Controlled Invariant Subspaces for Linear Impulsive Systems

Douglas A. Lawrence

Abstract—In this paper, controlled invariant subspaces for a class of linear impulsive systems are investigated. Geometric conditions are developed that are necessary as well as sufficient for a subspace to be controlled invariant. These conditions reflect the asymmetric roles played by the continuous-time and discrete-time (impulsive) dynamics that together form the overall impulsive system dynamics. State feedback laws having a sampled-data flavor are derived for which the aforementioned subspace becomes a closed-loop invariant.

I. INTRODUCTION

Controlled invariant subspaces have played a central role in a variety feedback synthesis problems for linear time-invariant (LTI) systems treated from a geometric perspective. ([11], [11], [12]). The efficacy of the geometric approach motivated extensions to nonlinear systems involving controlled invariant distributions and submanifolds. ([2], [8]). The fundamental concepts born from this pioneering work have been generalized to a variety of system classes, including linear parameter-varying systems, polynomial systems, and hybrid systems. The focus of this paper is on the characterization of controlled invariant subspaces for a class of linear impulsive systems. As has been the case in with previous geometric analyses of linear impulsive systems of this type, parallels can be drawn with a class of switched linear systems. In this vein, we mention the work of [3] on controlled invariant sets for switched linear systems represented as finite unions of subspaces.

A notion of controlled invariance, and its dual conditioned invariance, was adopted in the compensator synthesis framework for linear impulsive systems presented in [6] that sought to generalize the approach introduced by Schumacher ([9], [10]) for LTI systems decades ago. The geometric conditions proposed in [6] turn out to be sufficient, but in general not necessary, for controlled invariance defined in terms of of a linear impulsive system’s controlled trajectories. The primary aim of this paper is to derive alternate geometric conditions that are both necessary and sufficient for controlled invariance. Once in place, these conditions allow for the development of several follow-on results that parallel the LTI theory.

The remainder of this paper is organized as follows. This section concludes with a brief review of salient facts from the geometric theory for LTI systems. The class of linear impulsive systems under consideration is presented in Section II. Controlled invariant subspaces are defined and characterized in geometric terms in Section IV. The important concept of a supremal controlled invariant subspace contained within another specified subspace is developed in Section IV. A feedback characterization of controlled invariant subspaces is investigated in Section V. Finally, the paper offers some closing remarks in Section VI.

A. Geometric Theory of LTI Systems

We recall some basic notions from the geometric theory of LTI systems ([1], [11], [12]). Letting the matrix pair \((A, B)\) refer to either a continuous-time or discrete-time LTI state equation, we let \(\langle A, B \rangle\) denote the smallest subspace of the state space \(\mathcal{X}\) that is invariant with respect to the linear map \(A\) and contains \(B := \text{Im} B\). This subspace characterizes the state equation’s reachable set. Next, we say that a subspace \(\mathcal{V} \subset \mathcal{X}\) is \((A, B)\)-controlled invariant if \(A\mathcal{V} \subset \mathcal{V} + B\), equivalently, \(\mathcal{V} \subset A^{-1}(\mathcal{V} + B)\). Here, the notation \(A^{-1}\mathcal{V}\) refers to the inverse image of the subspace \(\mathcal{V}\) under the linear map \(A\), which is well-defined for any linear map. \((A, B)\)-controlled invariance is equivalent to the existence of a feedback gain matrix \(F\) that renders \(\mathcal{V} (A + BF)\)-invariant; we let \(F(\mathcal{V})\) denote the set of all feedback gains with this property, the so-called “friends” of \(\mathcal{V}\).

The remaining geometric concepts we review primarily follow the terminology and notation of [12]. The supremal \((A, B)\)-controlled invariant subspace contained in a subspace \(K \subset \mathcal{X}\), denoted here as \(\mathcal{V}^{*}(K)\), is uniquely defined. \((A, B)\)-controllability subspaces are those that can be represented as \(R = \langle A + BF \mid B \cap R \rangle\) for some feedback gain \(F\). The supremal \((A, B)\)-controllability subspace contained in \(K\), denoted \(\mathcal{V}^{*}(K)\), can then be characterized as \(\mathcal{R}^{*}(K) = \langle A + BF \mid B \cap \mathcal{V}^{*}(K) \rangle\), independent of the choice of feedback gain \(F \in F(\mathcal{V}^{*}(K))\). For subspaces \(K_{1} \subset K_{2}, \mathcal{V}^{*}(K_{1}) \subset \mathcal{V}^{*}(K_{2})\) and \(\mathcal{R}^{*}(K_{1}) \subset \mathcal{R}^{*}(K_{2})\). For the special case in which \(\mathcal{K} = \mathcal{V}\), an \((A, B)\)-controlled invariant subspace, we have \(\mathcal{R}^{*}(\mathcal{V}) := \langle A + BF \mid B \cap \mathcal{V} \rangle\) for any \(F \in F(\mathcal{V})\).

II. LINEAR IMPULSIVE SYSTEMS

We consider linear impulsive systems described by \(n\)-dimensional state equations of the form

\[
\dot{x}(t) = A_{c}x(t) + B_{c}u(t) \quad t \in \mathbb{R} \setminus \mathcal{T},
\]

\[
x(\tau_{k}) = A_{z}x(\tau_{k}^{-}) + B_{z}u[k] \quad \tau_{k} \in \mathcal{T} \tag{1}
\]

where \(\mathcal{T}\) is a countably infinite set of strictly increasing impulse times. Also, \(x(t)\) is the \(n\)-dimensional continuous-time state that undergoes instantaneous changes at the impulse times, \(u(t)\) is a continuous-time control input assumed to be piecewise continuous, and \(u[k]\) is a discrete-time control input. The underlying premise is that the impulse
times cannot be influenced by the control signals, however knowledge of their values is available for control purposes. For \( \tau_k \in \mathcal{T} \), we denote \( x(\tau_k^-) = \lim_{\tau \to \tau_k^-} x(\tau_k - \epsilon) \) and \( x(\tau_k^+) = \lim_{\tau \to \tau_k^+} x(\tau_k + \epsilon) \). In our set-up, the impulsive state equation (1) produces right-continuous state trajectories, i.e., \( x(\tau_k) = x(\tau_k^-), \tau_k \in \mathcal{T} \). The state space for (1) is denoted by \( \mathcal{X} \).

To facilitate subsequent bookkeeping, we introduce the function \( \kappa : \mathbb{R} \to \mathbb{Z} \) defined by

\[
\kappa(t) = \sup \{ k \in \mathbb{Z} | \tau_k \in \mathcal{T} \text{ and } \tau_k \leq t \}
\]

For notational convenience, given a specified initial time \( t_0 \) we set \( k_0 = \kappa(t_0) \) and \( k_1 = \kappa(t_0) + 1 \). Also, letting \( \delta_k = \tau_{k+1} - \tau_k \) denote the spacing between consecutive impulse times we assume:

**Assumption 2.1:** For a countably infinite impulse time set \( \mathcal{T} \),

\[
\delta := \inf \delta_k > 0 \quad \text{and} \quad \bar{\delta} := \sup \delta_k < \infty
\]

The lower bound ensures that the impulse time set \( \mathcal{T} \) contains a finite number of elements on any finite time interval and that \( \tau_k \to \infty \) as \( k \to \infty \).

### III. Controlled Invariant Subspaces for Linear Impulsive Systems

Our point of departure is a system-theoretic specification of controlled invariant subspaces for (1) in terms of its controlled trajectories.

**Definition 3.1:** For the linear impulsive system (1), a subspace \( \mathcal{V} \subset \mathcal{X} \) is controlled invariant if given any initial time \( t_0 \), for any \( x(t_0) \in \mathcal{V} \) there exist control inputs \( u(t), u[k] \) for which the controlled trajectory satisfies \( x(t) \in \mathcal{V} \) for all \( t \geq t_0 \).

Towards a geometric characterization of impulsive controlled invariance in the aforementioned sense, we begin with an obvious necessary condition:

**Lemma 3.2:** For the linear impulsive system (1), a subspace \( \mathcal{V} \subset \mathcal{X} \) is controlled invariant only if it is \((A_C, B_C)\)-controlled invariant.

**Proof:** Necessity of the geometric condition can be established by applying results for continuous-time linear systems to the evolution of the impulsive system (1) on the interval between \( t_0 \) and the first impulse time thereafter, \( \tau_{k_1} \).

Joint \((A_C, B_C)\) and \((A_T, B_T)\)-controlled invariance yields an intuitive sufficient condition for controlled invariance with respect to the impulsive system (1):

**Lemma 3.3:** For the linear impulsive system (1), a subspace \( \mathcal{V} \subset \mathcal{X} \) is controlled invariant if it is both \((A_C, B_C)\)-controlled invariant and \((A_T, B_T)\)-controlled invariant.

**Proof:** Sufficiency of the geometric conditions can be established by interleaving results for continuous-time and discrete-time LTI systems.

The joint controlled invariance condition of Lemma 3.3 is not necessary for impulsive controlled invariance. Specifically, it may happen that \( \mathcal{V} \) fails to be \((A_T, B_T)\)-controlled invariant but the continuous-time control signal can be specified in such a way as to drive the state trajectory at the end of the time interval between adjacent impulse times to a proper subspace of \( \mathcal{V} \) that is, in turn, contained in \( A_T^{-1} (\mathcal{V} + B_T) \).

The next result establishes a geometric condition that is both necessary and sufficient for impulsive controlled invariance. Following Section I-A, we let \( \mathcal{P}_C(\mathcal{V}) \) denote the set of friends of \( \mathcal{V} \) associated with \((A_C, B_C)\)-controlled invariance and let \( \mathcal{R}_C^*(\mathcal{V}) \) denote the largest \((A_C, B_C)\)-controllability subspace contained in \( \mathcal{V} \).

**Theorem 3.4:** For the linear impulsive system (1), a subspace \( \mathcal{V} \subset \mathcal{X} \) is controlled invariant if and only if \( \mathcal{V} \) is \((A_C, B_C)\)-controlled invariant and, in addition, satisfies

\[
\mathcal{V} = \mathcal{V} \cap A_T^{-1} (\mathcal{V} + B_T) + \mathcal{R}_C^*(\mathcal{V})
\]

**Proof:** For necessity, if \( \mathcal{V} \subset \mathcal{X} \) is a controlled invariant subspace for the linear impulsive system (1) in the sense of Definition 3.1, then \( \mathcal{V} \) is \((A_C, B_C)\)-controlled invariant as per Lemma 3.2. We argue the necessity of the condition (2) as follows. Because we always have

\[
\mathcal{V} \cap A_T^{-1} (\mathcal{V} + B_T) + \mathcal{R}_C^*(\mathcal{V}) \subset \mathcal{V}
\]

we must show that

\[
\mathcal{V} \subset \mathcal{V} \cap A_T^{-1} (\mathcal{V} + B_T) + \mathcal{R}_C^*(\mathcal{V})
\]

Take any \( x \in \mathcal{V} \). By \((A_C, B_C)\)-controlled invariance of \( \mathcal{V} \), corresponding to any initial time \( t_0 \) there exists a continuous-time control signal \( u_b(t), t \in [t_0, \tau_{k_1}] \) for which the state trajectory \( x_b(t) \) that evolves backward in time on the interval \( t \in [t_0, \tau_{k_1}] \) along the continuous-time dynamics in (1) initialized with \( x_b(\tau_{k_1}) = x \), given by

\[
x_b(t) = e^{A_C(t-\tau_{k_1})} x + \int_{\tau_{k_1}}^{t} e^{A_C(t-\sigma)} B_C u_b(\sigma) d\sigma
\]

lies in \( \mathcal{V} \) for all \( t \in [t_0, \tau_{k_1}] \), in particular, \( x_b(t_0) \in \mathcal{V} \). Now, by impulsive controlled invariance of \( \mathcal{V} \), there exists a continuous-time control signal \( u_f(t), t \in [t_0, \tau_{k_1}] \) for which the state trajectory \( x_f(t) \) that evolves forward in time on the interval \( t \in [t_0, \tau_{k_1}] \) along the continuous-time dynamics in (1) initialized with \( x_f(t_0) = x_b(t_0) \), given by

\[
x_f(t) = e^{A_C(t-t_0)} x_b(t_0) + \int_{t_0}^{t} e^{A_C(t-\sigma)} B_C u_f(\sigma) d\sigma
\]

lies in \( \mathcal{V} \) for all \( t \in [t_0, \tau_{k_1}] \) and there exists a discrete-time control value \( u[k_1] \) for which

\[
x_f(\tau_{k_1}) = A_T x_f(\tau_{k_1}^-) + B_T u[k_1] \in \mathcal{V}
\]

It follows that \( x_f(\tau_{k_1}) \in \mathcal{V} \cap A_T^{-1} (\mathcal{V} + B_T) \).

Next, straightforward manipulations give

\[
x_f(t) - x_b(t) = \int_{t_0}^{t} e^{A_C(t-\sigma)} B_C [u_f(\sigma) - u_b(\sigma)] d\sigma
\]

for all \( t \in [t_0, \tau_{k_1}] \). Thus, the integral term above describes a state trajectory that starts at the origin at time \( t_0 \) and lies in \( \mathcal{V} \) for all \( t \in [t_0, \tau_{k_1}] \). In particular,

\[
\int_{t_0}^{\tau_{k_1}} e^{A_C(t-\sigma)} B_C [u_f(\sigma) - u_b(\sigma)] d\sigma
\]
represents a state that is reachable from the origin in finite time along a trajectory that remains in \( V \) and is therefore an element of \( R_c^+(V) \).

Using this, \( x = x_b(\tau_k) = x_b(\tau^-_k) \) can be represented as

\[
x = x_f(\tau^-_k) - \int_{t_0}^{\tau_k} e^{A(\tau^-_k - \sigma)} B C \left[ u_f(\sigma) - u_b(\sigma) \right] d\sigma
\]

and so \( x \in V \cap A^{-1}_T (V + B_T) + R_c^+(V) \). The desired subspace containment follows because \( x \in V \) was chosen arbitrarily.

For sufficiency, suppose the subspace \( V \subset \mathcal{X} \) is \((A_C, B_C)\)-controlled invariant and satisfies (2). Again, let \( t_0 \) denote an arbitrary initial time and \( \tau_k > t_0 \) denote the next impulse time. As a consequence of (2), any \( x_0 \in V \) can be decomposed as \( x_0 = r_0 + s_0 \) with \( r_0 \in R_c^+(V) \) and \( s_0 \in V \cap A^{-1}_T (V + B_T) \). Corresponding to the initial state \( s_0 \) there exists a continuous-time control signal \( u_R(t), t \in [t_0, \tau_k) \) that yields a state trajectory lying in \( V \) for all \( t \in [t_0, \tau_k) \) and, in particular,

\[
x_1 := e^{A(t \tau_k - t_0) s_0} + \int_{t_0}^{\tau_k} e^{A(t \tau_k - t_\sigma)} B C u_R(\sigma) d\sigma \in V
\]

As such, we also have the decomposition \( x_1 = r_1 + s_1 \) with \( r_1 \in R_c^+(V) \) and \( s_1 \in V \cap A^{-1}_T (V + B_T) \). Again using properties of \( R_c^+(V) \), there exists a continuous-time control signal \( u_R(t), t \in [t_0, \tau_k) \) for which

\[
- r_1 = e^{A(t \tau_k - t_0) r_0} + \int_{t_0}^{\tau_k} e^{A(t \tau_k - t_\sigma)} B C u_R(\sigma) d\sigma \in V
\]

along a state trajectory lying in \( V \). With \( x(t_0) = x_0 \), the continuous-time control signal \( u(t) = u_R(t) + u_s(t) \) yields a trajectory lying in \( V \) for all \( t \in [t_0, \tau_k) \) and

\[
x(\tau^-_k) = e^{A(t \tau_k - t_0) r_0} + \int_{t_0}^{\tau_k} e^{A(t \tau_k - t_\sigma)} B C u_R(\sigma) + u_s(\sigma) d\sigma
\]

\[
= e^{A(\tau^-_k - t_0)} r_0 + \int_{t_0}^{\tau_k} e^{A(\tau^-_k - \sigma)} B C u_R(\sigma) d\sigma
\]

\[
+ e^{A(\tau^-_k - t_0)} s_0 + \int_{t_0}^{\tau_k} e^{A(\tau^-_k - \sigma)} B C u_R(\sigma) d\sigma
\]

\[
= e^{A(\tau^-_k - t_0)} r_0 + \int_{t_0}^{\tau_k} e^{A(\tau^-_k - \sigma)} B C u_R(\sigma) d\sigma + r_1
\]

\[
+ s_1
\]

Thus \( x(\tau^-_k) \in V \) and there exists a discrete-time control value \( u[k] \) for which

\[
x(\tau^-_k) = A_T x(\tau^-_k) + B_T u[k] \in V
\]

Continuing with \( x(\tau^-_k) \in V \), the above constructions can be adapted to yield a continuous-time control signal on \( [\tau_k, \tau_{k+1}) \) yielding a state trajectory that lies in \( [\tau_k, \tau_{k+1}) \) and further satisfies \( x(\tau^-_{k+1}) \in V \cap A^{-1}_T (V + B_T) \). This, in turn, implies the existence of a discrete-time control value \( u[k+1] \) for which \( x(\tau^-_{k+1}) \in V \). Clearly, this procedure can be repeated indefinitely, and so we conclude that \( V \) is a controlled invariant subspace for the linear impulse system (1) in the sense of Definition 3.1.

**Corollary 3.5:** For the linear impulse system (1), a subspace \( V \subset \mathcal{X} \) is controlled invariant if and only if \( V \) is \((A_C, B_C)\)-controlled invariant and, in addition, satisfies

\[
V \subset A^{-1}_T (V + B_T) + R_c^+(V)
\]

**Proof:** Because \( R_c^+(V) = \langle A_C + B_C R_C | B_C \cap V \rangle \) is, by construction, a subspace of \( V \), the subspace relationship (2) is equivalent to \( V = V \cap (A^{-1}_T (V + B_T) + R_c^+(V)) \) which holds if and only if \( V \subset A^{-1}_T (V + B_T) + R_c^+(V) \).

**Corollary 3.6:** For the linear impulse system (1), a subspace \( V \subset \mathcal{X} \) is controlled invariant if and only if \( V \) is \((A_C, B_C)\)-controlled invariant and, in addition, satisfies

\[
A_T V \subset V + B_T + A_T R_c^+(V)
\]

These corollaries show explicitly how \((A_T, B_T)\)-controlled invariance is not necessary for impulse controlled invariance. Either subspace containment (3), (4) may hold when \( V \) fails to be \((A_T, B_T)\)-controlled invariant. Moreover, these characterizations serve to highlight the fact that, unlike the sufficient conditions in Lemma 3.3, the continuous-time and discrete-time dynamics in (1) play asymmetric roles in achieving impulse controlled invariance of the subspace \( V \).

To conclude this section, we introduce the following impulse system linked to (1)

\[
\dot{q}(t) = A_C q(t) + B_C u(t) \quad t \in \mathbb{R} \setminus T
\]

\[
q(\tau_k) = A_T q(\tau^-_k) + B_T u[k] + A_T R_C \mu[k] \quad \tau_k \in T
\]

involving an additional, fictitious discrete-time input \( \mu[k] \) that enters the discrete-time dynamics via \( A_T R_C \) with \( \text{Im } R_c^+ = R_c^+(V) \). Our geometric characterizations of impulse controlled invariance immediately yield:

**Lemma 3.7:** A subspace \( V \subset \mathcal{X} \) is controlled invariant for (1) if and only if it is controlled invariant for (5).

**Proof:** The lemma holds as a consequence of Corollary 3.6.

**IV. SUPREMAL CONTROLLED INVARIANT SUBSPACES**

Several synthesis problems for LTI systems previously treated from a geometric point-of-view involve the notion of the largest or supremal controlled invariant subspace contained in another specified subspace as referred to in Section I-A. This section is devoted to the existence, uniqueness, and computation of such subspaces for linear impulse systems.

**Lemma 4.1:** The class of subspaces that are controlled invariant for the linear impulse system (1) and contained in a subspace \( K \subset \mathcal{X} \) contains a unique supremal member.

**Proof:** It suffices to show (see [12, Lemma 4.4]) that the class of subspaces that are controlled invariant for the linear impulse system (1) and contained in a subspace \( K \subset \mathcal{X} \) is closed under subspace addition. Suppose \( V_1 \) and \( V_2 \) are two subspaces in this class. That \( V_1 + V_2 \) is an \((A_C, B_C)\)-controlled invariant subspace contained in \( K \) follows from the geometric theory for linear time-invariant
systems. Furthermore, $R^*_C(V_i) \subset R^*_C(V_{i+1})$, $i=1,2$ and so $R^*_C(V_1) + R^*_C(V_2) \subset R^*_C(V_{1+2})$. It now follows that

$$A_{\mathcal{F}}(V_1 + V_2) \subset A_{\mathcal{F}} V_1 + A_{\mathcal{F}} V_2$$

$$\subset V_1 + V_2 + B_{\mathcal{F}} + A_{\mathcal{F}} R^*_C(V_1) + A_{\mathcal{F}} R^*_C(V_2)$$

$$= V_1 + V_2 + B_{\mathcal{F}} + \left( R^*_C(V_1) + R^*_C(V_2) \right)$$

$$\subset V_1 + V_2 + B_{\mathcal{F}} + A_{\mathcal{F}} R^*_C(V_{1+2})$$

from which the desired conclusion follows.

**Theorem 4.2:** The subspace algorithm

$$V_0 = K$$

$$V_k = K \cap A_{\mathcal{F}} C(V_{k-1} + B_C)$$

$$\cap \left( A_{\mathcal{F}}^{-1} (V_{k-1} + B_{\mathcal{F}}) + R^*_C(V_{k-1}) \right)$$

$$k \geq 1$$

yields a non-increasing sequence of subspaces that converges to a controlled invariant subspace for the linear impulsive system (1) and contained in $K$, denoted $V^*(K)$.

**Proof:** Clearly $V_1 \subset K = V_0$. If $V_k \subset V_{k-1}$ then $R^*_C(V_k) \subset R^*_C(V_{k-1})$ and

$$V_{k+1} = K \cap A_{\mathcal{F}} C(V_k + B_{\mathcal{F}})$$

$$\cap \left( A_{\mathcal{F}}^{-1} (V_{k-1} + B_{\mathcal{F}}) + R^*_C(V_{k-1}) \right)$$

$$\subset K \cap A_{\mathcal{F}} C(V_{k-1} + B_{\mathcal{F}})$$

$$\cap \left( A_{\mathcal{F}}^{-1} (V_{k-1} + B_{\mathcal{F}}) + R^*_C(V_{k-1}) \right)$$

$$= V_k$$

and so the sequence defined by (6) is non-increasing. Since $K \subset C$ is finite dimensional, there exists a finite $k^*$ for which $V_{k^*+1} = V_{k^*}$ and thus $V_{k^*+j} = V_{k^*}$, $j \geq 0$. This, in turn, yields

$$V_{k^*} \subset K, \quad V_{k^*} \subset A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}})$$

and

$$V_{k^*} \subset A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}}) + R^*_C(V_{k^*})$$

so that $V_{k^*}$ is a controlled invariant subspace for (1) that is contained in $K$. To show that $V_{k^*}$ is the largest such subspace, suppose $\hat{V}$ is another controlled invariant subspace for (1) that is contained in $K$. That is,

$$\hat{V} \subset K \quad \hat{V} \subset A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}}) \quad \hat{V} \subset A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}}) + R^*_C(\hat{V})$$

By definition, $\hat{V} \subset V_0$. If $\hat{V} \subset V_{k^*}$ then $R^*_C(\hat{V}) \subset R^*_C(V_{k^*})$ and

$$\hat{V} = K \cap A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}})$$

$$\cap \left( A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}}) + R^*_C(V_{k^*}) \right)$$

$$\subset K \cap A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}})$$

$$\cap \left( A_{\mathcal{F}}^{-1} (V_{k^*} + B_{\mathcal{F}}) + R^*_C(V_{k^*}) \right)$$

$$= V_{k^*}$$

from which it follows that $\hat{V} \subset V_{k^*}$ for all $k \geq 0$ and hence $\hat{V} \subset V_{k^*}$. Consequently, $V^{*}(K) = V_{k^*}$.

**V. Feedback Characterization**

The geometric conditions of Lemma 3.3 are equivalent to the existence of non-empty sets of feedback gain matrices $F_C(\mathcal{V})$, $F_{\mathcal{F}}(\mathcal{V})$ such that $F_C \in F_C(\mathcal{V})$ renders $\mathcal{V}$ $(A_C + B_C F_C)$–invariant and $F_{\mathcal{F}} \in F_{\mathcal{F}}(\mathcal{V})$ renders $\mathcal{V}$ $(A_{\mathcal{F}} + B_{\mathcal{F}} F_{\mathcal{F}})$–invariant. Towards realizing a feedback-type characterization of the conditions of Theorem 3.4 that incorporates the effect of the continuous-time control on finite intervals of the form $[\tau_k, \tau_{k+1}]$, we begin with the following.

**Lemma 5.1:** Suppose the subspace $\mathcal{V} \subset \mathcal{X}$ is controlled invariant for the linear impulsive system (1). Then, the set of feedback gain pairs given by

$$K(\mathcal{V}) = \{ (K_C, K_{\mathcal{F}}) \mid (A_{\mathcal{F}} + B_{\mathcal{F}} K_{\mathcal{F}} + A_C R_C K_C) \mathcal{V} \subset \mathcal{V} \}$$

and $K_{\mathcal{F}} R_C K_C = 0$ is nonempty.

**Proof:** Let $\mathcal{V} = [V_1 \ V_2 \ V_3 \ V_4]$ be a nonsingular matrix with $\text{Im} [V_1 \ V_2] = \mathcal{V} \cap A_{\mathcal{F}}^{-1} (\mathcal{V} + B_{\mathcal{F}})$, $\text{Im} [V_2 \ V_3] = R_C^*(\mathcal{V})$, and $\text{Im} V_2 = A_{\mathcal{F}}^{-1} (\mathcal{V} + B_{\mathcal{F}}) \cap R_C^*(\mathcal{V})$. The subspace relationship

$$A_{\mathcal{F}} (\mathcal{V} \cap A_{\mathcal{F}}^{-1} (\mathcal{V} + B_{\mathcal{F}})) \subset \mathcal{V} + B_{\mathcal{F}}$$

holds if and only if the matrix identity

$$A_{\mathcal{F}} [V_1 \ V_2 \ V_3] = \left[ \begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{array} \right] + B_F \left[ \begin{array}{c} U_1 \\ U_2 \end{array} \right]$$

holds for suitable matrices $X_{ij}$, $U_k$ appearing therein. We also have

$$[V_2 \ V_3] = R_C \left[ \begin{array}{cc} T_2 & T_3 \end{array} \right]$$

in which $[T_2 \ T_3]$ is nonsingular.

In order to illustrate some of the flexibility available in specifying feedback gain pairs in (7), we let $d := \dim (A_{\mathcal{F}}^{-1} (\mathcal{V} + B_{\mathcal{F}}) \cap R_C^*(\mathcal{V}))$, $\mathcal{W}$ be any subspace of $\mathbb{R}^d$, and $P$ denote the orthogonal projection of $\mathbb{R}^d$ onto $\mathcal{W}$. Consequently, $P_L := I - P$ is the orthogonal projection onto the orthogonal complement $\mathcal{W}^\perp$. Each projection is idempotent (e.g., $PP = P$) and $PP_L = P_L P = 0$. In terms of this, we define feedback gains $K_C$, $K_{\mathcal{F}}$

$$K_C [V_1 \ V_2 \ V_3 \ V_4] = - \left[ \begin{array}{cccc} 0 & T_2 P_L & T_3 & 0 \\ 0 & U_2 P L & 0 & 0 \end{array} \right]$$

A direct computations gives

$$K_{\mathcal{F}} R_C K_{\mathcal{F}} [V_1 \ V_2 \ V_3 \ V_4] = - \left[ \begin{array}{cccc} 0 & V_2 P_L & V_3 & 0 \\ 0 & U_2 P L & 0 & 0 \end{array} \right]$$

and so $K_{\mathcal{F}} R_C K_{\mathcal{F}} = 0$ because $V$ is nonsingular. As a result, we obtain the factorization

$$A_{\mathcal{F}} + B_{\mathcal{F}} K_{\mathcal{F}} + A_C R_C K_C = (A_{\mathcal{F}} + B_{\mathcal{F}} K_{\mathcal{F}})(I + R_C K_C)$$

and so we first consider

$$(I + R_C K_C) [V_1 \ V_2 \ V_3] = \left[ \begin{array}{c} V_1 \ V_2(I - P_L) \ (V_3 - V_3) \end{array} \right] = \left[ \begin{array}{c} V_1 \ V_2 P \ 0 \end{array} \right]$$
We then obtain
\[(A_I + B_T K_I + A_I R_C K_C) \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \]
\[= (A_I + B_I K_I) \begin{bmatrix} V_1 & V_2P & 0 \end{bmatrix} \]
\[+ B_I \begin{bmatrix} U_1 & U_2P & 0 \end{bmatrix} - B_I \begin{bmatrix} U_1 & U_2(P) & 0 \end{bmatrix} \]
\[= \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12P} & 0 \\ X_{21} & X_{22P} & 0 \\ X_{31} & X_{32P} & 0 \end{bmatrix} \]

which establishes that \( \mathcal{V} \) is \((A_I + B_T K_I + A_I R_C K_C)\)-invariant.

Our aim now is to incorporate the feedback gains \( F_C \in \mathbb{F}_C(\mathcal{V}) \) and \((K_C, K_T) \in \mathbb{K}(\mathcal{V})\) into the construction of state feedback laws for (1) that render the subspace \( \mathcal{V} \) invariant for the resulting closed-loop impulsive system. For \( F_C \in \mathbb{F}_C(\mathcal{V}) \), the map \( A_C + B_C F_C \) has well-defined restrictions to both \( \mathcal{V} \) and \( R_C^2(\mathcal{V}) \), denoted \( \hat{A}_C \) and \( \tilde{A}_C \), respectively. In terms of injective insertion maps \( V \) and \( R_C \) that satisfy \( \text{Im } V = \mathcal{V} \) and \( \text{Im } R_C = R_C^2(\mathcal{V}) \), we can factor \( R_C = V R_C \) for suitable \( R_C \). We also have the commutative relationships
\[(A_C + B_C F_C)V = V \hat{A}_C \quad \text{and} \quad (A_C + B_C F_C)R_C = R_C \tilde{A}_C \]

In addition, there are well-defined maps \( \hat{B}_C \) and \( \tilde{B}_C \) that satisfy \( \text{Im } \hat{B}_C \cap \mathcal{V} = V \hat{B}_C = R_C \hat{B}_C \) in which \( \hat{B}_C = \text{Im } \hat{B}_C \) and \( \tilde{B}_C = \text{Im } \tilde{B}_C \). In fact, once \( B_C \) has been determined, we can take \( B_C = R_C \hat{B}_C \). Finally, there exists a map \( G_C \) for which
\[B_C G_C = V \hat{B}_C = R_C \tilde{B}_C \] (8)
The pair \((\hat{A}_C, \hat{B}_C)\) is controllable/reachable with symmetric reachability gramian
\[\hat{W}_C(t_0, t_f) = \int_{t_0}^{t_f} e^{\hat{A}_C(t_f - \sigma)} \hat{B}_C \hat{B}_C^T e^{\hat{A}_C^T(t_f - \sigma)} d\sigma \]
that is positive definite for all \( t_f > t_0 \).

Now, for any \( F_C \in \mathbb{F}_C(\mathcal{V}) \) and \((K_C, K_T) \in \mathbb{K}(\mathcal{V})\), we consider the impulsive state feedback law
\[u(t) = F_C x(t) + H_C(t) x(\tau_k) + v(t) \]
\[u[k] = K_T x(\tau_k^-) + v[k] \] (9)
in which
\[H_C(t) = G_C \hat{B}_C^T e^{\hat{A}_C^T(t_k - t)} \hat{W}_C^{-1}(t_k, t_{k+1}) K_C \times e^{(A_C + B_C F_C)\delta_k} \]
for \( t \in [\tau_k, \tau_{k+1}) \), \( k > k_0 \) with straightforward modifications for \( t \in [t_0, \tau_k) \). This state feedback law incorporates a sampled-state term that contributes to the continuous-time control signal via the time-varying gain \( H_C(t) \) that effectively plays the role of a generalized hold device reminiscent of the approach in [4] for sampled-data control of LTI systems and in [5] for output feedback control of linear impulsive systems, and also resembles the technique employed in [7] for gramian-based control of linear periodic systems.

When applied to (1), we obtain the closed-loop impulsive system
\[\dot{x}(t) = (A_C + B_C F_C) x(t) + B_C H_C(t) x(\tau_k) + B_C v(t) \quad t \in [\tau_k, \tau_{k+1}) \]
\[x(\tau_k) = (A_I + B_T K_I) x(\tau_k^-) + B_T v[k] \quad \tau_k \in \mathcal{T} \] (11)

In order to verify that the feedback law (9) has the intended effect on (11), we consider in parallel the application of the feedback law
\[u(t) = F_C q(t) + v(t) \]
\[u[k] = K_T q(\tau_k^-) + v[k] \quad \mu[k] = K_C q(\tau_k^-) + v[k] \] (12)
to (5) yielding the closed-loop impulsive system
\[\dot{q}(t) = (A_C + B_C F_C) q(t) + B_C v(t) \quad t \in \mathbb{R} \setminus \mathcal{T} \]
\[q(\tau_k) = (A_I + B_T K_I + A_I R_C K_C) q(\tau_k^-) + B_T v[k] \]
\[+ A_C R_C v[k] \quad \tau_k \in \mathcal{T} \] (13)

Lemma 5.2: For \( F_C \in \mathbb{F}_C(\mathcal{V}) \) and \((K_C, K_T) \in \mathbb{K}(\mathcal{V})\), the closed-loop impulsive systems (11) and (13) are related as follows. For any input signals \( v(t), v[k] \) along with
\[\nu[k] := -K_C \int_{\tau_{k-1}}^{\tau_k} e^{(A_C + B_C F_C)\delta_k} B_C v(\sigma) d\sigma \]
(14)

\[x(t_0) - q(t_0) \in \mathcal{V} \]
\[x(t) - q(t) \in \mathcal{V} \quad t \geq t_0 \]
If, in addition, \( x(t_0) = q(t_0) \), then \( x(t) - q(t) \in R_C^2(\mathcal{V}) \) for all \( t \geq t_0 \) and \( x(\tau_k) = q(\tau_k) \) for all \( \tau_k \in \mathcal{T} \).

Proof: The difference \( x(t) - q(t) \) evolves according to
\[\frac{d}{dt} (x(t) - q(t)) = (A_C + B_C F_C) (x(t) - q(t)) \]
\[+ B_C H_C(t) x(\tau_k) \quad t \in [\tau_k, \tau_{k+1}) \]
\[x(\tau_k) - q(\tau_k) = (A_I + B_T K_T) (x(\tau_k^-) - q(\tau_k^-)) \]
\[- A_C R_C \mu[k] \quad \tau_k \in \mathcal{T} \] (15)

As a consequence of (14), we have
\[\mu[k] = K_C q(\tau_k^-) + v[k] = K_C e^{(A_C + B_C F_C)\delta_k} q(\tau_k^-) \]
for \( \tau_k \in \mathcal{T} \).

We again set \( k_1 = k_0 + 1 \) and also let \( \delta_0 = \tau_{k_0+1} - t_0 \). Then, on the interval \([t_0, \tau_{k_1})\),
\[x(t) - q(t) = e^{(A_C + B_C F_C)(t-t_0)} \]
\[x(t_0) - q(t_0) + R_C \lambda(t) \]
in which
\[\lambda(t) = \hat{W}_C(t_0, t) e^{\hat{A}_C(t_k - t)} \hat{W}_C^{-1}(t_0, t_0) K_C \times e^{(A_C + B_C F_C)\delta_k} x(t_0) \]
(10)

We see that \( x(t_0) - q(t_0) \in \mathcal{V} \) yields \( x(t) - q(t) \in \mathcal{V} \) for \( t \in [t_0, \tau_{k_1}) \).

In particular,
\[x(\tau_{k_1}^-) - q(\tau_{k_1}^-) = e^{(A_C + B_C F_C)\delta_k} (x(t_0) - q(t_0)) + R_C K_C e^{(A_C + B_C F_C)\delta_k} x(t_0) \]
\[= (I + R_C K_C) e^{(A_C + B_C F_C)\delta_k} x(t_0) \]
\[+ e^{(A_C + B_C F_C)\delta_k} q(t_0) \]
which, using \( K_C R_C K_C = 0 \), leads to
\[
x(\tau_k^-) - q(\tau_k^-) = (A_I + B_I K_I) (x(\tau_k^-) - q(\tau_k^-)) \\
- A_T R_C \mu[k]
\]
\[
= (A_I + B_I K_I + A_T R_C K_C) \\
\times e^{(A_C + B_C F_C)\delta_0} (x(t_0) - q(t_0))
\]
which lies in \( \mathcal{V} \). When \( x(t_0) = q(t_0) \), the preceding computations reduce to
\[
x(t) - q(t) = R_C \lambda(t) \in \mathcal{R}_C(\mathcal{V}) \quad t \in [t_0, \tau_k]
\]
and \( x(\tau_k^-) - q(\tau_k^-) = 0 \) as desired.

Next, suppose \( x(\tau_{k-1}) - q(\tau_{k-1}) \in \mathcal{V} \). On the interval \([\tau_{k-1}, \tau_k)\),
\[
x(t) - q(t) = e^{(A_C + B_C F_C)(t-\tau_{k-1})} (x(\tau_{k-1}) - q(\tau_{k-1})) \\
+ R_C \lambda(t)
\]
now with
\[
\lambda(t) = \tilde{W}_C(\tau_{k-1}, t) e^{\tilde{X}_C(\tau_{k-1} - t)} \tilde{W}_C^{-1}(\tau_{k-1}, \tau_k) K_C \\
\times e^{(A_C + R_C F_C)\delta_{k-1}} x(\tau_{k-1})
\]
Thus, \( x(\tau_{k-1}) - q(\tau_{k-1}) \in \mathcal{V} \) yields \( x(t) - q(t) \in \mathcal{V} \), \( t \in [\tau_{k-1}, \tau_k) \). In particular,
\[
x(\tau_k^-) - q(\tau_k^-) = e^{(A_C + B_C F_C)\delta_{k-1}} (x(\tau_{k-1}) - q(\tau_{k-1})) \\
+ R_C K_C e^{(A_C + B_C F_C)\delta_{k-1}} x(\tau_{k-1}) \\
= (I + R_C K_C) e^{(A_C + B_C F_C)\delta_{k-1}} x(\tau_{k-1}) \\
- e^{(A_C + B_C F_C)\delta_{k-1}} q(\tau_{k-1})
\]
As before, this leads to
\[
x(\tau_k^-) - q(\tau_k^-) = (A_I + B_I K_I) (x(\tau_k^-) - q(\tau_k^-)) \\
- A_T R_C \mu[k]
\]
\[
= (A_I + B_I K_I + A_T R_C K_C) \\
\times e^{(A_C + B_C F_C)\delta_0} (x(\tau_{k-1}) - q(\tau_{k-1}))
\]
which lies in \( \mathcal{V} \). Moreover, when \( x(\tau_{k-1}) = q(\tau_{k-1}) \), we obtain \( x(t) - q(t) = R_C \lambda(t) \in \mathcal{R}_C(\mathcal{V}) \) for \( t \in [\tau_{k-1}, \tau_k) \) and \( x(\tau_k^-) - q(\tau_k^-) = 0 \) as desired.

These relationships between the closed-loop impulsive systems (11) and (13) enables a quick confirmation that the controlled invariant subspace \( \mathcal{V} \) is invariant for (11).

**Theorem 5.3:** Suppose the subspace \( \mathcal{V} \subset \mathcal{X} \) is controlled invariant for the linear impulsive system (1). Then, for any \( F_C \in \mathbb{F}_C(\mathcal{V}) \) and \( (K_C, K_I) \in \mathbb{K}(\mathcal{V}) \), the impulsive state feedback law specified by (9), renders \( \mathcal{V} \) invariant for the closed-loop impulsive system (11) with \( v(t) \equiv 0 \) and \( v[k] \equiv 0 \).

**Proof:** For any \( F_C \in \mathbb{F}_C(\mathcal{V}) \) and \( (K_C, K_I) \in \mathbb{K}(\mathcal{V}) \), \( \mathcal{V} \) is an invariant subspace for (13) with \( v(t) \equiv 0 \), \( v[k] \equiv 0 \), and \( v[k] \equiv 0 \). Noting that for \( v(t) \equiv 0 \), \( v[k] \equiv 0 \) satisfies (14), we apply Lemma 5.2 as follows. For any \( x(t_0) \in \mathcal{V} \), we set \( q(t_0) = x(t_0) \) yielding \( q(t) \in \mathcal{V} \) for all \( t \geq t_0 \). This, along with \( x(t) - q(t) \in \mathcal{R}_C(\mathcal{V}) \subset \mathcal{V} \), \( t \geq t_0 \) implies that \( x(t) \in \mathcal{V} \) for all \( t \geq t_0 \).

**VI. CONCLUDING REMARKS**

This paper has presented an in-depth characterization of controlled invariant subspaces for a class of linear impulsive systems. Geometric conditions have been derived that are necessary as well as sufficient for a subspace to be controlled invariant. As noted earlier, these conditions reflect the asymmetric roles played by the continuous-time and the discrete-time (impulsive) dynamics that together form the overall impulsive system dynamics. State feedback laws have been specified for which the subspace becomes a closed-loop invariant. It is interesting to note that whereas controlled invariant subspaces themselves are independent of the impulse time set, the open-loop control signals and state feedback laws that maintain the state trajectory within the subspace do depend on the impulse times in general.

Future work includes characterizing internal and external stabilizability of impulsive controlled invariants, applying this analysis to state feedback synthesis problems, dualizing these results to treat conditioned invariant subspaces, and exploring their combined role in output feedback synthesis problems for linear impulsive systems.

**REFERENCES**