ON THE K_{-i}-FUNCTORS

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0. Statement of results

The functors K_0 and K_1 have well-known descriptions in terms of projectives and automorphisms, respectively. The purpose of this paper is to give analogous descriptions of the K_{-i}-functors of Bass [1]. In fact given a ring R, we give two descriptions of $K_{-i}(R)$, $i \geq 0$; one as a Whitehead construction on a certain category of R-modules (elements represented by automorphisms), and one as a Grothendieck construction on a related category of R-modules (elements represented by objects). The categories in question are associated with the category of $\mathbb{Z}^i$-graded R-modules and bounded homomorphisms in the following sense:

0.1**. Definition.** Let R be a ring. $\mathcal{A}_i(R)$ denotes the category of $\mathbb{Z}^i$-graded R-modules and bounded homomorphisms. This means an object A is a direct sum $\bigoplus_{j_1,\dots,j_i} A(j_1,\dots,j_i)$ of R-modules, and a morphism $f : A \to B$ is an R-module morphism, such that there exists $k = k(f)$ satisfying

$$f(A(j_1,\dots,j_i)) \subseteq \bigoplus_{\substack{h_s=-k \\ s=1,\dots,i}}^{k} (B(j_1+h_1,\dots,j_i+h_i).$$

Remark. $\mathcal{A}_0(R)$ is just the category of R-modules.

We shall be more interested in the full subcategory of $\mathcal{A}_i(R)$ with objects A satisfying $A(j_1,\dots,j_i)$ are finitely generated free R-modules. This category we denote $\mathcal{C}_i(R)$. We shall some times write $\mathcal{A}_i$ and $\mathcal{C}_i$ instead of $\mathcal{A}_i(R)$ and $\mathcal{C}_i(R)$ when it is clear from the context which ring we are working with. Since $\mathcal{C}_0(R)$ is the category of finitely generated free R-modules, it is helpful to think of $\mathcal{C}_i(R)$ as the category of $\mathbb{Z}^i$-graded finitely generated free R-modules and bounded homomorphisms.

In $\mathcal{A}_i$ as well as $\mathcal{C}_i$ we have an obvious notion of direct sum (degree-wise). We define a sequence $0 \to A \to B \to C$ to be *exact* if it is split-exact, i. e. , it is equivalent (in the category) to the sequence $0 \to A \to A \oplus B \to B \to 0$. We may now define K_1 of the category $\mathcal{C}_{i+1}(R)$. That is the Abelian group generated by $[A, \alpha]$ where A is an object of $\mathcal{C}_{i+1}(R)$ and α an automorphism of A, subject to the relations

$$[A, \alpha\beta] = [A, \alpha] + [A, \beta] \quad \text{and} \quad [B, \beta] = [A, \alpha] + [C, \gamma]$$

when there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with exact rows. Note that this implies that $[A, 1] = 0$ and further that

$$A \oplus B \xrightarrow{\left\{\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix}\right\}} A \oplus B,$$

where $\eta : B \to A$ any bounded morphism, represents 0. An isomorphism of this type we shall call an *elementary* isomorphism.

0.2. **Definition.** $K'_{-i}(R) = K_1(\mathcal{C}_{i+1}(R))$.

Given a category $\mathfrak{A}$ we define the category $P\mathfrak{A}$ as follows: An object is an idempotent in $\mathfrak{A}$, i. e. $p : A \to A$ with $p^2 = p$, and a morphism $\phi : (A_1, p_1) \to (A_2, p_2)$ is a morphism $\phi : A_1 \to A_2$ so that $\phi p_1 = p_2 \phi$. In $P\mathcal{C}_i(R)$ we have an induced notion of direct sum, so we may form the Grothendieck group of $P\mathcal{C}_i(R)$.

0.3. **Definition.** We define $K''_{-i}(R)$ to be the Grothendieck group on the category $P\mathcal{C}_i(R)$ with the additional relation $[A, 0] = 0$ if $i = 0$.

We may now state our

Main Theorem. *Let R be a ring. Then there are natural isomorphisms*

$$K_{-i}(R) \cong K'_{-i}(R) \cong K''_{-i}(R).$$

This result indicates that $\mathcal{C}_{i+1}(R)$ is some kind of nonconnective delooping of the category of finitely generated free R-modules. This is indeed the case, and it is the subject of a forthcoming joint work with C. Weibel.

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1. The Isomorphism $K'_{-i}(R) \simeq K''_{-i}(R)$

In this section we define isomorphism $\phi^s : K'_{-i}(R) \cong K''_{-i}(R)$, where $s = 1, \dots, i+1$. The construction we employ to define ϕ^s is based on a variation of a well-known construction due to Bass, Heller and Swan[2]. First we need some definitions. Let A be an object of $\mathcal{C}_i(R)$. We define

$$s^+(A)(j_1, \dots, j_s, \dots, j_i) = A(j_1, \dots, j_s - 1, \dots j_i) \tag{1.1}$$

and

$$s^-(A)(j_1, \dots, j_s, \dots, j_i) = A(j_1, \dots, j_s + 1, \dots j_i) \tag{1.2}$$

for $1 \leq s \leq i$. There are obvious bounded isomorphism $A \cong s^+(A)$ and $A \cong s^-(A)$ induced by the identity, which we denote by s^+ and s^- respectively. Also define

$$A^{s^+}(j_1, \dots, j_i) = \begin{cases} A(j_1, \dots, j_i) & \text{if} \quad j_s \geq 0 \\ 0 & \text{if} \quad j_s < 0, \end{cases} \tag{1.3}$$

$$A^{s^-}(j_1, \dots, j_i) = \begin{cases} 0 & \text{if } j_s \geq 0 \\ A(j_1, \dots, j_i) & \text{if } j_s < 0, \end{cases} \tag{1.4}$$

for $1 \leq s \leq i$. The following lemma was pointed out by the referee.

1.5. **Lemma.** *Let A be an object in $\mathcal{C}_i(R)$. Then $[A, 1]$ and $[A, 0]$ represent 0 in $K''_{-i}(R)$ if $i > 0$.*

Proof. Clearly $[A, 1] = [A^{1^+}, 1] + [A^{1^-}, 1]$ but

$$A^{1^+} \oplus \bigoplus_{k=1}^{\infty} (1^+)^k (A^{1^+}) = \bigoplus_{k=0}^{\infty} (1^+)^k (A^{1^+})$$

and 1^+ is an isomorphism

$$\bigoplus_{k=0}^{\infty} (1^+)^k (A^{1^+}) \cong \bigoplus_{k=1}^{\infty} (1^+)^k (A^{1^+})$$

Note that even though the sum is infinite, it is finite in each degree since $A^{1^+}(j_1, \dots, j_i) = 0$ if $j_1 < 0$. This proves $[A^{1^+}, 1] = 0$ in $K''_{-i}(R)$ and $[A^{1-}, 1]$ is dealt with similarly. Clearly $[A, 0]$ can be treated the same way. $\square$

1.6. *Remark.* Lemma 1.5 proves that any two objects of $\mathcal{C}_i(R)$, $i > 0$, are stably isomorphic.

We now proceed to define the isomorphism $\phi^s : K'_{-i}(R) \cong K''_{-i}(R)$.

Given an object A of $\mathcal{C}_{i+1}(R)$, we have a direct sum decomposition $A = A^{s^-} \oplus A^{s^+}$ (by 1.3 and 1.4 above). We denote the projection on the first factor by

$$p^s_- : A \to A \tag{1.7}$$

(the projection on the negative s-half space). Given an automorphism $\alpha : A \to A$ in $\mathcal{C}_{i+1}(R)$, consider $\alpha p^s_- \alpha^{-1}$. Assuming α is bounded by k, this is a projection of A which is the identity on $A(j_1, \dots, j_s, \dots, j_{i+1})$ if $j_s < -2k$, and the 0-map if $j_s > 2k$. We define $\phi^s[A, \alpha]$ by

$$\begin{gathered} \overline{A}(j_1, \dots, \widehat{j_s}, \dots, j_{i+1}) = \bigoplus_{j_s=-2k}^{2k} A(j_1, \dots, j_{i+1}), \\ \phi^s([A, \alpha]) = [\overline{A}, \alpha p^s_- \alpha^{-1}] - [\overline{A}, p^s_-]. \end{gathered} \tag{1.8}$$

Several comments are in order here. The term $[\overline{A}, p^s_-]$ represents 0 unless $i = 0$. A nice way to think of $\phi^s([A, \alpha])$ is to keep the $\mathbb{Z}^{i+1}$-grading and notice we are looking at $\alpha p^s_- \alpha^{-1}$ in a certain band around $j_s = 0$, the width of the band at least from $-2k$ to $2k$. Outside this band $\alpha p_-^s\alpha^{-1}$ is equal to p_-^s; hence, when we subtract the restrictions of $\alpha p_-^s\alpha^{-1}$ and p_-^s, the width of the band does not matter. Actually it is useful to notice that widening the band corresponds to stabilization.

1.9. **Theorem.** *ϕ^s defines an isomorphism $K'_{-i}(R) \to K''_{-i}(R)$*

To prove this theorem we first need to see that ϕ^s respects the relations in the definition of $K'_{-i}(R)$.

1.10. **Lemma.** *Let A and B be objects of $\mathcal{C}_{i+1}(R)$ and $\psi : A \oplus B \to A \oplus B$ a bounded projection satisfying*

$$\psi|(A \oplus B)(j_1, \dots, j_{i+1}) = \begin{cases} 0 & \text{if } j_s > k \\ 1 & \text{if } j_s < -k \end{cases}$$

for some k. Let $\gamma : A \oplus B \to A \oplus B$ be an elementary isomorphism with matrix

$$\gamma = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \qquad \eta : B \to A$$

Then ψ and $\gamma\psi\gamma^{-1}$ restricted to a sufficiently big band around $j_s = 0$ represent the same element of $K''_{-i}(R)$.

Proof. Assume η is bounded by $l > k$. Define B' and B'' by

$$B'(j_1, \dots, j_{i+1}) = \begin{cases} B(j_1, \dots, j_{i+1} & \text{if } |j_s| \leq 2l \\ 0 & \text{if } |j_s| > 2l \end{cases}$$

and $B = B' \oplus B''$. Also define $\eta', \eta'' : B \to A$ as the composites $B \to B' \oplus 0 \to B \xrightarrow{\eta} A$ and $B \to 0 \oplus B'' \to B \xrightarrow{\eta} A$. Letting

$$\gamma' = \begin{pmatrix} 1 & \eta' \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma'' = \begin{pmatrix} 1 & \eta'' \\ 0 & 1 \end{pmatrix}$$

it is clear that $\gamma = \gamma' \cdot \gamma'' = \gamma'' \cdot \gamma'$. But

$$\gamma\psi\gamma^{-1} = \gamma'\gamma''\psi(\gamma'')^{-1}(\gamma')^{-1} = \gamma'\psi(\gamma')^{-1}.$$

This follows since ψ is only nontrivial in a small band around $j_s = 0$ and γ'' is the identity in a bigger band around $j_s = 0$. But $\gamma'\psi(\gamma')^{-1}$ and ψ are equivalent projections in the band $|j_s| \leq 3l$ since γ' restricts to an isomorphism of that band, □

This immediately leads to

1.11. **Lemma.** *The construction* 1.8 *gives a well-defined homomorphism*

$$\phi^s : K'_{-i}(R) \to K''_{-i}(R).$$

Proof. If we have a diagram in $\mathcal{C}_{i+1}(R)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A \oplus B & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus B & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

one easily sees that $\gamma \cdot (\alpha^{-1} \oplus \beta^{-1})$ is an elementary isomorphism. Since ϕ^s commutes with direct sum, Lemma 1.10 shows

$$\phi^s([A \oplus B, \gamma]) = \phi^s([A, \alpha]) + \phi^s([B, \beta]). \tag{1.12}$$

It follows directly from the definition that

$$\phi^s([A, 1]) = 0 \tag{1.13}$$

and now the standard identity

$$\begin{aligned} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & \beta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{1.14}$$

and repeated application of Lemma 1.10 shows that if α and β are isomorphisms of A in $\mathcal{C}_{i+1}(R)$, then

$$\begin{aligned} \phi^s([A, \alpha\beta]) &= \phi^s([A \oplus A, \alpha\beta \oplus 1]) = \phi^s([A \oplus A, \alpha \oplus \beta]) \\ &= \phi^s([A, \alpha]) + \phi^s([A, \beta]). \end{aligned}$$

□

Here we used (1.13) in the first equality. It is easy to see

1.15. **Lemma.** *The map ϕ^s is onto.*

Proof. Let B be an object of $\mathcal{C}_i(R)$ and $p : B \to B$ a projection. Define $A(j_1, \ldots, j_s, \ldots, j_{i+1}) = B(j_1, \ldots, j_{s-1}, j_{s+1}, \ldots, j_{i+1})$ and $\alpha : A \to A$ by

$$\begin{array}{ccccccccc} j_s & = & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ & & & & & & & & \\ A & & & B & B & B & B & B & \\ \downarrow{\scriptstyle \alpha} & = & \cdots \searrow & \downarrow{\scriptstyle 1-p} \searrow^{p} & \downarrow{\scriptstyle 1-p} \searrow^{p} & \downarrow{\scriptstyle 1-p} \searrow^{p} & \downarrow{\scriptstyle 1-p} \searrow^{p} & \downarrow{\scriptstyle 1-p} \searrow & \cdots \\ A & & & B & B & B & B & B & \end{array} \tag{1.16}$$

This is a standard construction of Swan [5]. We compute $\alpha p^s_- \alpha^{-1}$ to be

$$\begin{array}{ccccccccc}
j_s & = & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\\
A & & & B & B & B & B & B & \\
\Big\downarrow \alpha p_-^s \alpha^{-1} & = & \cdots & \Big\downarrow 1 & \Big\downarrow 1 & \Big\downarrow p & \Big\downarrow 0 & \Big\downarrow 0 & \cdots \\
A & & & B & B & B & B & B &
\end{array}$$

so $\phi^s([\alpha]) = ([B, p])$. □

The most difficult part of Theorem 1.9 is to prove ϕ^s is 1-1.

For this we need some condition to ensure elements represent 0 in $K'_{-i}(R)$.

1.17**. Definition.** Let α be an automorphism of A in $\mathcal{C}_{i+1}(R)$. We say α is split at s-degree m if the following holds:

$$j_s \geq m \text{ implies } \alpha(A(j_1, \ldots, j_s, \ldots, j_{i+1})) \subset \bigoplus_{\substack{k_l; l \neq s \\ k_s \geq m}} A(k_1, \ldots, k_s, \ldots, k_{i+1})$$

and

$$j_s < m \text{ implies } \alpha(A(j_1, \ldots, j_s, \ldots, j_{i+1})) \subset \bigoplus_{\substack{k_l; l \neq s \\ k_s < m}} A(k_1, \ldots, k_s, \ldots, k_{i+1})$$

Heuristically the point of the definition is that α preserves the two halves of A given by $j_s \geq m$ and $j_s < m$, respectively.

1.18**. Lemma.** *Let α be an automorphism of A in $\mathcal{C}_{i+1}(R)$, which is split at s-degree m. Then $[A, \alpha]$ represents the trivial element of $K'_{-i}(R)$.*

Proof. Define A' and A'' in $\mathcal{C}_{i+1}(R)$ by

$$A'(j_1, \ldots, j_s, \ldots, j_{i+1}) = \begin{cases} A(j_1, \ldots, j_s, \ldots, j_{i+1}), & j_s \geq m \\ 0, & j_s < m \end{cases}$$

and $A = A' \oplus A''$.

α restricts to isomorphisms $\alpha' : A' \to A'$ and $\alpha'' : A'' \to A''$ and $(A' \oplus A'', \alpha' \oplus \alpha'') = (A, \alpha)$ so

$$[A, \alpha] = [A', \alpha'] + [A'', \alpha''].$$

We show these two terms are 0. Consider $[A', \alpha']$. We define

$$B' = \bigoplus_{l=0}^{\infty} (s^+)^l (A') \qquad \text{and} \qquad B'' = \bigoplus_{l=0}^{\infty} (s^+)^{2l} (A')$$

and note that $1 \oplus s^+$ is an isomorphism

$$1 \oplus s^+ : B'' \oplus B'' \cong B' \tag{1.19}$$

in $\mathcal{C}_{i+1}(R)$. As usual these infinite sums are finite in each degree, so they do make sense. Conjugating $\alpha' : A' \to A'$ by $(s^+)^l$ gives an automorphism $(s^+)^l(A') \to (s^+)^l A'$ which we will denote by $(\alpha')_l$. We now use a trick due to Farrell and Wagoner, really just the so-called "Eilenberg swindle":

$$[A', \alpha'] = [B', \alpha' \oplus 1]$$

by stabilization and

$$\begin{aligned} \alpha' \oplus 1 &= (\alpha' \oplus 1 \oplus 1 \oplus 1 \oplus \ldots) \\ &= (\alpha' \oplus (\alpha')_1^{-1} \oplus (\alpha')_2 \oplus (\alpha')_3^{-1} \oplus \ldots)(1 \oplus (\alpha')_1 \oplus (\alpha')_2^{-1} \oplus \ldots), \end{aligned}$$

so we shall show these two automorphisms represent 0. But conjugating $(\alpha' \oplus (\alpha')_1^{-1} \oplus \cdots)$ by the isomorphism 1.19 gives $(B'' \oplus B'', \beta \oplus \beta^{-1})$ where $\beta = (\alpha' \oplus (\alpha')_2 \oplus (\alpha')_4 \oplus \cdots)$.

We finish off using 1.14 and the fact that elementary automorphisms represent 0 in $K'_{-i}(R)$. The other factor is dealt with similarly. □

Next we investigate what it means for some elements to be 0 in $K''_{-i}(R)$.

1.20. **Lemma.** *Let A be an object of $\mathcal{C}_i(R)$ and p_1,p_2 projections on A. Then $[A, p_1] - [A, p_2] = 0 \in K''_{-i}(R)$ if and only if there are objects A' and A'' in $\mathcal{C}_i(R)$ and an automorphism ϕ of $A \oplus A' \oplus A''$ so that $(p_2 \oplus 1 \oplus 0) \cdot \phi = \phi(p_1 \oplus 1 \oplus 0)$.*

Proof. The if part is trivial, so assume $[A, p_1] = [A, p_2]$. In case $i > 0$ we conclude there is a projection $q : A' \to A'$ of some object in $\mathcal{C}_i(R)$ so that $(A \oplus A', p_1 \oplus q)$ is isomorphic to $(A \oplus A', p_2 \oplus q)$. But then $(A \oplus A' \oplus A', p_1 \oplus q \oplus (1-q)$ is isomorphic to $(A \oplus A' \oplus A', p_2, \oplus q \oplus (1-q))$. Conjugating $(A' \oplus A', q \oplus (1-q))$ by $\left\{ \begin{smallmatrix} q & 1-q \\ 1-q & q \end{smallmatrix} \right\}$ gives $(A' \oplus A', 1 \oplus 0)$ so we obtain the desired result by letting $A'' = A'$. In case $i = 0$ we only conclude $(A \oplus A' \oplus B', p_1 \oplus q \oplus 0)$ is isomorphic to $(A \oplus A' \oplus B'', p_2 \oplus q \oplus 0)$ since in this case we divide out by terms of the form $(B, 0)$. But then B' and B'' are stably isomorphic and we are reduced to considerations as above. □

The proof of Theorem 1.9 is completed by

1.21. **Lemma.** *The map ϕ^s is monic.*

Proof. Assume $\phi^s([A, \alpha]) = 0$. In the terminology of 1.8 we have $[\overline{A}, \alpha p^s_- \alpha^{-1}] - [\overline{A}, p^s_-] = 0$ in $K''_{-i}(R)$. Thus we may use Lemma 1.20 to determine A' and A'' in $\mathcal{C}_i(R)$ so that $(\overline{A} \oplus A' \oplus A'', \alpha p^s_- \alpha^{-1} \oplus 1 \oplus 0)$ is isomorphic to $(\overline{A} \oplus A' \oplus A'', p^s_- \oplus 1 \oplus 0)$. However $(\overline{A} \oplus A' \oplus A'', \alpha p^s_- \alpha^{-1} \oplus 1 \oplus 0) = (\overline{A} \oplus A' \oplus A'', (\alpha \oplus 1 \oplus 1)(p^s_- \oplus 1 \oplus 0)(\alpha \oplus 1 \oplus 1)^{-1})$. Since we can replace (A, α) by $(A \oplus B, \alpha \oplus 1)$ where

$$B(j_1, \ldots, j_s, \ldots, j_{i+1}) = \begin{cases} A'(j_1, \ldots, \widehat{j_s}, \ldots, j_{i+1}) & j_s = -1 \\ A''(j_1, \ldots, \widehat{j_s}, \ldots, j_{i+1}) & j_s = 0 \\ 0, & \text{otherwise,} \end{cases}$$

we may thus assume there is an isomorphism $\beta : \overline{A} \to \overline{A}$ so that $\beta\alpha p^s_- \alpha^{-1} = p^s_- \beta$. Extending β to all A by the identity, we get on all A that $\beta\alpha p^s_- = p^s_- \beta\alpha$. This means that $\beta\alpha$ is split at s-degree 0 so $[A, \beta\alpha] = 0$ by Lemma 1.18. However β is the identity outside $\overline{A}$, so β is split at s-degree $2k+1$ where k is the bound for α, hence $[A, \beta] = 0$ and thus $[A, \alpha] = 0$. □

This ends the proof of Theorem 1.9

2. The Bass-Heller-Swan homomorphisms

In this section we define homomorphisms

$$\lambda^s_t : K'_{-i}(R) \to K'_{-i}(R[t, t^{-1}]), \qquad s = 1, 2, \dots, i+1,$$

which will eventually be the Bass-Heller-Swan homomorphisms.

Let $[A, \alpha]$ represent an element of $K'_{-i}(R)$. Consider the automorphism $p^s_t : A[t, t^{-1}] \to A[t, t^{-1}]$ given by

$$p^s_t = tp^s_- + (1 - p^s_-) \tag{2.1}$$

(with inverse $t^{-1}p^+_- + (1 - p^s_-)$). Consider the commutator between α (extended to a map of $A[t, t^{-1}]$) and p^s_t, $[\alpha, p^s_t]$. Since α is bounded and commutes with multiplication by t, this is the identity on $A(j_1, \dots, j_{i+1})$ away from a band $-k \le j_s \le k$, where k is a bound for α. By restriction as in 1.8 we obtain that $[\alpha, p^s_t]$ is an i-graded bounded $R[t, t^{-1}]$ automorphism of $\overline{A}[t, t^{-1}]$ and we define

$$\lambda^s_t([A, \alpha]) = [\overline{A}[t, t^{-1}], [\alpha, p^s_t]]. \tag{2.2}$$

2.3. **Theorem.** *λ^s_t is a well-defined homomorphism of $K'_{-i}(R) \to K'_{-i+1}(R[t, t^{-1}])$.*

To prove λ^s_t respects the relations, we need to consider the following situation: Let A be an object of $\mathcal{C}_{i+2}(R)$ and γ an isomorphism of A which is 1 except for some band around $j_s = 0$. Then γ may be thought of as an isomorphism of a $\mathbb{Z}^{i+1}$-graded object by restriction and thus defines an element of $K'_{-i}(R)$. If β is a bounded isomorphism of A, then $\beta\gamma\beta^{-1}$ is also the identity outside a sufficiently big band around $j_s = 0$ and we have

2.4. **Lemma.** $[\gamma] = [\beta\gamma\beta^{-1}] \in K'_{-i}(R)$.

Proof. We stabilize $\beta\gamma\beta^{-1}$ to $\beta\gamma\beta^{-1} \oplus 1$ on $A \oplus A$ and note that $\beta\gamma\beta^{-1} \oplus 1 = (\beta \oplus \beta^{-1})(\gamma \oplus 1)(\beta^{-1} \oplus \beta)$. Now we proceed exactly as in Lemma 1.10 using 1.14 to complete the proof. □

Proof of Theorem 2.3. It is clear that λ^s_t sends exact sequences a to exact sequences. Now consider α and β, two automorphisms of an object $A \in \mathcal{C}_{i+1}(R)$.

$$\begin{aligned}\lambda^s_t[\beta\alpha] &= [\beta\alpha, p^s_t] = \beta\alpha p^s_t \alpha^{-1}\beta^{-1}(p^s_t)^{-1} \\ &= \beta[\alpha, p^s_t]\beta^{-1} \cdot [\beta, p^s_t]\end{aligned}$$

and, by Lemma 2.4, $\beta[\alpha, p^s_t]\beta^{-1}$ represents the same element as $[\alpha, p^s_t]$. □

2.5**. Theorem.** *If $s_1 < s_2$, then the diagram*

$$\begin{array}{ccccc} & & K'_{-i+1}(R[t_1,t_1^{-1}]) & & \\ & \overset{\lambda_{t_1}^{s_1}}{\nearrow} & & \overset{\lambda_{t_2}^{s_2-1}}{\searrow} & \\ K'_{-i}(R) & & & & K'_{-i+2}(R[t_1,t_1^{-1},t_2,t_2^{-1}]) \\ & \underset{\lambda_{t_2}^{s_2}}{\searrow} & & \underset{\lambda_{t_1}^{s_1}}{\nearrow} & \\ & & K'_{-i+1}(R[t_2,t_2^{-1}])) & & \end{array}$$

is commutative.

Proof. Let p_i represent p_{t_i} for $i = 1, 2$. Given $[A, \alpha] \in K'_{-i}(R)$ we have to compare

$$[[\alpha, p_1], p_2] \qquad \text{and} \qquad [[\alpha, p_2], p_1]$$

on $A[t_1, t_1^{-1}, t_2, t_2^{-1}]$ in a band around $j_{s_1} = 0$ and $j_{s_2} = 0$.

Using Lemma 2.4 we see that

$$[\alpha, p_2] = \alpha p_2 \alpha^{-1} p_2^{-1} \sim \alpha^{-1} p_2^{-1} \alpha p_2$$

as an element of $K'_{-i+1}(R[t_2, t_2^{-1}])$, hence

$$[\alpha, p_2] = [\alpha^{-1}, p_2^{-1}]$$

. Now

$$[[\alpha, p_1], p_2] = \alpha p_1 \alpha^{-1} p_2 \alpha p_1^{-1} \alpha^{-1} p_2^{-1}$$

since p_1 and p_2 commute, whereas $[[\alpha^{-1}, p_2^{-1}], p_1] = \alpha^{-1} p_2^{-1} \alpha p_1 \alpha^{-1} p_2 \alpha p_1^{-1}$. This last expression however may be conjugated by $p_2\alpha$ to give $\alpha p_1 \alpha^{-1} p_2 \alpha p_1^{-1} \alpha^{-1} p_2^{-1}$. So applying Lemma 2.4 once again shows the two terms represent the same element, and we are done. □

2.6**. Proposition.** *There is a standard identification of K'_1 with K_1 and of $K'_0 = K''_0$ with K_0 under which $\lambda_t : K_0(R) \to K_1(R[t, t^{-1}])$ is the usual Bass-Heller-Swan homomorphism.*

Proof. $K'_1(R)$ is equal to $K_1(R)$ by definition. We have seen $K'_0(R) \simeq K''_0(R)$ and, if we send a projection $p : R^n \to R^n$ to $\mathrm{im}(p) \subset R^n$, this gives a direct summand (the other summand being $\mathrm{im}(1 - p)$) and thus a finitely generated projective. On the other hand, if P is projective, we may find Q such that $P \oplus Q = R^n$ and $1_P \oplus 0_Q$ will define the appropriate projection. If $\phi : P_1 \to P_2$ is an isomorphism of projectives, the diagram

$$\begin{array}{ccc} P_1 \oplus Q_1 \oplus P_2 \oplus Q_2 & \xrightarrow{\phi\oplus 1\oplus\phi^{-1}\oplus 1} & P_2 \oplus Q_1 \oplus P_1 \oplus Q_2 \\ \downarrow{\scriptstyle 1\oplus 0\oplus 0\oplus 0} & & \downarrow{\scriptstyle 1\oplus 0\oplus 0\oplus 0} \\ P_1 \oplus Q_1 \oplus P_2 \oplus Q_2 & \xrightarrow{\phi\oplus 1\oplus\phi^{-1}\oplus 1} & P_2 \oplus Q_1 \oplus P_1 \oplus Q_2 \end{array} \tag{2.7}$$

shows that the corresponding projections are equivalent in $K''_0(R)$, so $K''_0(R)$ is isomorphic to $K_0(R)$. The usual Bass-Heller-Swan homomorphism is described as follows: Let P be a finitely generated projective R-module. Choose Q so that $P \oplus Q = R^n$ and send $[P]$ to $[P[t,t^{-1}] \oplus Q[t,t^{-1}], t \oplus 1]$. In terms of projections this means that $p : R^n \to R^n$ is sent to $tp+1-p : R[t,t^{-1}]^n \to R[t,t^{-1}]^n$. If we are given $[A,\alpha] \in K_0'(R)$, the corresponding element in $K_0''(R)$ is given by $[\overline{A}, \alpha p_- \alpha^{-1}] - [\overline{A}, p_-]$ where $\overline{A} = \oplus_{l=-k}^{k} A(i)$.

The Bass-Heller-Swan construction thus gives $t\alpha p_- \alpha^{-1} + 1 - \alpha p_- \alpha^{-1})(tp_- + 1 - p_-)^{-1} = \alpha(t \cdot p_- + 1 - p_-)\alpha^{-1}(tp_- + 1 - p_-)^{-1} = [\alpha, p_t]$. This completes the proof. □

We wish to show that the λ_t^s-homomorphisms we have constructed are split monomorphisms. The idea is to define a map $K'_{-i+1}(R[t,t^{-1}]) \to K'_{-i}(R)$ using the t-powers to give an extra grading. However given a $\mathbb{Z}^i$-graded $R[t,t^{-1}]$ isomorphism $\alpha : A[t,t^{-1}] \to A[t,t^{-1}]$, we do not have a bound on the powers of t that may occur in expressing α. Hence we may not get a bounded isomorphism when we use the t-powers as gradings. This is the reason for the following slightly artificial step.

2.8. **Definition.** Let R be a ring. We define $\mathcal{C}_i(R)[J,J^{-1}]$ where $J = (t_1, \ldots, t_r)$ as follows: We denote $R[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$ by $R[J,J^{-1}]$, and given an R-module A, we denote the $R[J,J^{-1}]$-module $A[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$ by $A[J,J^{-1}]$. An object of $\mathcal{C}_i(R)[J,J^{-1}]$ is just an object of $\mathcal{C}_i(R)$, but a morphism $A \to B$ is an $R[J,J^{-1}]$-morphism $A[J,J^{-1}] \to B[J,J^{-1}]$ that can be written as a *finite* sum $\alpha = \sum t_1^{n_1} \cdots t_r^{n_r} \alpha_{n_a, \cdots, n_r}$, where $\alpha_{n_1, \cdots, n_r}$ are morphisms in $\mathcal{C}_i(R)$.

We also need the category $P\mathcal{C}_i(R)[J,J^{-1}]$ and the result analogous to Section 1:

2.9. **Lemma.** *$K_1(\mathcal{C}_i(R)[J,J^{-1}]) \cong K_0(P\mathcal{C}_i(R)[J,J^{-1}])$ for $i > 0$ and $K_1(\mathcal{C}_i(R)[J,J^{-1}])$ is isomorphic to $K_0(P\mathcal{C}_i(R)[J,J^{-1}])$ with the extra relations induced by $[A,0] = 0$.*

Proof. We define ϕ^s as in section 1, and note there are no infinite compositions, so everything we do in Section 1 (which corresponds to $J = \emptyset$) goes right through. □

We define
$$K_{-i}^J(R[J,J^{-1}]) = K_1(\mathcal{C}_{i+1}(R)[J,J^{-1}]). \tag{2.10}$$

2.11. *Remark.* There is an obvious map $K_{-i}^J(R[J,J^{-1}]) \to K'_{-i}(R[J,J^{-1}])$ induced by sending $[A,\alpha]$ to $[A[J,J^{-1}],\alpha]$. By construction $\lambda_t^s : K'_{-i}(R) \to K'_{-i+1}(R[t,t^{-1}])$ factors through $K_{-i+1}^t(R[t,t^{-1}]) \to K'_{-i+1}(R[t,t^{-1}])$. It is also clear that λ_t^s generalizes to
$$\lambda_t^s : K_{-i}^J(R[J,J^{-1}]) \to K_{-i}^{J,t}(R[J,J^{-1},t,t^{-1}])$$
. Thinking of λ_t^s in this way as a homomorphism $K'_{-i}(R) \to K_{-i+1}^t(R[t,t^{-1}])$ we will be able to define a left inverse, and we shall then eventually show $K_{-i+1}^t(R[t,t^{-1}]) \cong K'_{-i+1}(R[t,t^{-1}])$.

Consider an element $[B,\beta]$ of $K_{-i+1}^{J,t}(R[J,J^{-1},t,t^{-1}])$ where $J = (t_1, \ldots, t_r)$ as above. We define
$$C(j_1, \ldots, j_s, \ldots, j_{i+1}) = B(j_1, \ldots, \widehat{j_s}, \ldots, j_{i+1})[J,J^{-1}](t^{j_s}),$$

the $R[J,J^{-1}]$-submodule of $B(j_1,\dots,\widehat{j_s},\dots,j_{i+1})[J,J^{-1}][t,t^{-1}]$ generated by t^{j_s}. We may clearly consider β an $R[J,J^{-1}]$-module isomorphism of C. Since the condition that β (and β^{-1}) may be written as a sum only involving finitely many t-powers will ensure that β is a bounded $\mathbb{Z}^{i+1}$-graded automorphism of C, we may define $\mu_t^s : K^{J,t}_{-i+1}(R[J,J^{-1},t,t^{-1}]) \to K^{J}_{-i}(R[J,J^{-1}])$ by

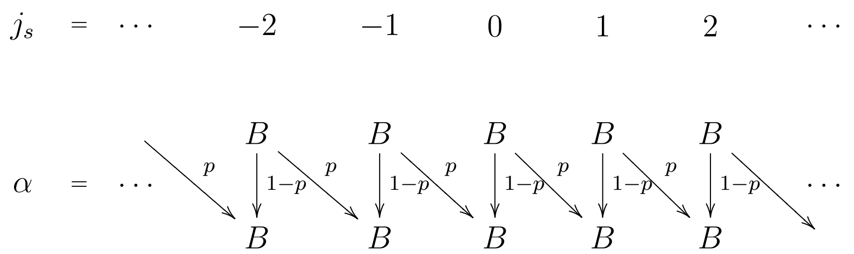
$$\mu_t^s([B,\beta]) = [C,\beta]. \tag{2.12}$$

Popularly speaking μ_t^s is the identity, only we consider the t-powers an extra grading placed at the sth coordinate.

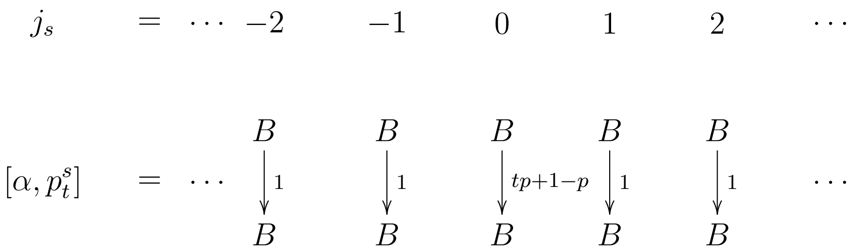
2.13. **Proposition.** *μ_t^s is a well-defined homomorphism and $\mu_t^s \cdot \lambda_t^s = 1$*

Proof. We consider the case $K'_{-i}(R) \xrightarrow{\lambda_t^s} K^t_{-i}(R[t,t^{-1}]) \xrightarrow{\mu_t^s} K'_{-i}(R)$ and note the argument we give carries over to the general case as in the proof of Lemma 2.9.

The fact that β may be written as a finite sum $\sum t^j\beta_j$ ensures that β becomes a bounded automorphism of C. Since μ_t^s is essentially the identity, it will respect all relations. To prove that $\mu_t^s \cdot \lambda_t^s$ is the identity, consider an element of $K'_{-i}(R)$. Using Theorem 1.9 (see in particular Lemma 1.15) we may assume α is of the form



where B is an object of $\mathcal{C}_i(R)$ and $p : B \to B$ a projection. It is easy to compute the commutator $[\alpha, p_t^s]$, and we get



so $\lambda_t^s([\alpha]) = [B[t,t^{-1}], tp+1-p]$.

When we turn the t-powers into gradings we get back α on the nose, so we are done. $\square$

As mentioned above (Remark 2.11) we get a map $K^J_{-i}(R[J,J^{-1}]) \to K'_{-i}(R[J,J^{-1}])$ sending $[A,\alpha]$ to $[A[J,J^{-1}],\alpha]$. This map induces an isomorphism.

2.14. **Proposition.** *$K^J_{-i}(R[J,J^{-1}]) \cong K'_{-i}(R[J,J^{-1}])$.*

Proof. The proof is by induction on i, the induction starting with $i = -1$. In this case the $R[J, J^{-1}]$-module is finitely generated so it is no restriction to require a bound on the powers of t_i (remember $J = (t_1, \ldots, t_r)$). The slight difference between $A[J, J^{-1}]$ where A is a finitely generated free R-module, and a finitely generated free $R[J, J^{-1}]$-module causes no trouble. Assume inductively that $K^J_{-i+1}(R[J, J^{-1}]) \to K'_{-i+1}(R[J, J^{-1}])$ is an isomorphism for all rings R. In the commutative diagram

$$\begin{array}{ccccc} K^J_{-i}(R[J,J^{-1}]) & & \longrightarrow & & K'_{-i}(R[J,J^{-1}]) \\ \downarrow \lambda^1_t & & & & \lambda^t_1 \downarrow \\ K^{J,t}_{-i+1}(R[J,J^{-1},t,t^{-1}]) & \longrightarrow & K'_{-i+1}(R[J,J^{-1},t,t^{-1}]) & \longleftarrow & K^t_{-i+1}(R[J,J^{-1}][t,t^{-1}]) \\ \downarrow \mu^1_t & & & & \downarrow \mu^1_t \\ K^J_{-i}(R[J,J^{-1}]) & & \longrightarrow & & K'_{-i}(R[J,J^{-1}]) \end{array} \tag{2.15}$$

the middle horizontal maps are isomorphisms by induction hypothesis. Since $\mu^1_t \cdot \lambda^1_t = 1$, it follows that $K^J_{-i}(R[J, J^{-1}]) \to K'_{-i}(R[J, J^{-1}])$ is an isomorphism. □

In view of Proposition 2.13 and 2.14 we have proved the following:

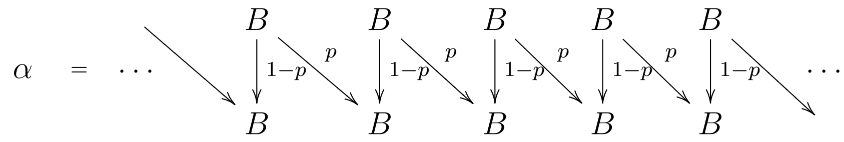
2.16. **Theorem.** *$\lambda^s_t : K'_{-i}(R) \to K'_{-i}(R[t, t^{-1}])$ is a split monomorphism with left inverse given by $K'_{-i}(R[t, t^{-1}]) \cong K^t_{-i}(R[t, t^{-1}]) \xrightarrow{\mu^s_t} K'_{-i}(R)$.*

We have not discussed how λ^s_t and μ^s_t depend on s. Note, that if $g \in Gl(i + 1, \mathbb{Z})$ is used to regrade an object A of $\mathcal{C}_{i+1}(R)$ by $A^g(j_1, \ldots, j_{i+1}) = A(g(j_1, \ldots, j_{i+1}))$, the identity $1_g : A^g \to A$ is not a *bounded* automorphism of A^g. But if α is a bounded automorphism of A, $1_g \alpha 1_g^{-1}$ is a bounded automorphism of A^g. This defines an action of $Gl(i + 1, \mathbb{Z})$ on $K_{-i}(R)$ which is given by

2.17. **Lemma.** *$g \in Gl(i + 1, \mathbb{Z})$ acts on $K'_{-i}(R)$ by multiplication by* $\det(g)$.

Proof. First we show that if g is elementary, $g = E_{rs}(a)$, the action is trivial. If $[A, \alpha]$ is regraded by g we get the composite $A^g \xrightarrow{1_g} A \xrightarrow{\alpha} A \xrightarrow{1_g^{-1}} A^g$. But $\lambda^s_t([A^g, 1_g^{-1}\alpha 1_g]) = [A[t, t^{-1}]^g, [1_g^{-1}\alpha 1_g, p^s_t]]$ and since 1_g and p^s_t commute, we get $[A[t, t^{-1}]^g, 1_g^{-1}[\alpha, p^s_t]1_g]$ and restrict to a band around $j_s = 0$. But $g = E_{rs}(a)$, so 1_g is a bounded isomorphism when restricted to a band around $j_s = 0$, hence this last element is equivalent to $[A[t, t^{-1}], [\alpha, p^s_t]]$ which represents $\lambda^s_t([A, \alpha])$. Since λ^s_t is a monomorphism, we are done. We now only need to see how g, which acts on $\mathbb{Z}^{i+1}$ by multiplying the sth coordinate by -1, acts. A typical element $[\alpha] \in K'_{-i}(R)$ may be written (by Theorem 1.9)

$$j_s \quad = \quad \cdots \qquad -2 \qquad -1 \qquad 0 \qquad 1 \qquad 2 \qquad \cdots$$



which is mirrored about $j_s = 0$ by g to give α^{-1}, so g acts by -1. This completes the proof. ☐

2.18**. Corollary.** *$\lambda_t^{s+1} = -\lambda_t^s : K'_{-i}(R) \to K'_{-i+1}(R[t,t^{-1}])$ and $\mu_t^{s+1} = -\mu_t^s : K'_{-i+1}(R[t,t^{-1}]) \to K'_{-i}(R)$.*

Proof. Let τ be the transposition that interchanges the sth and $s+1$th coordinate. Then $\lambda_t^s \cdot \tau = \lambda_t^{s+1}$ and $\mu_t^{s+1} = \tau \cdot \mu_t^s$. ☐

In view of 2.18 we shall denote λ_t^1, μ_t^1 by λ_t, μ_t, respectively, and note that

$$\mu_t^s = (-1)^{s-1}\mu_t, \qquad \lambda_t^s = (-1)^s \lambda_t. \tag{2.19}$$

Proof of Main Theorem. One possible definition of $K_{-i-1}(R)$ is by induction as the intersection of the images of the Bass-Heller-Swan homomorphisms

$$K_{-i}(R[t_1,t_1^{-1}]) \to K_{-i+1}(R[t_1,t_1^{-1},t_2,t_2^{-1}])$$

and

$$K_{-i}(R[t_2,t_2^{-1}]) \to K_{-i+1}(R[t_1,t_1^{-1},t_2,t_2^{-1}]).$$

Our proof will be by induction on the statement $K'_{-i}(R) \cong K_{-i}(R)$ by an isomorphism under which the image of $\lambda_t : K'_{-i}(R) \to K_{-i+1}(R[t,t^{-1}])$ is sent to the image of the usual Bass-Heller-Swan homomorphism. The start of the induction is proposition 2.6. Denoting $\lambda_{t_1}, \lambda_{t_2}, \mu_{t_1}, \mu_{t_2}$ by $\lambda_1, \lambda_2, \mu_1, \mu_2$ respectively, we have a homomorphism $\lambda_2 \cdot \lambda_1 : K'_{-i-1}(R) \to K_{-i+1}(R[t_1,t_1^{-1},t_2,t_2^{-1}])$. Theorem 2.5 implies $\lambda_2\lambda_1 = -\lambda_1\lambda_2$ so $\operatorname{im}(\lambda_2\lambda_1)$ is contained in $(\operatorname{im}\lambda_1 \cap (\operatorname{im}\lambda_1) = K_{-i-1}(R)$ (by induction hypothesis). $\lambda_2\lambda_1$ is a monomorphism by Proposition 2.13. To show it is epic let $a \in K_{-i+1}(R)$. Consider a as an element of $K_{-i+1}(R[t_1,t_1^{-1},t_2,t_2^{-1}])$. By induction hypothesis $a = \lambda_1(b_1) = \lambda_2(b_2)$ and, by 2.13,

$b_1 = \mu_1(a), b_2 = \mu_2(a)$. The diagram

$$
\begin{array}{ccccc}
 & & K'_{-i-1}(R) & & \\
 & \overset{\mu^{s+1}_{t_2}}{\nearrow} & & \overset{\lambda^{s}_{t_1}}{\searrow} & \\
K'_{-i}R[t_2, t_2^{-1}] & & & & K'_{-i}(R[t_1, t_1^{-1}]) \\
 & \underset{\lambda^{s}_{t_1}}{\searrow} & & \underset{\mu^{s}_{t_2}}{\nearrow} & \\
 & & K'_{-i+1}R[t_1, t_1^{-1}, t_2, t_2^{-1}] & &
\end{array}
\tag{2.20}
$$

is commutative since one way sends $[\alpha]$ to $[\alpha, p^s_{t_1}]$ and then turns t_2-powers into a grading, whereas the other way around turns t_2-powers into a grading and then takes the commutator with $p^s_{t_1}$. Hence $\mu_2\lambda_1 = -\lambda_1\mu_2$ and we get $\lambda_2\lambda_1(-\mu_2\mu_1(a)) = \lambda_2\mu_2\lambda_1\mu_1(a) = a$, so $\lambda_2\lambda_1$ is onto. This completes the proof of the Main Theorem. □

3. Final Remarks

In geometric applications one is usually not considering an automorphism $\alpha : A \to A$, but rather an isomorphism $\alpha : A \to B$. This, however, only makes a difference in the category $\mathcal{C}_0(R)$ since if $p^s_- : A \to A$ is the projection 1.7 we may consider $\alpha p^s_- \alpha^{-1}$. The restriction to a band around $j_s = 0$ gives a projection, which is 1 if $j_s \ll 0$ and 0 if $j_s \gg 0$. Hence the width of the band only matters in case $i = 0$; in all other cases we have a well-defined invariant in $K_{-i}(R)$. For $i = 0$ we have to divide out by identity projections to get a well-defined invariant.

This amounts to getting an invariant in $\widetilde{K}_0(R)$. With the obvious notion of a contractible chain-complex in the category $\mathcal{C}_{i+1}(R)$ we thus get an associated $\widetilde{K}_{-i}(R)$-invariant ($\widetilde{K}_{-i}(R) = K_{-i}(R)$ for $i > 0$). Also associated to a homotopy projection of a $\mathcal{C}_i(R)$ chain complex, we get a $\widetilde{K}_{-i}(R)$-invariant (using the methods of [4], see [3] for a proof). These ideas are further developed in [3]. Altogether a number of results due to Quinn can be given "standard" proofs using this description of the K_{-i}-functors.

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