Pergamon

Appl. Math. Lett. Vol. 10, No. 2, pp. 123–127, 1997

0893-9659/97 $7.00 + 0.00

Sturm Sequences and the Number of Zeros of a Real Polynomial in the Unit Disk: Numerical Computation

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(Received and accepted April 1996)

Communicated by E. L. Ortiz

Abstract—We present an efficient and precise numerical method in computational complexity and reliability to compute the number of zeros of a real polynomial in the unit disk.

Keywords—Polynomials, Root-counting method, Sturm sequence, Round-off error analysis, Numerical computation.

1. INTRODUCTION

In this paper, a method is proposed to determine the precise number of zeros in the unit disk (counted with their multiplicities) of a real polynomial defined in $R[x]$ using the argument's principle in terms of Cauchy indices [1–3] and a Sturm sequence of polynomials in Chebychev form. This algorithm evaluates an estimate of the location of moduli of the zeros of a polynomial with control on rounding-off errors. These results are used in studies on stability or convergence of mathematical models or numerical schemes [4].

2. STURM SEQUENCE

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n(\deg(p) = n)$, T_k, and U_k be Chebyshev polynomials of the first and second kind, respectively. We consider the sequence (p_k) of polynomials defined by following recurrence relation:

$$p_{k+1} = U_1 p_k - p_{k-1}, \qquad k = 2, 3, \dots, \qquad s = \left[\frac{n}{2}\right]. \tag{1}$$

$$p_0 = \sum_{i=0}^{n} a_i T_i, \qquad p_1 = \sum_{i=1}^{n} a_i U_{i-1}, \qquad p_2 = T_1 p_1 - p_0.$$

In this form, p_k is not constructed by Euclid's algorithm, which would have given the last term of the sequence equal to a constant (no zero if a common factor does not exist). We have the following properties:

$$p_k = \sum_{i=0}^{k-2} (a_{k+i} - a_{k-i-2}) U_i + \sum_{i=k-1}^{n-k} a_{k+i} U_i, \qquad k = 2, 3, \dots, s + 1.$$

By this explicit formula it is easy to compute the coefficients of the polynomials p_k, with a minimal use of memory, but for $k > s$ the $\deg(p_{k+1})$ is greater than the $\deg(p_k)$ in the recurrent relation (1).

2.1. Regular Case

The argument's principle on the unit circle gives a relation for the number of zeros of p with a Cauchy index on the interval $[-1,1]$ of a rational function of consecutive terms in the (p_k) sequence: Let us denote by $N_D(p)$ the number of zeros of p in the unit disk,

$$N_D(p) = I_{-1}^{+1} \frac{p_1}{p_0}.$$

We assume that there are no zeros on the unit circle. Let $\Delta(p_h, p_l)$ be the difference of the numbers of variations in sign of the sequences from $p_h(-1)$ to $p_l(-1)$, and from $p_h(1)$ to $p_l(1)$. Let $\xi_0 = -p_1(-1)/p_0(-1) = -p'(-1)/p(-1)$ and $\eta_0 = -p_1(1)/p_0(1) = -p'(1)/p(1)$ be the Sturm constants. If $\xi_0 \neq s$ and $\eta_0 \neq s$, we have the following: $p_0, p_1, \dots, p_{s+1}$ is a Sturm sequence,

$$\Delta(p_0, p_{s+1}) = \begin{cases} s+1, & \text{if } \xi_0 \text{ and } \eta_0 > s, \\ s, & \text{if Min } (\xi_0, \eta_0) < s < \text{Max } (\xi_0, \eta_0), \\ s-1, & \text{if } \xi_0 \text{ and } \eta_0 < s, \end{cases}$$

and hence, by Sturm's theorem, we have

$$\Delta(p_0, p_{s+1}) = I_{-1}^{+1} \frac{p_1}{p_0} + I_{-1}^{+1} \frac{p_s}{p_{s+1}}.$$

REMARKS. If the p_{s+1} have the same sign on $[-1,1]$ (for example if p_{s+1} is a not zero constant), then

(1) $I_{-1}^{+1} p_s/p_{s+1} = 0$,
(2) $N_D(p) = \Delta(p_0, p_{s+1})$.

But, in general, for polynomials of high degree, the value of the Cauchy index on $[-1,1]$ of the rational function p_s/p_{s+1} is not easy to obtain directly.

EXAMPLE.

$$p(x) = 4x^6 + 17x^4 + 20x^2 + 4, \quad p_{s+1} = p_4 = -1.5U_0, \qquad \deg(p_4) \neq 2 \quad \text{and} \quad N_D(p) = 2.$$

2.2. Singular Case

To continue the sequence (p_k), we use the Schelin transformation [5]:

$$p_{k+1} = \alpha_k (q_k p_k - p_{k-1}), \quad k \geq s+1, \qquad p_k = \sum_{i=0}^{n_k} a_i^{(k)} U_i,$$

where $q_k = q_1^{(k)} U_1 + q_0^{(k)} U_0$ and $\alpha_k > 0$, for the normalization of the leading coefficient of p_k. We assume that p_k is regular ($\deg(p_k)$ decreases one unit, $n_k = n - k$) and $p_k(\pm 1) \neq 0$, $k \geq s+1$,

$$p_{k+1} = \sum_{i=0}^{n-(k+1)} \left(q_1^{(k)} \left(a_{i+1}^{(k)} + a_{i-1}^{(k)} \right) + q_0^{(k)} a_i^{(k)} - a_i^{(k-1)} \right) U_i,$$

with $a_{-1}^{(k)} = 0$ and such that

$$a_{n-k}^{(k)} q_1^{(k)} = a_{n-k+1}^{(k-1)},$$
$$a_{n-k}^{(k)} q_0^{(k)} + a_{n-k-1}^{(k)} q_1^{(k)} = a_{n-k}^{(k-1)},$$

p_{k+1} is regular ($\deg(p_{k+1}) = n-(k+1)$) if $a_{n-(k+1)}^{(k+1)} \neq 0$. Let $\xi_k = p_{k+1}(-1)/p_k(-1)$ and $\eta_k = p_{k+1}(1)/p_k(1)$, $k = s+1,\ldots,m-1$ be the Schelin's constants. We have with

$$\Delta_{(p_{s+k},p_{s+k+1})} = \begin{cases} 1, & \text{if } \xi_k < 0 < \eta_k, \\ -1, & \text{if } \eta_k < 0 < \xi_k, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta\left(p_{s+1},p_m\right) = \sum_{k=s+1}^{m-1} \Delta_{(p_{s+k},p_{s+k+1})},$$

(1) $p_0, p_1, \ldots, p_m$ is a Sturm sequence, and
(2) $\Delta(p_{s+1},p_m) = I_{-1}^{+1} p_{m-1}/p_m + I_{-1}^{+1} p_s/p_{s+1}$.

We have a complexity of $n^2/4 + O(n)$, multiplications and divisions.

3. CONTROL ON ROUNDING-OFF ERRORS

The numerical round-off errors caused by the computer floating point arithmetic appear in the implementation. The computation of the Schelin constants ξ_k, η_k, $k = s+1,\ldots,m$ require arithmetic operations such as multiplication, division, addition, and subtraction by the formulae

$$\xi_k = \alpha_k \left(\frac{q_0^{(k)} - 2q_1^{(k)} - 1}{\xi_{k-1}} \right) \quad \text{and} \quad \eta_k = \alpha_k \left(\frac{q_0^{(k)} + 2q_1^{(k)} - 1}{\eta_{k-1}} \right), \qquad k = s+1,\ldots,m-1.$$

To control these errors and their propagation, which affect these constants, we propose the use of the Vignes permutation-perturbation method [6].

It provides an analysis of the propagation of data and computing errors for any algorithm. We consider an algebraic procedure defined by $y = \varphi(u,+,-,\times,\div,\omega)$ in which $u \subset R$ is the set of data, $y \in R$ is the result, and $+,-,\times,\div,\omega$ are exact mathematical operators and functions. This expression is implemented in the high-level programming language with $Y = \Phi(U,+,-,*,/,\Omega)$ in which $U \subset F$ is the set of data, $Y \in R$ is the result, and $+,-,*,/,\Omega$ are data processing operators and functions. F is the set of floating point values encoded in the computer.

In floating point arithmetic, the associativity of algebraic addition is not verified, so the various data processing images of the expression φ are a finite set $\{\Phi_i,\, i \in I\}$ obtained by combinations corresponding to all possible arrangements of the permutable operations in the algebraic procedure.

For each data processing image Φ_i run, the result of a arithmetic operation or assignment is chopped or rounded off, so we must take into account two possible results by default or by excess. Therefore, with j elementary operations we have a set of results $\{Y_N,\, Y_N \in F,\, N = 2^j\}$, as representive as the exact result y.

Assuming the hypothesis of independency of round-off errors, the best estimate of Y_i is given by its mean $\overline{Y}$ and the standard deviation $\delta(Y)$ of three random results Y_i, $i = 1,2,3$ is obtained by applying the permutation-pertubation method (random perturbation of the last bit of the mantissa of numbers after an arithmetic operation and permutation the order of the permutable operators). The number of decimal significant figures for the mean $\overline{Y}$ is the integer $c(Y)$ closest to $\log_{10}|\overline{Y}|/\delta(Y)$, with a probability of 95% (as given in Student's table) and a eventual error of unity.

A finite process can be interrupted when some values are smaller than *a priori* positive epsilon. Such a stopping control in this way is not satisfactory, as it depends on the arbitrary value of epsilon. An adequate test is to compare the data processing Sturm's or Schelin's constants with zero (by translation of s for Sturm's constants).

We can compute the integer c for each constants by random permutation of the order of additions of terms in Schelin's constants, or of sums or alternated sums of coefficients of the polynomial p in the Sturm's constants and perturbation of these results in the assignment.

If $c < 1$ the constant is numerically singular (considered as zero), otherwise it is numerically regular. So if the criterion is positive, then we break off the run.

4. DESCRIPTION OF THE ALGORITHM

INPUT. A polynomial p in $R[x]$ of degree n in x ($n \geq 1$).

OUTPUT. Its number of zeros in the unit disk (with multiplicity).

ALGORITHM.

For $k = 1, 2, \ldots, s+1$.
Computation of p_k.
Test of ξ_0 and η_0.
Evaluation of $N_D(p)$.

For $k = s+1, \ldots, m-1$.
Computation of p_k.
Test of ξ_k and η_k.
Regular criterion.
Evaluation of $N_D(p)$.

5. EXAMPLE

The experiments were run on the Sun Sparc workstation with Fortran compiler to find the number $N_D(p)$ of zeros of p in the unit disk. We consider the polynomial

$$x^3 - 4.01x^2 + 5.029989x - 2.019978, \qquad \text{which has zeros } 0.999,\ 1.011, \text{ and } 2.$$

The mean and standard deviation of Sturm's constants are

$$\overline{\xi}_0 = 1.330848500663393219, \qquad \overline{\eta}_0 = 908.090909089293290890,$$
$$\delta(\xi_0) = 0.00000000000000471, \qquad \delta(\eta_0) = 0.000000028076888858.$$

The number of decimal signifiant figures for these means are, respectively, $c(\xi_0) = 14$ and $c(\eta_0) = 9$. So the tests are negative, and we can continue the computation of the sequence p_k, so p_3. The mean and standard deviation of Schelin's constants are

$$\overline{\xi}_2 = 0.250625184522877464, \qquad \overline{\eta}_2 = -100.220485067167160764,$$
$$\delta(\xi_2) = 0.000000000000516840, \qquad \delta(\eta_2) = 0.000000000012019200.$$

The number of decimal signifiant figures for these means are, respectively, $c(\xi_2) = 10$ and $c(\eta_2) = 12$. So the tests are negative, and we have $N_D(p) = 1$.

6. CONCLUSION

We can control efficiently by the permutation-perturbation method, the numerical stability, and tests to zero of the counting algorithm. For polynomials of degree $n > 4$, the number of the comparisons and runs is important; so the time required to determine the number of zeros is larger than a simple implementation, but suitable for a standard double-precision arithmetic. We can compute more exactly to check the accuracy of Sturm's or Schelin's constants, using multiple-precision floating point arithmetic.

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