Social Context Games

Itai Ashlagi[1], Piotr Krysta[2], and Moshe Tennenholtz[3,4]

[1] Harvard Business School, Harvard University, USA
[2] CS Department, University of Liverpool, UK
[3] Industrial Engineering & Management, Technion, Israel
[4] Microsoft Israel R&D Center

Abstract. We introduce the study of *social context games*. A social context game is defined by an underlying game in strategic form, and a social context consisting of an undirected graph of neighborhood among players and aggregation functions. The players and strategies in a social context game are as in the underlying game, while the players' utilities in a social context game are computed from their payoffs in the underlying game based on the graph of neighborhood and the aggregation functions. Examples of social context games are ranking games and coalitional congestion games. A significant challenge is the study of how various social contexts affect various properties of the game. In this paper we consider resource selection games as the underlying games, and four basic social contexts. An important property of resource selection games is the existence of pure strategy equilibrium. We study the existence of pure strategy Nash equilibrium in the corresponding social context games. We also show that the social context games possessing pure strategy Nash equilibria are not potential games, and therefore are distinguished from congestion games.

1 Introduction

Game theory has become a standard tool for the analysis of social interactions of self-motivated agents.[5] Naturally, social interactions may be complex and refer to issues such as competition, coordination, and collaboration among agents. The attitude of agents towards other agents is typically captured by their utility functions. However, it may be of interest to separate the payoff the agent receives, by means of e.g. delay, cost, etc., from his social attitude; this will allow to study the possible effects that various social contexts have. Consider for example several service providers who act on behalf of a set of customers. Each service provider suffers a cost, caused by the need for sharing resources with other service providers. This type of situations are typically modelled as a form of congestion game. Indeed, such congestion games have been a central topic of study in the interplay between computer science and game theory. However, while these games capture the underlying situation, they implicity assume a very particular social context, where the actual goals of each service provider is

[5] We use the terms agents and players interchangeably.

to minimize its own cost. In some situations it may be the case that the aim of each agent is to have its payoff ranked as high as possible comparing to other agents' payoffs, as studied in [2]. Another example is when each agent cares about the sum (or average) of payoffs of a set of agents in a coalition it belongs to, as studied in coalitional congestion games [4, 5]. In our terminology the above are examples of *social contexts*. Our aim is to study the effects of social contexts on basic properties of fundamental types of underlying games.

Consider a resource selection game. In a resource selection game we have a set of n agents, and a set of m resources. Each agent chooses a resource from among the set of resources, and his cost is a non-decreasing function of the number of agents who have chosen his selected resource. Needless to say, resource selection games are central to work in various communities, such as operations research, computer science, game theory and economics. A resource selection game is the most famous type of congestion games [8]. A fundamental property of congestion games is that they possess a pure strategy equilibrium. This result is implied by the fact that congestion games possess a potential function. Indeed, the classes of potential games and congestion games coincide [6]. Given the importance of resource selection games, and the desire to consider various social contexts, we will consider the existence of pure strategy Nash equilibrium when the resource selection games are embodied in the following basic social contexts:

1. Rank competition: the agents are partitioned into cliques, where at each clique the agents compete on the their relative payoff. This significantly extends upon ranking games, where the whole graph is a single clique.
2. Best-Member Collaboration: there is a given social network, and each agent cares about the highest payoff obtained by him or by one of his neighbors. This is in the spirit of work in congestion games where an agent can choose several resources and cares about the one with the best performance [7]; here the agent cares that either himself or one of his friends will behave (e.g. provide a service) as good as possible.
3. Min-Max Collaboration: there is a given social network, and each agent cares that the worst case payoff obtained by him or by one of his friends will be maximized. This requirement is in the spirit of minmax fairness[6] ; however, it is stated as a social attitude rather than as a system requirement.
4. Surplus Collaboration: there is a social network, and each agent wishes to maximize the average payoff of himself and his friends. This is in the spirit of coalitional congestion games [4, 5]; however, here we allow arbitrary graphs rather than a partition of the nodes into cliques.

In the following sections we deal with the social context games generated by resource selection games and the above social contexts. We show:

- When the resources are identical then any Rank Competition resource selection game possesses a pure strategy equilibrium. This is no longer true when the resources are non-identical.

[6] Our particular treatment extends upon [3], by allowing arbitrary networks.

- Any Best-Member Collaboration game possesses a pure strategy equilibrium whenever the size of the largest independent dominating set of the corresponding graph is smaller than $\frac{n}{2}$; this condition is necessary.
- Any pure strategy Nash equilibrium is an equilibrium also in the corresponding Min-Max collaboration game; the converse is not true.
- Pure strategy equilibrium does not always exist in surplus collaboration resource selection games, even with identical resources and when the graph is a tree. We show a subset of the resource selection games where a pure strategy equilibrium exists.
- We show that all social contexts games above do not possess a potential function, and therefore they are not isomorphic to congestion games.

2 Basic Definitions

A *game in strategic form* (or, for short, a *game*) is a tuple $H = (N, (A_i)_{i\in N}, (c_i)_{i\in N})$ where $N = \{1, 2, \ldots, n\}$ is a finite set of players (aka agents), for every player $i \in N$, A_i is the action set of player i, and $c_i : \times_{i\in N} A_i \to \mathbb{R}$ is player i's cost function. We denote by $\mathbf{A} = \times_{i\in N} A_i$ the set action profiles. For every subset of the players $S \subseteq N$ and every vector $b = (b_1, \ldots, b_n)$ we denote by b_S the vector $(b_i)_{i\in S}$ where $b_{-i} = b_{N\setminus\{i\}}$. An action profile $\mathbf{a} \in \mathbf{A}$ is a *Nash equilibrium* (or just an equilibrium[7]) if for every player i, $c_i(a_i, \mathbf{a}_{-i}) \leq c_i(b_i, \mathbf{a}_{-i})$ for every $b_i \in A_i$.

Given an underlying game H, a social context game is generated by considering a neighborhood graph over the players, and aggregation functions that determine how the game is effected by that graph. Formally, a *social context* is a tuple $F = (G, (f_i)_{i\in N})$, where $G = (N, E)$ is an undirected graph, and for every i, $f_i : G \times \mathbb{R}^N \to \mathbb{R}$ is an aggregation function. The aggregation function maps a payoff profile of the underlying game into a utility profile, as a function of the graph structure. The aggregation function captures the agent's social attitude. Given an underlying game $H = (N, (A_i)_{i\in N}, (c_i)_{i\in N})$, and a social context $F = (G, (f_i)_{i\in N})$, a social context game $S = S(H, F) = (N, (A_i)_{i\in N}, (t_i)_{i\in N})$ is a game in strategic form, where N is the set of players, A_i is the set of actions available to player i, and $t_i : \mathbf{A} \to \mathbb{R}$ satisfies that $t_i(a) = f_i(c_1(a), \ldots, c_n(a))$ for every $a \in \mathbf{A}$.

To distinguish between the costs in the games H and S we will refer from now on to the costs in H as *immediate* costs. Notice that the set of players and the set of actions in the social context game are as in H. The following notation will be useful for us. Denote by $v(i)$ the set of neighbors of i in the graph G and let $g(i) = \{i\} \cup v(i)$ be the *group* of player i.

Throughout this paper we study social context games with the following aggregation functions:

1. Best-Member Collaboration: a player's cost is the minimal immediate cost in her group. Formally, $f_i(G, c = (c_1, \ldots, c_n)) = \min_{j\in g(i)} c_j$.

[7] We do not consider mixed strategies in this paper

2. MinMax Collaboration: a player's cost is the maximal immediate cost in her group. Formally, $f_i(G, c = (c_1, \ldots, c_n)) = \max_{j \in g(i)} c_j$.
3. Surplus Collaboration: a player's cost is the average of the immediate costs of her group. Formally, $f_i(G, c = (c_1, \ldots, c_n)) = \frac{1}{|g(i)|} \sum_{j \in g(i)} c_j$.
4. Competitive Ranking: a player cares about his ranking among the players within her group. We assume that each group is a clique of nodes, and the graph is partitioned into cliques. We need a few notations. Let $c = (c_1, c_2, \ldots, c_n)$ be a tuple of immediate costs. Let $l_i(c)$ denote the number of players in $g(i)$ that have a lower immediate cost than i, and let $m_i(c)$ denote the number of players in $g(i)$ with identical immediate cost as i in c. For every player i, $f_i(G, c) = l_i(c) + \lfloor \frac{m_i(c))}{2} \rfloor$.
Notice that the aggregation function simply counts the number of players who obtain lower immediate costs assuming that ties are broken randomly. This is a standard practice in work on ranking systems [1].

For convenience, we will overload notation, and write $f_i(G, \mathbf{a})$ to refer to $f_i(G, (c_1(\mathbf{a}), \ldots, c_n(\mathbf{a})))$.

In this paper we focus on social context games in which the underlying game H is a *resource selection game*. In a resource selection game there is a set of resources $R = \{1, \ldots, m\}$ and for every resource $k \in R$, $w_k : \{1, \ldots, n\} \rightarrow \mathbb{R}_+$ is resource k's cost function. Every player i's action set is $A_i = R$. Finally, the cost of a player in the game who chooses resource k is $w_k(l)$, where l is the number of players that choose resource k; Formally, $c_i(\mathbf{a}) = w_{a_i}(\sigma_{a_i}(\mathbf{a}))$ where $\sigma_k(\mathbf{a}) = |\{j \in N : a_j = k\}|$ denotes the number of players that choose resource k in $\mathbf{a}$. We assume the resource cost functions are non-decreasing.

Resource selection games belong to a larger class of games called congestion games. It is well-known [8] that every congestion game possesses a (pure strategy) Nash equilibrium .

3 The Competitive Ranking Game

In this section we study social context games in which the aggregation function is the competitive ranking function. We call such a game a competitive ranking game. We will assume that the graph is a partition into cliques. Hence, every player cares about her ranking (with respect to immediate costs) in her clique. We assume there is some order over the cliques, $t = 1, 2, \ldots, T$, where $T \geq 1$. Notice that the case $T = 1$ is the special case of ranking games [2].

Our main result is the following:

Theorem 1. *Let $S = (H, F)$ be a competitive ranking game in which H is a resource selection game with identical resources, and G is a partition into cliques. Then, there exists a Nash equilibrium in S.*

Before the proof we need some preparations.

Definition 1. *Let $\mathbf{a}$ be an action profile. Let $i \in N$. Resource $j > (<) a_i$ is a* right (left) improvement *for player i in $\mathbf{a}$ if $f_i(G, \mathbf{a}) > (<) f_i(G, (j, \mathbf{a}_{-i}))$. We* *say that player i has a right (left) improvement in* $\mathbf{a}$ *if there exist a resource j which is a right (left) improvement for i in* $\mathbf{a}$*.*

Resource j is a minimal *left improvement for i in* $\mathbf{a}$*, if j is a left improvement for i in* $\mathbf{a}$ *and there is no resource $j' < j$ such that j' is left improvement for i in* $\mathbf{a}$*.*

We say that there is a right (left) improvement in $\mathbf{a}$ *if there exist a player i which has a right (left) improvement in* $\mathbf{a}$*.*

Definition 2. *An action profile* $\mathbf{a}$ *is* ordered *if for every pair of players $i, i' \in N$ such that $g(i) < g(i')$ we have that $a_i \leq a_{i'}$.*

As we deal with identical resources we let $w() = w_j()$ for every resource j. Let $v(i, \mathbf{a}, j)$ denote the number of neighbors of player i that choose resource j in the action profile $\mathbf{a}$; that is, $v(i, \mathbf{a}, j) = |\{i' \in v(i) : a_{i'} = j\}|$.

The following Lemma is the key for the proof of Theorem 1.

Lemma 1. *Let S be a social context game as in Theorem 1. Let $k = \lfloor \frac{n}{m} \rfloor$. Let* $\mathbf{a}$ *be an ordered action profile such that $\sigma_j(\mathbf{a}) \in \{k, k+1\}$ for every resource j.*

1. *$f_i(G, (j, \mathbf{a}_{-i})) < f_i(G, (j', \mathbf{a}_{-i}))$ only if $v(i, \mathbf{a}, j) > v(i, \mathbf{a}, j')$, $\sigma_{j'}(\mathbf{a}) = k + 1$ and $\sigma_j(\mathbf{a}) = k$.*
2. *Suppose there is no right improvement in* $\mathbf{a}$*. If j' is a minimal left improvement for some player i, then there is no right improvement in $(j', \mathbf{a}_{-i})$.*

Proof. Note that part 1 follows immediately from the definitions. We next prove part 2.

Let $j = a_i$ and let $\mathbf{b} = (j', \mathbf{a}_{-i})$. Note that $\mathbf{b}$ is ordered. Moreover since j' is a left improvement for i by part 1 we have that $\sigma_{j'}(\mathbf{b}) = k + 1$ and $\sigma_j(\mathbf{b}) = k$. Suppose in negation that there exists a right improvement q' for some player i' in $\mathbf{b}$. Let $q = b_i$. Hence $q < q'$.

It must be that $q = j'$ or $q' = j$ since no other resource has changed in the action profile $\mathbf{a}$ after i's deviation to j'. We distinguish between the following two cases:

1. $q = j'$: Suppose that $i' \in g(i)$. Therefore $q' \neq j$ otherwise j' wouldn't be a left improvement for i in $\mathbf{a}$. Assume $q' > j$. Since q' is a right improvement for i', by part 1 $v(i', \mathbf{b}, q') > 0$. Therefore since $\mathbf{b}$ is ordered $v(i', \mathbf{b}, j) = k$, which implies that $v(i', \mathbf{a}, j) = k$. By part 1 this is a contradiction to i having a left improvement in $\mathbf{a}$. Assume that $q' < j$. Since q' is a right improvement for i' in $\mathbf{b}$ by part 1 $\sigma_{q'}(\mathbf{b}) = k$. Note that $\sigma_{q'}(\mathbf{b}) = \sigma_{q'}(\mathbf{a})$. Since $\mathbf{a}$ is ordered and $q' > j'$ $v(i, \mathbf{a}, q') = k$. Therefore q' is a left improvement for i in $\mathbf{a}$, contradicting that j' is minimal.
   Suppose that $i' \notin g(i)$. Since $\mathbf{b}$ is ordered we have that $v(i', \mathbf{b}, q') = 0$, contradicting part 1.
2. $q' = j$: Suppose that $i' \in g(i)$. Assume that $q > j'$. Since $\mathbf{a}$ is ordered and $j' < q < j$, $v(i', \mathbf{a}, q) = k$. But $v(i', \mathbf{a}, q) = v(i', \mathbf{b}, q)$. By part 1 this is a contradiction to i' having a right improvement in $\mathbf{b}$. Assume that $q < j'$. We have that $v(i', \mathbf{a}, q) = v(i', \mathbf{b}, q)$. Note that

$$v(i', \mathbf{a}, q) = v(i', \mathbf{b}, q) < v(i', \mathbf{b}, j) = v(i, \mathbf{a}, j) < v(i, \mathbf{a}, j'), \quad (1)$$

where the equalities follow since $q < j' < j$, the first inequality follows from part 1 since j is a right improvement for i' in $\mathbf{b}$ and the last inequality follows since j' is left inequality for i in $\mathbf{a}$.
Moreover, since j is a right improvement for i' in $\mathbf{b}$, $\sigma_q(\mathbf{b}) = k+1$. Therefore, since $\sigma_q(\mathbf{a}) = k+1$, together with (1) it follows that j' is a right improvement for i' in $\mathbf{a}$ - a contradiction.
Suppose $i' \notin g(i)$. Since $\mathbf{b}$ is ordered $v(i', \mathbf{b}, j) = 0$, contradicting part 1.

□

Proof of Theorem 1: Our proof is by construction. Let $k = \lfloor \frac{n}{m} \rfloor$. Note that there exist an ordered Nash equilibrium in H in which on every resource there are either k or $k+1$ players, and let $\mathbf{a}$ be such an equilibrium. Since the resources are identical every permutation of the players in $\mathbf{a}$ is an equilibrium in H. Rename the resources such that if $j < j'$ then $\sigma_j(\mathbf{a}) \leq \sigma_{j'}(\mathbf{a})$.

If $\sigma_j(\mathbf{a}) = k$ for every j then by part 1 of Lemma 1 $\mathbf{a}$ is an equilibrium in S and we are done. Observe by part 1 of Lemma 1 that there is no right improvement in $\mathbf{a}$. If there is no left improvement in $\mathbf{a}$ then $\mathbf{a}$ is an equilibrium in S. Suppose there exist a player i that has left improvement in $\mathbf{a}$. Let j be a minimal left improvement for i in $\mathbf{a}$ and let $\mathbf{a}^1 = (j, \mathbf{a}_{-i})$. By Lemma 1 part 2 there is no right improvement in $\mathbf{a}^1$. If there is no left improvement in $\mathbf{a}^1$ then $\mathbf{a}^1$ is an equilibrium in S. Assume otherwise. Note that $\mathbf{a}^1$ is ordered. Construct $\mathbf{a}^2$ from $\mathbf{a}^1$ in a similar fashion as $\mathbf{a}^1$ was constructed from $\mathbf{a}$. After each iteration s there is no right improvement in $\mathbf{a}^s$ and $\mathbf{a}^s$ is ordered. Therefore since the number of players and the number of resources are finite this process is finite and we will eventually end with an equilibrium in S. □.

In the next example we show that Theorem 1 is not true when H does not have identical resources.

Example 1. Suppose there is a single clique with 5 players. Let the underlying resource selection game have 3 resources with the following resource cost functions: $w_1 \equiv (0, 8, 10, 100, 100)$, $w_2 \equiv (7, 100, 100, 100, 100)$ and $w_3 \equiv (0, 10, 100, 100, 100)$. An action profile in which some resource is not chosen is not an equilibrium since any player on a resource with 3 or more players is better of by choosing the empty resource. In addition no action profile in which resource 2 is chosen by at least two players or resource 3 is chosen by at least three players is not an equilibrium. Let $\mathbf{a}$ be such that $\sigma_1(\mathbf{a}) = 2$, $\sigma_2(\mathbf{a}) = 1$ and $\sigma_1(\mathbf{a}) = 2$. Then a player i that chooses resource 3 is better off choosing 1. Finally let $\mathbf{b}$ be an action profile in which $\sigma_1(\mathbf{b}) = 3$, $\sigma_2(\mathbf{a}) = 1$ and $\sigma_1(\mathbf{a}) = 1$. In this case a player that chooses resource 2 is better off choosing resource 3.

□

4 The Best-Member Collaboration Game

In this section we study social context games with the best-member collaboration aggregation functions. We call such a game a best-member collaboration game.

Recall that in such games every player wishes to minimize the minimal immediate cost in its group. Before we state our main result we need the following definition:

Definition 3. *Let $G = (V, E)$ be an undirected graph. A subset of nodes $Q \subseteq V$ is called a* dominating independent set *if every node $v \in V \setminus Q$ has an edge connecting to a node in Q and no two nodes in Q are connected by a single edge. The cardinality of the* minimum dominating independent set *is denoted by $i(G)$; i.e. the minimum dominating independent set in G is of size $i(G)$ if there exists a dominating independent set $Q, |Q| = i(G)$ and every other dominating independent set T satisfies $|T| \geq |Q|$.*

Theorem 2. *Let $S = (H, F)$ be a best-member collaboration game in which the underlying game H is a resource selection game. If $i(G) < \frac{n}{2}$ then there exists a Nash equilibrium in S.*

Proof. Let $T \subseteq N$ be a dominating independent set in G such that $|T| = i(G)$. Let $z = |N \setminus T|$. By our assumption $z > \frac{n}{2}$. Let $j' \in \arg\max_j w_j(z-1)$. Let $H(j', T)$ be the resource selection game with all resources in H excluding resource j' and the set of players is T. Let $\mathbf{b}$ be an equilibrium in $H(j', T)$. Let $\mathbf{a}$ be the action profile in which all players in T choose the same resource as they choose in $\mathbf{b}$ and for every $i \in N \setminus S$, $a_i = j'$. We claim that $\mathbf{a}$ is an equilibrium in S. Let i be a player such that $a_i = j'$. Since $z > \frac{n}{2}$, for every resource $j \neq j'$, $\sigma_j(\mathbf{a}) < z$. Note that since T is a dominating independent set there exist a player $i' \in g(i)$ such that $a_{i'} \neq j'$. Since $\mathbf{a}_T$ is an equilibrium in $H(j', T)$ then $c_i((j, \mathbf{a}_{-i})) > c_{i'}(\mathbf{a})$ for every resource $j \neq j'$. In addition by the definition of j', $w_{j'}(\sigma_{j'}((j, \mathbf{a}_{-i})) > c_{i'}(\mathbf{a})$. Therefore i is not better off by deviating. Suppose $a_i = j$ where $j \neq j'$. Since i has no neighbor in T and $\mathbf{a}_T$ is an equilibrium in $H(j', T)$ deviating to a resource $j'' \neq j'$ is not better off for i. Deviating to j' is also not better off for i by the definition of j' and since $z > \sigma_j(\mathbf{a})$. □.

We next provide an example for a social context game for which $i(G) = \frac{n}{2}$, which does not posses an equilibrium in the best-member collaboration game. This example implies that our theorem is tight.

Example 2. Consider the following best-member collaboration game: the set of players is $N = \{1, 2, 3, 4\}$ and in the underlying game H there are two identical resources with strictly increasing cost functions. In the social context the graph G has two cliques: players 1,2 and 3 form one clique, and player 4 is a singleton. Clearly, $i(G) = 2$. W.l.o.g. let player 4 choose resource 2. If two or more of the players from the first clique choose resource 1 then any one of these players will benefit from deviating to resource 2. In any other case player 4 is better off deviating to resource 1.

□

The following example shows that even if G is connected, the condition in Theorem 2 is still necessary for the existence of Nash equilibrium.

Example 3. Consider the following social context game. The set of players is $\{1, 2, 3, 4, 5, 6\}$, H has two identical resources with strictly increasing cost functions, and G has a 3-clique on players $1, 2, 3$, and each of these players $i \in \{1, 2, 3\}$ has an additional edge to player $i + 3$; note that the degree of each vertex $i \in \{1, 2, 3\}$ is 3 and the degree of each vertex $4, 5, 6$ is 1. Clearly, $i(G) = 3$.

We first observe that it is not possible that all three players $1, 2, 3$ choose the same resource. If they all choose resource 1, then no player among $4, 5, 6$ can also go to resource 1. Thus $4, 5, 6$ choose resource 2. But then one of $1, 2, 3$, say 3 has an incentive to change to resource 2.

Thus, we can assume w.l.o.g., that players $1, 2$ choose resource 1, and player 3 is on resource 2. But then players connected to players $1, 2$, namely 4 and 5 will necessarily go to resource 2, and player 6 connected to player 3 has to choose resource 1. Let us call this configuration (and all such symmetric configurations) a special configuration.

Having a special configuration as above, player 1 or 2, say 2, can improve his cost by changing to resource 2. But having now players 2 and 5 together on resource 2, player 5 will want to change to resource 1. This new configuration is again special.

□

5 The MinMax Collaboration Game

In this section we study social context games in which the aggregation function is the minmax collaboration. We call such game a minmax collaboration game. Hence, every player wishes to minimize the maximal immediate cost in its group. Let S be a minmax collaboration game where the underlying game H is a resource selection game. It can be observed that a necessary condition for a deviation by a single player i to be beneficial is that player i has the maximal immediate cost in $g(i)$ before the deviation and strictly reduces its own immediate cost after deviating. Therefore, we get:

Theorem 3. *Let $S = (H, F)$ be a minmax collaboration game in which H is a resource selection game. Then, $NE(H) \subseteq NE(S)$.*

However, there may exist $\mathbf{a} \in NE(S)$ such that $\mathbf{a} \notin NE(H)$. To see this consider three identical resources with strictly increasing cost functions, and a graph which consists of two cliques both of size 4. Let $\mathbf{a}$ be the action profile in which two players from each clique are on resource 1 and all other players are on resource 2. Let i be a player on resource 1. Note that i has the maximal cost in $g(i)$. Clearly, deviating to resource 2 will increase i's immediate cost. In addition, note that deviating to resource 3 is not beneficial to i since there are still two players in $g(i)$ on resource 2. Similar arguments hold for a player i on resource 2.

6 The Surplus Collaboration Game

In this section we study social context games in which the aggregation function is the surplus collaboration. We call such games surplus collaboration games. In such games every player i wishes to minimize the average immediate costs in $g(i)$. We begin with a couple of negative results.

Proposition 1. *There exist a resource selection game, and a surplus collaboration social context, such that the corresponding social context game does not possess a Nash equilibrium.*

Proof. We will define a social context game with 4 players $\{1, 2, 3, 4\}$ and 2 identical resources $\{1, 2\}$. Each resource has the same cost function $(1, 5, 6, 6)$. The graph G has a 3-clique on the vertices $1, 2, 3$ and vertex 4 is an isolated vertex (singleton).

We will show now that any assignment of the four players to the two resources does not define a Nash equilibrium of this game. Suppose, w.l.o.g., that player 4 is assigned to resource 2 (the other case is symmetric). Now we consider the following cases:

- Players $1, 2, 3$ are on resource 2 together with player 4. This obviously is not a Nash equilibrium.
- One of the players $1, 2, 3$, say 1, is on resource 1, and players $2, 3$ are on resource 2. Then, the cost of player 4 is 6, and if 4 changes to resource 1, then its cost drops down to 5. So this case is also not a Nash equilibrium.
- Two among the players $1, 2, 3$, say $1, 2$, are on resource 1, and players 3 and 4 are on resource 2. Then, the cost of player 2 is 5, but if 2 changes to resource 2 then its cost would be $\frac{1+6+6}{3} < 5$.
- All the players $1, 2, 3$ are on resource 1, and player 4 is on resource 2. Then the cost of any of the players $1, 2, 3$ is 6. But if any of them changes to resource 2 its cost would drop to $5 < 6$. Thus, again the starting configuration is not a Nash equilibrium.

Observe that each of the new configurations after a player changes its strategy leads again to one of the above cases. By symmetry this implies that there always is a cost improving move. □

The example given in the proof of Proposition 1 is given for identical resources, but uses a disconnected graph. The following example shows that there may not be a NE even when the graph is connected and in particular is a tree. The proof that the corresponding social context game does not possess a pure strategy equilibrium is left to the full paper.

Example 4. Let SCG be a social context game with the following structure. Let G be an undirected tree with one root and 6 children, and let H have 2 identical resources with the cost function $(1, 1, 2.9, 5, 5, 5, 5)$. W.l.o.g. let the root be on resource 1. Let $(k, 7-k)$ denote the partition of the players on the resource where k is the number of players in resource 1 (including the root). For every

$k = 4, 5, 6, 7$ any child on resource 1 will benefit from deviating to resource 2 since he will get at most $\frac{2.9+5}{2}$ which is lower than 5. If $k = 3$ the root benefit from deviating since $5 * 5 + 2 * 1 < 2.9 * 3 + 4 * 5$. If $k = 2$ then a child from resource 2 will benefit by deviating to resource 1 since $2 * 2.9 < 5 + 1$ and finally for $k = 1$ a child will benefit by joining the root on resource 1 since $2 * 1 < 5 + 1$.

□

Notice that while work on coalitional congestion games has shown the non-existence of pure strategy equilibria (when a coalition may be of size greater than two), in our study this is shown for the case where only unilateral deviations (and not deviations by a whole coalition) are considered. In the next theorem we provide a family of resource selection games and social contexts, which posses a Nash equilibrium in the surplus collaboration social context.

Theorem 4. *Let H be a resource selection game with m identical resources and let G be a tree with maximal degree $m-2$. Then, there exists a Nash equilibrium in the corresponding surplus collaboration social context game SCG.*

The proof of the above result appears in the full paper.

7 (The lack of) Potential Functions

It is well known that every potential game is a congestion game and vice-versa. Although the underlying game in each of the social context games we have studied is a potential game we show in this section that none of the games in our positive results is a potential game. The following characterization lemma will be the main tool in our proof:

Lemma 2 (MondererShapley96). *Let $H = (N, (A_i)_{i \in N}, (c_i)_{i \in N})$ be a game in strategic form. H is a potential game if and only if for every pair of players $i, j \in N$, for every $\mathbf{a} \in \mathbf{A}_{N \setminus \{i,j\}}$ and every $a_i, b_i \in A_i$ and $a_j, b_j \in A_j$*

$$(c_i(b_i, a_j, \mathbf{a}) - c_i(a_i, a_j, \mathbf{a})) + (c_j(b_i, b_j, \mathbf{a}) - c_j(b_i, a_j, \mathbf{a})) +$$
$$(c_i(a_i, b_j, \mathbf{a}) - c_i(b_i, b_j, \mathbf{a})) + (c_j(a_i, a_j, \mathbf{a}) - c_j(a_i, b_j, \mathbf{a})) = 0.$$

Theorem 5. *Resource selection games do not possess a potential function in the competitive ranking, best member collaboration, and minmax collaboration social contexts.*

Proof. 1. Let $S = (H, F)$ be the following competitive ranking game. The set of players is $\{1, 2, 3\}$. H is resource selection selection game with 2 identical resources with resource cost functions $w(x) = x$. The graph G is partitioned into two cliques; players 1 and 2 form a clique and player 3 is a singleton. The following cycle of action profiles in which only players 2 and 3 change their actions will provides that there is no potential by Lemma 2: $(1, 1, 2)$,$(1, 2, 2)$, $(1, 2, 1)$ and $(1, 1, 1)$. Note that $f_2(G, (1, 1, 2)) - f_2(G, (1, 2, 2)) + f_3(G, (1, 2, 2)) - f_3(G, (1, 2, 1)) + f_2(G, (1, 2, 1)) - f_2(G, (1, 1, 1)) + f_3(G, (1, 1, 1)) - f_3(G, (1, 1, 2)) = 1/2 - 0 + 0 - 0 + 1 - 1/2 + 0 - 0 = 1 \neq 0$.

2. Let $S = (H, F)$ be the following best-member collaboration game. The set of players is $\{1, 2, \dots, 10\}$. H is resource selection selection game with three identical resources with resource cost functions $w(x) = x$. The graph G is partitioned into two cliques; players 1 and 2 form a clique, player 3 is a singleton and players 4-10 form a clique. Note that $i(G) = 3 < \frac{10}{2}$. Let $a \in A_{N\setminus\{1,2,3\}}$ be the action profile in which all players 4-10 choose resource 3. The following cycle of action profiles in which only players 2 and 3 change their actions will provides that there is no potential by Lemma 2: $(1, 1, 2, a)$,$(1, 2, 2, a)$, $(1, 2, 1, a)$ and $(1, 1, 1, a)$. Note that $f_2(G, (1, 1, 2, a)) - f_2(G, (1, 2, 2, a)) + f_3(G, (1, 2, 2, a)) - f_3(G, (1, 2, 1, a)) + f_2(G, (1, 2, 1, a)) - f_2(G, (1, 1, 1, a)) + f_3(G, (1, 1, 1, a)) - f_3(G, (1, 1, 2, a)) = 2 - 1 + 2 - 2 + 1 - 3 + 3 - 1 = 1 \neq 0$.
3. Let $S = (H, F)$ be the following minmax collaboration game. The structure is as in part 1, but the resources are not identical. The resource cost functions are: $w_1(x) = 5$ and $w_2(x) = x$ for every $x = 1, 2, 3$. The following cycle of action profiles in which only players 2 and 3 change their actions provides that there is no potential by Lemma 2: $(1, 1, 2)$,$(1, 2, 2)$, $(1, 2, 1)$ and $(1, 1, 1)$. Note that $f_2(G, (1, 1, 2)) - f_2(G, (1, 2, 2)) + f_3(G, (1, 2, 2)) - f_3(G, (1, 2, 1)) + f_2(G, (1, 2, 1)) - f_2(G, (1, 1, 1)) + f_3(G, (1, 1, 1)) - f_3(G, (1, 1, 2)) = 5 - 5 + 2 - 5 + 5 - 5 + 5 - 1 = 1 \neq 0$.

□

8 Summary and Future Work

In this work we introduced social context games. Social context games make explicit the separation between the underlying game, where agents' costs are measured by some objective terms, and the social context that is used in order to interpret these costs as actual utilities. Technically, a social context is defined using a neighborhood graph and aggregation functions, which capture the agents' social attitudes. We consider resource selection games, a fundamental class of games in the interplay between computer science and game theory; an important property of these games is the existence of pure strategy equilibrium; this property is implied by the existence of a potential function. We consider four basis social contexts, and study the existence of pure strategy equilibrium in the related social context games. Our results illustrate the power of the social context games framework; social context games allow for the understanding of how social contexts effect game-theoretic properties. In particular, we extend upon previous work on ranking games and coalitional congestion games, and prove the existence of pure Nash equilibrium in several rich social contexts of resource selection games.

By introducing social context games, we offer the community a rich set of problems to look at. In fact, given any game of study, and any property of this game, we suggest to consider the effects of social contexts (as defined in our work) on the existence of that property in the corresponding social context games. Some immediate goals are the study of existence of pure strategy equilibrium in other forms of congestions game, as well as in e.g. position auctions, when various social contexts are taken into account. Following this, the study of the effects of various social contexts on the surplus obtained in the related equilibria is also of significant importance.

References

1. Alon Altman and Moshe Tennenholtz. Quantifying incentive compatibility of ranking systems. In *Proc. of AAAI-06*, 2006.
2. F. Brandt, F. Fischer, and Y. Shoham. On strictly competitive multi-player games. In *AAAI-06Proceedings of the 27th IEEE Conf. Decision and Control*, pages 605–612, 2006.
3. Dimitris Fotakis, Spyros C. Kontogiannis, and Paul G. Spirakis. Atomic congestion games among coalitions. In *ICALP-06*, pages 572–583, 2006.
4. Ara Hayrapetyan, Eva Tardos, and Tom Wexler. The effect of collusion in congestion games. In *STOC-06*, pages 89–98, 2006.
5. Sergey Kuniavsky and Rann Smorodinsky. Coalitional Congestion Games. Technical report, Technion, Israel, 2007.
6. D. Monderer and L.S. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.
7. M. Penn, M. Polukarov, and M. Tennenholtz. Asychronous congestion games. In *AAMAS-08*, 2008.
8. R.W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.