 & 1 - c_{42} & 1 - c_{43} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c_{42} & c_{43} & 1 \end{pmatrix}, \quad (82)$$

where each of  $c_{42}$  and  $c_{43}$  are either  $D_R$  or  $D_L$ . Taking  $A_1 \equiv A$  and  $A_2 \equiv B$ , the interactions introduced by (79), for instance, are

$$\begin{array}{ccc} A\emptyset & \xrightarrow{D_R} & \emptyset A \\ B\emptyset & \xrightarrow{D_R} & \emptyset B \\ \emptyset A & \xrightarrow{D_L} & A\emptyset \\ \emptyset B & \xrightarrow{D_L} & B\emptyset \end{array}$$



The model built on the reactions (83) is integrable.

Assuming that the solvability condition (70) is satisfied, it is easy to see that the conditional probability (the propagator) is

$$U(\mathbf{x}; t|\mathbf{y}; 0) = \int \frac{d^N p}{(2\pi)^N} e^{-E_N t} e^{-i\mathbf{p}\cdot\mathbf{y}} \sum_{\sigma} \mathbf{A}_{\sigma} e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}, \tag{84}$$

where the integration region for each  $p_i$  is  $[0, 2\pi]$  and  $A_1 = 1$ . The singularity in  $A_{\sigma}$  is removed by setting  $p_j \rightarrow p_j + i\varepsilon$ , where one should consider the limit  $\varepsilon \rightarrow 0^+$ . Using this propagator, one can write the probability at the time  $t$  in terms of the initial value of probability:

$$|P(\mathbf{x}; t)\rangle = \sum_{\mathbf{y}} U(\mathbf{x}; t|\mathbf{y}; 0)|P(\mathbf{y}; 0)\rangle. \tag{85}$$

Note that although  $S^{(1)}$  and  $S^{(2)}$  are similar to ones considered in [24] and [23], but the propagators  $U^{(1)}$  and  $U^{(2)}$  are different since the energy spectrum of our models differs from those considered there. In  $D_L = 0$ , our results must coincide with those obtained in [23, 24].

For the two-particle sector, there is only one matrix in the expression of  $U^{(i)}$ s ( $b$  in  $U^{(1)}$  and  $c$  in  $U^{(2)}$  and  $U^{(3)}$ ). So it can be treat as a  $c$ -number. Using calculation similar to what has been done in [23]-[25], one arrives at:

$$\begin{aligned}
U^{(1)}(x_1, x_2; t|y_1, y_2; 0) &= e^{-2t} \sum_{n,m=0}^{\infty} \left\{ \frac{D_L^n D_R^{x_1-y_1+n} t^{x_1-y_1+2n}}{n!(x_1-y_1+n)!} \frac{D_L^m D_R^{x_2-y_2+m} t^{x_2-y_2+2m}}{m!(x_2-y_2+m)!} \right. \\
&+ \sum_{l=0}^{\infty} \frac{D_L^n D_R^{x_2-y_1+n} t^{x_2-y_1+2n}}{n!(x_2-y_1+n)!} \frac{D_L^m D_R^{x_1-y_2-l+m} t^{x_1-y_2-l+2m}}{m!(x_1-y_2-l+m)!} \\
&\left. \times b^l \left[ -1 + \frac{x_2-y_1+n}{D_R t} b \right] \right\}, \tag{86}
\end{aligned}$$

and

$$\begin{aligned}
U^{(2)}(x_1, x_2; t|y_1, y_2; 0) &= e^{-2t} \sum_{n,m=0}^{\infty} \left\{ \frac{D_L^n D_R^{x_1-y_1+n} t^{x_1-y_1+2n}}{n!(x_1-y_1+n)!} \frac{D_L^m D_R^{x_2-y_2+m} t^{x_2-y_2+2m}}{m!(x_2-y_2+m)!} \right. \\
&+ \sum_{l=0}^{\infty} \frac{D_L^n D_R^{x_2-y_1+l+n} t^{x_2-y_1+l+2n}}{n!(x_2-y_1+l+n)!} \frac{D_L^m D_R^{x_1-y_2+m} t^{x_1-y_2+2m}}{m!(x_1-y_2+m)!} \\
&\left. \times c^l \left[ -1 + \frac{D_R t}{x_1-y_2+m+1} c \right] \right\}. \tag{87}
\end{aligned}$$

Similarly one can obtain a more lengthy expression for  $U^{(3)}$ . Note that at  $D_L = 0$ , eqs.(86) and (87) lead eqs.(38) of [24] and (30) of [23], respectively.

To investigate the large-time behaviours of the probabilities  $U^{(1)}$ ,  $U^{(2)}$ , and  $U^{(3)}$ , it is useful to decompose the vector spaces on which  $b$  (in type 1 model)

and  $c$  (in types 2 and 3 models) act, in two subspaces invariant under the action of  $b(c)$ : the first subspace corresponding to eigenvalues with modulus one, and another invariant subspace. For types 1 and 2 models with conditions (46) and (50), as all the elements of matrix  $b(c)$  are non-negative, the second subspace corresponds to eigenvalues with modulus less than one. By focusing on type 1 model, this decomposition can be done by introducing two projectors  $Q$  and  $R$ , satisfying

$$\begin{aligned} Q + R &= 1, \\ QR = RQ &= 0, \\ [b, Q] = [b, R] &= 0. \end{aligned} \quad (88)$$

$Q$  projects on the first subspace and  $R$  projects on the second. Following [23], we multiply  $U^{(1)}$  by  $Q + R = 1$ :

$$U^{(1)}(\mathbf{x}; t | \mathbf{y}; 0) = U^{(1)}Q + U^{(1)}R. \quad (89)$$

In the terms multiplied by  $R$ , one can treat  $b$  as a number with modulus different from one. So the integrand in (84) is non-singular at points  $p_j = 0$ , which have the main contributions at large times. Putting  $p_j = 0$ , we have  $S^{(1)} \approx -1$  and  $\mathbf{A}_\sigma \approx (-1)^{|\sigma|}$ , and eq.(84) results

$$\begin{aligned} \text{the second term of } U^{(1)} &= \frac{1}{2\pi t} \{ e^{-\{[x_1-y_1-(D_R-D_L)t]^2+[x_2-y_2-(D_R-D_L)t]^2\}/(2t)} \\ &\quad - e^{-\{[x_1-y_2-(D_R-D_L)t]^2+[x_2-y_1-(D_R-D_L)t]^2\}/(2t)} \} R, \quad t \rightarrow \infty, \end{aligned} \quad (90)$$

which is independent of  $b$ . So at large time, the second term of  $U^{(1)}$  tends to zero faster than  $t^{-1}$  and the leading term in  $U^{(1)}$ , which is order  $t^{-1}$ , does not involve the second term.

If the only eigenvalue of  $b$  with modulus 1 is 1, then  $bQ = Q$  and  $U^{(1)}$  has a simple behaviour at  $t \rightarrow \infty$ :

$$\begin{aligned} U^{(1)}(x_1, x_2; t | y_1, y_2; 0) &= e^{-2t} \sum_{n,m=0}^{\infty} \left\{ \frac{D_L^n D_R^{x_1-y_1+n} t^{x_1-y_1+2n}}{n!(x_1-y_1+n)!} \frac{D_L^m D_R^{x_2-y_2+m} t^{x_2-y_2+2m}}{m!(x_2-y_2+m)!} \right. \\ &\quad + \sum_{l=0}^{\infty} \frac{D_L^n D_R^{x_2-y_1+n} t^{x_2-y_1+2n}}{n!(x_2-y_1+n)!} \frac{D_L^m D_R^{x_1-y_2-l+m} t^{x_1-y_2-l+2m}}{m!(x_1-y_2-l+m)!} \\ &\quad \left. \times \left[ -1 + \frac{x_2-y_1+n}{D_R t} \right] \right\} Q. \end{aligned} \quad (91)$$

This is simply the propagator of a single-species model with diffusions to right and left and drop-push to right (i.e. the  $\lambda = 0$  case of the reactions studied in [26] and [30]), multiplied by  $Q$ . In fact eq.(91) is  $\lambda = 0$  case of eq.(30) of [26], times  $Q$ .

For  $U^{(2)}$ , the same decomposition leads the eq.(90) for its second term and in the case  $cQ = Q$ ,  $U^{(2)}$  tends to (87), with  $c = 1$ , times  $Q$ , at  $t \rightarrow \infty$ . The resulting one-species model is  $\mu = 0$  case of the reactions studied in [26] and [30]. For  $U^{(3)}$ , we again find (90) and the one-species partially asymmetric simple exclusion process of [17].

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