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Coefficient Bounds for Certain Subclasses of Bi-univalent Functions

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Abstract. In this paper, we introduce and investigate two new subclasses of the function class Σ of bi-univalent functions. Also, we find estimates of $|a_2|$ and $|a_3|$. Some related consequences of the results are also pointed out.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha;\ z \in U;\ \ 0 \leq \alpha < 1 \right\} \tag{1.1.2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha;\ z \in U;\ \ 0 \leq \alpha < 1 \right\}. \tag{1.1.3}$$

It readily follows from the definitions (1.1.2) and (1.1.3) that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z,\ z \in \mathbb{U}$ and $f(f^{-1}(w)) = w,\ |w| < r_0(f);\ r_0(f) \geq \frac{1}{4}$, where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots . \tag{1.1.4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1.1). Examples of functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are also not members of Σ (see [5, 12]).

In 1967, Lewin [7] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [10], on the other hand, showed that $\max\limits_{f\in\Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} := \{1, 2, 3, \dots\}$ is presumably still an open problem.

A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_\Sigma^\alpha$ of strongly bi-starlike of order α $(0 < \alpha \leq 1)$, if each of the following condition is satisfied:

$$f \in \Sigma,\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \text{ and } \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2},\ z, w \in \mathbb{U};\ 0 < \alpha \leq 1,$$

where the function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{1.1.5}$$

the extension of f^{-1} to $\mathbb{U}$.

The classes $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ defined by (1.1.2) and (1.1.3), were also introduced analogously. For each of the function classes $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, Brannan and Taha [4] found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details see [4, 14]). Following Brannan and Taha [4], Srivastava et al. [12]

introduced certain subclass $\mathcal{H}_{\Sigma}^{\alpha}$, $0 < \alpha \leq 1$ of the bi-univalent functions class Σ, a function $f(z)$ given by (1.1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{\alpha}$, $0 < \alpha \leq 1$, if the following conditions are satified:

$$f \in \Sigma,\ |\arg(f'(z))| < \frac{\alpha\pi}{2}, \text{ and } |\arg(g'(w))| < \frac{\alpha\pi}{2},\ z, w \in \mathbb{U};\ 0 < \alpha \leq 1,$$

where the function g is given

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \tag{1.1.6}$$

Then later many researchers (see [1, 6, 15, 16]) studied extensively the same class $\mathcal{H}_{\Sigma}^{\alpha}$, by different techniques and found the non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. It is interest to note that the estimates were found are improved but not sharp. Further, Frasin and Aouf [5] extended the class $\mathcal{H}_{\Sigma}^{\alpha}$, and obtained the non-sharp bounds (see also [9, 13]).

Motivated by the aforementioned works, we introduce the following subclasses of the function class Σ.

Definition 1.1. A function $f(z)$ given by (1.1.1) is said to be in the class $\mathcal{S}_{\Sigma}(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \qquad \left|\arg\left(\frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)}\right)\right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leqq 1;\ 0 \leqq \lambda \leqq 1;\ z \in \mathbb{U}) \tag{1.1.7}$$

and

$$\left|\arg\left(\frac{wg'(w) + (2\lambda^2 - \lambda)w^2 g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)}\right)\right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leqq 1;\ 0 \leqq \lambda \leqq 1;\ w \in \mathbb{U}), \tag{1.1.8}$$

where the function g is given by 1.1.6.

We note that for $\lambda = \frac{1}{2}$, the class $\mathcal{S}_{\Sigma}(\alpha, \lambda)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}$ introduced and studied by Srivastava et al. [12]. Putting $\lambda = 0$, the class $\mathcal{S}_{\Sigma}(\alpha, \lambda)$ reduces to the class of strongly bi-starlike functions of order $\alpha(0 < \alpha \leqq 1)$ and denoted by $\mathcal{S}_{\Sigma}^{*}(\alpha)$.

Definition 1.2. A function $f(z)$ given by (1.1.1) is said to be in the class $\mathcal{M}_{\Sigma}(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \qquad \Re\left(\frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)}\right) > \beta$$

$$(0 \leqq \beta < 1;\ 0 \leqq \lambda \leqq 1;\ z \in \mathbb{U}) \tag{1.1.9}$$

and

$$\Re\left(\frac{wg'(w)+(2\lambda^2-\lambda)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)}\right)>\beta$$

$$(0\leqq\beta<1;\ 0\leqq\lambda\leqq 1;\ w\in\mathbb{U}), \tag{1.1.10}$$

where the function g is given by (1.1.6).

It is interesting to note that, for $\lambda=\frac{1}{2}$ the class $\mathcal{M}_\Sigma(\beta,\lambda)$ reduces to the class $\mathcal{H}_\Sigma^\beta$ introduced and studied by Srivastava et al. [12]. Putting $\lambda=0$, the class $\mathcal{M}_\Sigma(\beta,\lambda)$ reduces to the class of bi-starlike functions of order $\beta(0<\beta\leqq 1)$ and denoted by $\mathcal{S}_\Sigma(\beta)$. When $\lambda=1$, the class $\mathcal{K}_\Sigma(\beta,\lambda)$ reduces to the class of bi-convex functions of order $\beta(0<\beta\leqq 1)$ and denoted by $\mathcal{K}_\Sigma(\beta)$.

The object of the present paper is to find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{S}_\Sigma(\alpha,\lambda)$ and $\mathcal{M}_\Sigma(\alpha,\lambda)$ of the function class Σ.

In order to derive our main results, we shall need the following lemma.

Lemma 1.3. ([11]) *If $h\in\mathcal{P}$, then $|c_k|\leqq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\}>0 \qquad (z\in\mathbb{U}),$$

where

$$h(z)=1+c_1z+c_2z^2+\cdots \qquad (z\in\mathbb{U}).$$

2. Coefficient Bounds for the Function Class $\mathcal{S}_\Sigma(\alpha,\lambda)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{S}_\Sigma(\alpha,\lambda)$.

Theorem 2.1. *Let the function $f(z)$ given by* (1.1.1) *be in the following class*:

$$\mathcal{S}_\Sigma(\alpha,\lambda) \qquad (0<\alpha\leqq 1;\ \ 0\leqq\lambda\leqq 1).$$

Then

$$|a_2|\leqq\frac{2\alpha}{\sqrt{\alpha(1-2\lambda+25\lambda^2-44\lambda^3+20\lambda^4)+(1+3\lambda-2\lambda^2)^2}} \tag{2.2.1}$$

and

$$|a_3|\leqq\frac{\alpha}{1+2\lambda^2}+\frac{4\alpha^2}{(1+3\lambda-2\lambda^2)^2}. \tag{2.2.2}$$

Proof. It follows from (1.1.7) and (1.1.8) that

$$\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}=[p(z)]^\alpha \tag{2.2.3}$$

and

$$\frac{wg'(w)+(2\lambda^2-\lambda)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)}=[q(w)]^\alpha, \tag{2.2.4}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{2.2.5}$$

and

$$q(z) = 1 + q_1 w + q_2 w^2 + \cdots , \tag{2.2.6}$$

respectively. Now, equating the coefficients in (2.2.3) and (2.2.4), we get

$$(1 + 3\lambda - 2\lambda^2)a_2 = \alpha p_1, \tag{2.2.7}$$

(2.2.8)

$$(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1)a_2^2 + (4\lambda^2 + 2)a_3 = \frac{1}{2}\left[\alpha(\alpha-1)p_1^2 + 2\alpha p_2\right],$$

$$-(1 + 3\lambda - 2\lambda^2)a_2 = \alpha q_1 \tag{2.2.9}$$

and

(2.2.10)

$$(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3)a_2^2 - (4\lambda^2 + 2)a_3 = \frac{1}{2}\left[\alpha(\alpha-1)q_1^2 + 2\alpha q_2\right].$$

From (2.2.7) and (2.2.9), we get

$$p_1 = -q_1 \tag{2.2.11}$$

and

$$2(1 + 3\lambda - 2\lambda^2)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.2.12}$$

From (2.2.8), (2.2.10) and (2.2.12), we obtain

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{\alpha(1 - 2\lambda + 25\lambda^2 - 44\lambda^3 + 20\lambda^4) + (1 + 3\lambda - 2\lambda^2)^2}.$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha(1 - 2\lambda + 25\lambda^2 - 44\lambda^3 + 20\lambda^4) + (1 + 3\lambda - 2\lambda^2)^2}}.$$

This gives the bound on $|a_2|$ as asserted in (2.2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.2.10) from (2.2.8), we get

$$2(2 + 4\lambda^2)a_3 - (8\lambda^2 + 4)a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2). \tag{2.2.13}$$

It follows from (2.2.11), (2.2.12) and (2.2.13) that

$$a_3 = \frac{\alpha(p_2 - q_2)}{2(2 + 4\lambda^2)} + \frac{\alpha^2(p_1^2 + q_1^2)(3\lambda^2 + 1)}{2(2\lambda^2 + 1)(1 + 3\lambda - 2\lambda^2)^2}. \tag{2.2.14}$$

Applying Lemma 1.3 once again, we readily get

$$|a_3| \le \frac{\alpha}{1 + 2\lambda^2} + \frac{4\alpha^2}{(1 + 3\lambda - 2\lambda^2)^2}.$$

This completes the proof of Theorem 2.1. □

In the following section we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{M}_\Sigma(\beta,\lambda)$.

3. Coefficient bounds for the function class $\mathcal{M}_\Sigma(\beta,\lambda)$

Theorem 3.1. *Let $f(z)$ given by (1.1.1) be in the class $\mathcal{M}_\Sigma(\beta,\lambda)$, $0\le\beta<1$ and $0\le\lambda<1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}} \tag{3.3.1}$$

and

$$|a_3| \le \frac{2(1-\beta)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}. \tag{3.3.2}$$

Proof. It follows from (1.1.9) and (1.1.10) that there exists $p,q\in\mathcal{P}$ such that

$$\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} = \beta+(1-\beta)p(z) \tag{3.3.3}$$

and

$$\frac{wg'(w)+(2\lambda^2-\lambda)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)} = \beta+(1-\beta)q(w), \tag{3.3.4}$$

where $p(z)$ and $q(w)$ have the forms (2.2.5) and (2.2.6), respectively. Equating coefficients in (3.3.3) and (3.3.4), we get

$$(1+3\lambda-2\lambda^2)a_2 = (1-\beta)p_1 \tag{3.3.5}$$

$$(12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1)a_2^2+(2+4\lambda^2)a_3 = (1-\beta)p_2 \tag{3.3.6}$$

$$-(1+3\lambda-2\lambda^2)a_2 = (1-\beta)q_1 \tag{3.3.7}$$

and

$$(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3)a_2^2-(2+4\lambda^2)a_3 = (1-\beta)q_2. \tag{3.3.8}$$

From (3.3.5) and (3.3.7), we get

$$p_1 = -q_1 \tag{3.3.9}$$

and

$$2(1+3\lambda-2\lambda^2)^2a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{3.3.10}$$

Also, from (3.3.6), (3.3.8) and (3.3.10), we obtain

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{2(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)}.$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}}.$$

This gives the bound on $|a_2|$ as asserted in (3.3.1).

Next, in order to find the bound on $|a_3|$, by subtracting (3.3.8) from (3.3.6), we get

$$4(1+2\lambda^2)a_3 - 4(1+2\lambda^2)a_2^2 = (1-\beta)(p_2 - q_2). \tag{3.3.11}$$

It follows from (3.3.9), (3.3.10) and (3.3.11) that

$$4(1+2\lambda^2)a_3 = \frac{4(1+2\lambda^2)(1-\beta)(p_2+q_2)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1} + (1-\beta)(p_2 - q_2). \tag{3.3.12}$$

Applying Lemma 1.3 once again, we readily get

$$|a_3| \leq \frac{2(1-\beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}.$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Taking $\lambda = 0$ in Theorem 2.1 and 3.1, the estimates on the coefficients $|a_2|$ and $|a_3|$ are improvement of the estimates on the first two Taylor-Maclaurin coefficients obtained in [8]. Also, for the choice of $\lambda = \frac{1}{2}$, the results stated in Theorem 2.1 and Theorem 3.1 would improve bounds stated in [12].

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