Inefficient Equilibria of Second-Price/English Auctions with Resale

Rod Garratt[*], Thomas Tröger[†], and Charles Zheng[‡]

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Abstract

In second-price or English auctions involving symmetric, independent, private-value bidders the equilibrium outcome may not be efficient if resale is allowed. In addition to the efficient, symmetric equilibrium there exist inefficient, asymmetric equilibria in which the bidder who wins the item offers it for resale to the losers. For the case where the reserve price in the initial auction equals zero, we show that any surplus between the first best and that of a market where one bidder is a monopolist seller is supported by an equilibrium in undominated strategies. In addition, we show that an efficiency-minded seller may find it optimal to set a positive reserve price.

KEYWORDS: second-price auction, English auction, inefficiency, resale, asymmetric equilibria

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1 Introduction

Efficiency is often stated to be an important objective when allocating goods. However, this objective is sometimes ignored on the grounds that any inefficiencies in the initial allocation will be corrected in the resale market. The

[*]Department of Economics, University of California, Santa Barbara, CA 93106-9210, U.S.A. Email: garratt@econ.ucsb.edu.

[†]Department of Economics, University of Bonn, Adenauerallee 24-26, 53113 Bonn, Germany

[‡]Department of Economics, Northwestern University, U.S.A.

allocation of radio spectrum rights, for example, now perceived as one of the biggest successes of market design, used to be allocated randomly, by using a lottery. However, such a system can create a rather inefficient "bidder monopoly" where the happy winner of the lottery sells the good at a monopoly price in the resale market.[1] Moreover, it can take considerable time to sort out allocative inefficiencies.[2]

Second-price and English auctions with zero reserve price have been suggested as a remedy. Indeed, over the last decade the FCC has used an English-auction type mechanism, the simultaneous ascending auction, to allocate spectrum worth billions of dollars. It is well known that second-price and English auctions with zero reserve price allocate a good efficiently in any market with private values if resale is not possible. We show that the efficiency property in fact depends on the absence of a resale market. With a resale opportunity, inefficiencies cannot be excluded. While the efficient, "value-bidding" equilibrium still exists in a second-price auction market where resale is permitted (See Haile 1999, Theorem 1), inefficient, asymmetric equilibria also emerge.

In the equilibria we construct, one bidder bids aggressively and drives out other bidders. She wins the item at a low price and then offers it for resale. The other bidders are better off waiting for the resale market than topping the aggressive bid. Our initial results are for 2-bidder auctions, although later in the paper we extend the our equilibrium construction to auctions with more than 2 bidders.[3] We show that in any two-bidder second-price auction (which is strategically equivalent to a two-bidder English auction), the surplus may be reduced to the level of a "bidder monopoly" where the initial seller first allocates the good to an arbitrary bidder who then acts as a monopolist in the resale market. I.e., a second-price auction may not be a better way to allocate a good than randomly assigning the good to an arbitrary bidder. More generally, we prove that any surplus in between the first best and that of a bidder monopoly is supported by an equilibrium in undominated strategies.

We provide a second result that goes against conventional wisdom: an

[1]See McMillan (1994).

[2]See the discussion on the aftermath of the cellular lotteries in Crampton (1998).

[3]Our extension applies to the equilibrium where the aggressive bidder wins the initial auction with probability 1. We are currently working on an alternative equilibrium construction that establishes the existence of a continuum of inefficient equilibrium for n SIPV bidders.

initial seller who is efficiency-minded may want to set a positive reserve price. This result is an immediate corollary of our equilibrium construction. By setting a positive reserve price, the seller may be able to eliminate the asymmetric equilibria and thus may be able to avoid an active resale market with an inefficient outcome. The inefficiency caused by the reserve price may be smaller than the inefficiency caused by aggressive bidding.

Our results build on earlier results by Zheng (2000, Section 5.2) and Garratt and Tröger (2003, 2006). Zheng constructs an equilibrium in a second-price-auction-type mechanism that produces a 'bidder monopoly." Garratt and Tröger (2003, 2006) construct a continuum of equilibria similar to those constructed here, that apply to second-price auctions where the aggressive bidder is commonly known to have zero value for the good on sale, i.e., this bidder acts as a pure speculator. Apart from the speculator, an arbitrary number of "regular" bidders with symmetric independent private values participate in the market.

Previous studies of second-price auctions with resale have mostly focussed on the impact of a resale opportunity on seller revenue. Haile (1999, 2000, 2003) analyzes resale in symmetric environments where at the time of the initial auction each bidder is uncertain about her own use value.[4] The resale opportunity increases the initial seller's revenue if the resale seller has sufficient bargaining power in the resale market.

In Section 2 we outline the model and establish the equilibrium conditions. In Section 2 we state the main equilibrium results for 2-bidder auctions. Section 4 contains some examples that illustrate the equilibrium construction for the case where values are uniformly distributed. Section 5 describes a single asymmetric equilibrium that exists in the case of $n > 2$ bidders. Concluding remarks are contained in Section 6. A statement and proof of a simplified version of the main proposition with no discounting is in the Appendix.

2 Model

We consider two risk-neutral bidders who are interested in consuming a single indivisible item. The item is initially owned by a seller who has no (use) value

[4] In Haile's terminology, the term "use value" is used in the same way as our term "value." His term "valuation" refers to the opportunity cost of not winning the initial auction.

for it. Bidder $i = 1, 2$ has the random (use) value $\tilde{\theta}_i \in [0, \bar{\theta}_i]$ for the item. Let F_i denote the distribution function for $\tilde{\theta}_i$. We assume that the value distribution of at least one bidder, say bidder 1, has a positive continuous density and a weakly increasing hazard rate.

We consider a two-period interaction. Before period 1, bidder i privately learns the realization of her value, $\tilde{\theta}_i = \theta_i$. In period 1, the item is offered via a sealed-bid second-price auction with reserve price $r \geq 0$. Each bidder submits a nonnegative bid. The highest bidder wins the item at a price equal to the maximum of the loser's bid and r, unless both bidders submit bids below r. Our equilibria are valid with any tie breaking rule. The bidder who wins in period 1 either consumes the item or waits until period 2 and makes a take-it-or-leave-it offer to the period-one loser; if she fails to resell the item in period 2 she consumes it. Bidder i's discount factor for period-two payoffs is $\delta_i \in (0, 1]$.

Actions taken in period 2 may depend on information that is revealed during period 1. Because the winner pays the maximum of the loser's bid and the reserve price, denote this value by p_1, it is natural to assume that this information is revealed to the winner.[5] Our equilibria are valid whether or not the winning bid is revealed to the loser. We will only consider equilibria in weakly increasing bid functions. This means that the observation of the maximum of the losing bid and reserve price translates into information that the loser's value is in an interval (possibly a point) $I \subseteq [0, 1]$. Let $a = \inf I$ and $a' = \sup I$. According to Bayes rule, the resulting posterior distribution function $\hat{F}_I$ for the loser's value is given by

$$\hat{F}_I(\theta_i) = \begin{cases} \frac{F(\theta_i) - F(a)}{F(a') - F(a)} & \text{if } \theta_i \in [a, a'), \\ 1 & \text{if } \theta_i \geq a', \\ 0 & \text{if } \theta_i < a. \end{cases} \tag{1}$$

where F (minus the subscript) is used to denote the loser's value distribution. Bidder i's bid function is denoted $b_i(\cdot)$. Let $\Pi_i(\cdot \mid p_1)$ denote the posterior distribution function for bidder i's value (from the viewpoint of bidder $-i$)

[5]The initial seller can try to establish a deferred payment rule to keep the loser's bid private, but this runs into a number of practical and theoretical problems. In fact, real sellers usually try to avoid any payment deference to prevent defaults, renegotiations, and other problems.

if bidder $-i$ wins the initial auction at price p_1. Then

$$\Pi_i(\cdot \mid p_1) = \begin{cases} \hat{F}_{b_i{}^{-1}(p_1)} & \text{if } p_1 \in b_i([0,1]) \text{ and } p_1 > r, \\ \hat{F}_{b_i{}^{-1}([0,r])} & \text{if } b_i([0,1]) \cap [0,r] \neq \emptyset \text{ and } p_1 = r, \end{cases} \tag{2}$$

and is undetermined otherwise.

Consider bidder i, who has won the initial auction at price p_1 and has waited to period 2 to offer the item for resale to bidder $-i$. For any resale price p, the expected profit of bidder i is

$$\pi_i(p, \theta_i) = (1 - \Pi^-_{-i}(p \mid p_1))p + \Pi^-_{-i}(p \mid p_1)\theta_i \tag{3}$$

where $\Pi^-_{-i}(p \mid p_1) = \lim_{p' \nearrow p} \Pi_{-i}(p' \mid p_1)$ denotes the probability that the offer p is not accepted. An optimal take-it-or-leave-it offer of bidder i is denoted

$$T_i(p_1, \theta_i) \in \arg\max_{p \geq \theta_i} \pi_i(p, \theta_i). \tag{4}$$

Let $\rho_i(\cdot)$ denote bidder i's resale decision function. Given the resale choice $\rho \in \{\text{YES, NO}\}$, bidder i's payoff to winning the initial auction at price p_1 is

$$W_i(\rho, p_1, \theta_i) = -p_1 + \mathbf{1}_{\rho=NO}\theta_i + \mathbf{1}_{\rho=YES}\delta_i\pi_i(T_i(p_1, \theta_i), \theta_i). \tag{5}$$

Bidder i's payoff if she loses the auction with bid b while bidder $-i$ wins, equals

$$L_i(b, \tilde{\theta}_{-i}, \theta_i) = \max\{0, \delta_i[\theta_i - T_{-i}(\max\{b, r\}, \tilde{\theta}_{-i})]\}. \tag{6}$$

Hence, the realized payoff of bidder i with value $\theta_i \in [0,1]$, bid $b \geq 0$, and resale decision $\rho_i(\cdot)$ equals

$$\begin{aligned} \tilde{u}_i(b, \rho_i(\tilde{p}_1, \theta_i), \theta_i) \quad &= \quad \mathbf{1}_{\tilde{w}(b, b_{-i}(\tilde{\theta}_{-i}))=i}\ W_i(\rho_i(\tilde{p}^i_1, \theta_i), \tilde{p}^i_1, \theta_i) \\ &\quad + \mathbf{1}_{\tilde{w}(b, b_{-i}(\tilde{\theta}_{-i}))=-i}\ L_i(b, \tilde{\theta}_{-i}, \theta_i), \end{aligned}$$

where $\tilde{w}$ denotes the random variable for the auction winner as a function of the bid profile and $\tilde{p}^i_1 = \max\{b_{-i}(\tilde{\theta}_{-i}), r\}$ denotes the random variable for the period 1 price paid by (winning) bidder i.

Definition 1 *A perfect Bayesian equilibrium of the second-price auction with resale is a tuple* $(b_1, b_2, \rho_1, \rho_2, T_1, T_2, \Pi_1, \Pi_2)$ *such that for* $i = 1, 2,\ \theta_i \in [0,1]$, *(2), (4), and*

$$b_i(\theta_i), \rho_i(\cdot) \in \arg\max_{b \geq 0, \rho_i(\cdot)} E[\tilde{u}_i(b, \rho, \theta_i)]. \tag{7}$$

hold.

3 Results

Before we present the main equilibrium construction we introduce the following additional notation. For any $\theta, \theta^* \in [0,1]$, let $T^*(\theta)$ denote a solution to (4) when the posterior distribution for the other bidders value equals $\hat{F}_{[0,\theta^*]}$. In addition, let $R^* = (1 - \hat{F}_{[0,\theta^*]}(T^*(0)))T^*(0)$ denote the corresponding resale profit of type $\theta = 0$. We will assume in the results that follow that $r \leq \delta_2 R^*$. Hence, in equilibrium, bidder 2 will place a positive bid (above r) in the initial auction.

We will also make use of the following threshold variable that applies when bidder 2 wins the initial auction and her posterior distribution for bidder 1's value equals $\hat{F}_{[0,\theta^*]}$. Viewed from period 1, bidder 2's net gain from consuming in period 1 instead of waiting for possible resale in period 2 is equal to $\theta_2 - \delta_2 \pi_2(T^*(\theta_2), \theta_2)$. Setting this net gain to zero, we obtain bidder 2's threshold $\tau(\theta^*)$, which defines the critical value at which bidder 2 switches from being willing to wait and attempt resale in period 2 to consuming immediately in period 1. This value exists and is unique:

Lemma 1 *Fix* θ^* *and set* $\Pi_2^-(p \mid p_1) = \hat{F}_{[0,\theta^*]}(p)$ *in (3). If* $\theta^* > 0$, *there exists a unique* $\tau(\theta^*) \in (0, \theta^*]$ *such that* $\theta_2 - \delta_2 \pi_2(T^*(\theta_2), \theta_2)$ *is negative over the nondegenerate interval* $[0, \tau(\theta^*))$, *equal to zero at* $\theta_2 = \tau(\theta^*)$, *and nonnegative on* $[\tau(\theta^*), \bar{\theta}_2]$.

Proof. It suffices to prove that the net gain $\theta_2 - \delta_2 \pi_2(T^*(\theta_2), \theta_2)$ is negative if $\theta_2 = 0$, continuous and strictly increasing in θ_2 over $[0, \theta_2^*]$, and equal to $(1-\delta_2)\theta_2$ if $\theta_2 \geq \theta^*$. Negativity at $\theta_2 = 0$ follows from the assumption that $\delta_2 > 0$ and $\theta^* > 0$ (hence $T^*(0) > 0$, so $\pi_2(T^*(0), 0) > 0$).

Substituting the optimal resale price $T^*(\theta_2)$ into (3) yields bidder 2's indirect expected profit

$$\pi_2(T^*(\theta_2), \theta_2) = (1 - \hat{F}_{[0,\theta^*]}(T^*(\theta_2)))T^*(\theta_2) + \hat{F}_{[0,\theta^*]}(T^*(\theta_2))\theta_2 \tag{8}$$

Applying the envelope theorem to (8) we have

$$\frac{\partial \pi_2(T^*(\theta_2), \theta_2)}{\partial \theta_2} = 1 - (1 - \hat{F}_{[0,\theta^*]}(T^*(\theta_2))) = \hat{F}_{[0,\theta^*]}(T^*(\theta_2)), \tag{9}$$

which implies that for almost every $\theta_2 < \theta^*$, $\delta_2 \pi_2'(T^*(\theta_2), \theta_2) = \delta_2 \hat{F}_{[0,\theta^*]}(T^*(\theta_2)) < 1$, where the strict inequality uses the fact that $\theta_2 < \theta^*$ implies $T^*(\theta_2) < \theta^*$. The rest is trivial. ■

Next we construct the period-one bid $\bar{b}_2(\theta_2, \theta^*)$ for bidder 2 that makes bidder 1 of type θ^* indifferent between winning the initial auction and consuming the item and waiting for resale. This function is defined by the equation

$$\theta^* - \bar{b}_2(\theta_2, \theta^*) = \delta_1(\theta^* - T^*(\theta_2)), \tag{10}$$

where the left-hand-side gives bidder 1's payoff to placing a bid that exceeds $\bar{b}_2(\theta_2, \theta^*)$ and consuming the item in period 1 while the right-hand-side gives her payoff if she waits until period 2 and buys the good from bidder 2 in the resale market at price $T^*(\theta_2)$.

Lemma 2 *For any $\theta^* > 0$ and any θ_2 there exist a unique $\bar{b}_2(\theta_2, \theta^*)$ such that $T^*(\theta_2) \leq \bar{b}_2(\theta_2, \theta^*) \leq \theta^*$ and $\theta_1 - \bar{b}_2(\theta_2, \theta^*) - \delta_1(\theta_1 - T^*(\theta_2))$ is nonpositive if $\theta_1 \leq \theta^*$, zero if $\theta_1 = \theta^*$, and nonnegative if $\theta_1 \geq \theta^*$.*

Proof. Let $\bar{b}_2(\theta_2, \theta^*) := \theta^* - \delta_1(\theta^* - T^*(\theta_2))$. Then $T^*(\theta_2) \leq \bar{b}_2(\theta_2, \theta^*) \leq \theta^*$ since $\theta^* \geq T^*(\theta_2)$. Moreover, $\theta_1 - \bar{b}_2(\theta_2, \theta^*) - \delta_1(\theta_1 - T^*(\theta_2)) = (1-\delta_1)(\theta_1 - \theta^*)$ which is less than, equal to, or greater than zero as θ_1 is less than, equal to, or greater than θ^*, respectively. ■

Finally, let $T^{**}(\theta)$ denote an optimal resale offer by a bidder of type θ if her posterior distribution for the other bidder's value equals $\hat{F}_{[\tau(\theta^*), \theta^*]}$. We can now state our main result.

Proposition 1 *Fix $\theta^* \in (0, 1]$. Suppose $r \leq \delta_2 R^*$. Then there exists a perfect Bayesian equilibrium of the second-price auction with resale such that for all $\theta_1, \theta_2 \in [0, 1]$ and $p_1 \geq 0$,*

$$b_1(\theta_1) = \begin{cases} 0 & \text{if } \theta_1 \in [0, \theta^*] \\ \theta_1 & \text{if } \theta_1 \in (\theta^*, 1] \end{cases} \tag{11}$$

$$b_2(\theta_2) = \begin{cases} \bar{b}_2(\theta_2, \theta^*) & \text{if } \theta_2 \in [0, \tau(\theta^*)] \\ \max\{\theta^*, \theta_2\} & \text{if } \theta_2 \in (\tau(\theta^*), 1] \end{cases} \tag{12}$$

$$\rho_1(p_1, \theta_1) = NO \tag{13}$$

$$\rho_2(p_1, \theta_2) = \begin{cases} YES \quad if & p_1 \in [r, \theta^*), \theta_2 \in [0, \tau(\theta^*)] \\ NO & otherwise \end{cases} \tag{14}$$

$$T_1(p_1, \theta_1) = \begin{cases} \max\{\bar{b}_2^{-1}(\cdot, \theta^*), \theta_1\} & \text{if } p_1 \in \bar{b}_2([0, \tau(\theta^*)], \theta^*) \\ T^{**}(\theta_1) & \text{if } p_1 = \theta^* \\ \theta_1 & \text{if } p_1 \in [r, \bar{b}_2(0, \theta^*)) \cup (\bar{b}_2(\tau(\theta^*), \theta^*), \theta^*) \\ \max\{p_1, \theta_1\} & \text{if } p_1 \in (\theta^*, 1] \end{cases} \tag{15}$$

$$T_2(p_1, \theta_2) = \begin{cases} T^*(\theta_2) & \text{if } p_1 = r \\ \max\{\max\{\theta^*, p_1\}, \theta_2\} & \text{if } p_1 \in (r, 1] \end{cases} \tag{16}$$

where $\tau(\theta^*)$ and $\bar{b}_2(\theta_2, \theta^*)$ are defined in Lemmas 1 and 2, respectively. The equilibrium outcome is supported by the following off-path beliefs

$$\forall \theta_1 \in [0,1] : \ \Pi_1(\theta_1 \mid p_1) = \mathbf{1}_{\theta_1 \geq \theta^*} \quad \text{if } p_1 \in [r, \theta^*], \tag{17}$$

$$\forall \theta_2 \in [0,1] : \ \Pi_2(\theta_2 \mid p_1) = \mathbf{1}_{\theta_1 \geq 0} \quad \text{if } p_1 \in [r, \bar{b}_2(0, \theta^*)) \cup (\bar{b}_2(\tau(\theta^*), \theta^*), \theta^*). \tag{18}$$

Proof. The resale price functions (15) and (16) are straightforward from the off-path beliefs (17) and from the posterior beliefs implied by the equilibrium bid functions (11) and (12) through Bayesian updating.

The resale choice functions (13) are optimal given the resale price functions.

To show that bidder 1's bid function (11) is optimal we start with two observations:

Fact (i): If bidder 1 wins in period 1 then his expost payoff viewed from period 1 is equal to $\theta_1 - \bar{b}_2(\theta_2, \theta^*)$ if $\theta_2 \leq \tau(\theta^*)$ and is equal to $\theta_1 - \max\{\theta^*, \theta_2\}$ if $\theta_1 > \tau(\theta^*)$.

Fact (i) is true is because bidder 1 always consumes the item upon winning, because his payment conditional on winning is no less than bidder 2's value: if $\theta_2 > \tau(\theta^*)$, bidder 2's bid is $\max\{\theta^*, \theta_2\}$ and if $\theta_2 \leq \tau(\theta^*)$, bidder 2's bid is $b_2(\theta_2) = \bar{b}_2(\theta_2, \theta^*) \geq T^*(\theta_2) \geq \theta_2$.

Fact (ii): If bidder 1 loses in period 1, then his ex post payoff viewed from period 1 is equal to $[\delta_1(\theta_1 - T^*(\theta_2))]^+$ if he bid zero and $\theta_2 \leq \tau(\theta^*)$, and is equal to zero in any other case.[6]

The case where bidder 1 bids zero and $\theta_2 \leq \tau(\theta^*)$ in fact (ii) follows directly from bidder 2's expected resale strategy. If bidder 1 loses with a bid in $[r, \theta^*]$, bidder 2 has the posterior belief that her value is θ^* and hence makes an unacceptable resale offer. In any other case, bidder 2's posterior belief is that bidder 1's value is less than or equal to hers and hence will not offer the item for resale.

[6] We use the convention $[x]^+ \overset{\text{def}}{=} \max\{x, 0\}$ for any $x \in \mathbb{R}$.

Suppose bidder 1's type $\theta_2 \leq \theta^*$. We claim the bidding zero is optimal for bidder 1. Bidding above θ^* is dominated because his payoff is negative if he wins (fact (i)) and zero if he loses (fact (ii)). A positive bid below θ^* yields the same payoff as the zero bid does unless $\theta_2 \leq \tau(\theta^*)$. In that case, the former yields at most $\theta_1 - \bar{b}_2(\theta_2, \theta^*)$, by facts (i) and (ii), and the latter yields $[\delta_1(\theta_1 - T^*(\theta_2))]^+$. But, in this case, $\theta_1 - \bar{b}_2(\theta_2, \theta^*) < \delta_1(\theta_1 - T^*(\theta_2))$, by Lemma 2. Hence, bidder 1 bids zero.

Suppose bidder 1's type $\theta_1 > \theta^*$. Bidding θ_1 dominates bidding zero, because the former yields $\theta_1 - \bar{b}_2(\theta_2, \theta^*)$ if $\theta_2 \leq \tau(\theta^*)$ and $\theta_1 - \max\{\theta^*, \theta_2\}$ if $\theta_2 > \tau(\theta^*)$ (fact (i)), while the latter yields $[\delta_1(\theta_1 - T^*(\theta_2))]^+$ if $\theta_2 \leq \tau(\theta^*)$ and zero otherwise (fact (ii)). Since $\theta_1 > \theta^*$, the former is better than the latter. Thus, bidder 1 prefers positive bids to a zero bid. Conditional on submitting a positive bid, bidder 1's decision problem is the same as a second-price auction that bans resale (facts (i) and (ii)), hence value-bidding is optimal.

It remains to show that bidder 2's bid function (11) is optimal. The key fact is that bidder 2 knows that if she loses in period 1 then her payoff is zero. That is because she expects the winner, bidder 1, to update to the belief that $\theta_1 > \theta_2$ and hence will not offer the item for resale. Thus we only have to argue that bidder 2 cannot gain by deviating to an alternative winning bid.

Suppose bidder 2's type $\theta_2 > \tau(\theta^*)$. If she wins in period 1, then, by construction of $\tau(\theta^*)$, it is optimal to consume the item right away. Thus, bidder 2 with such a type faces the same decision problem as in a second-price auction that bans resale. Hence value-bidding is optimal. If her type is in $(\tau(\theta^*), \theta^*]$, bidding θ^* does as well as bidding θ_2, because bidder 2 knows that bidder 1 either bids zero or bids above θ^*.

Suppose $\theta_2 < \tau(\theta^*)$. Any bid in the range $(r, \theta^*]$ has the same payoff as bidding θ^*, since bidder 1 bids zero if her value lies in this range, and she bids an amount greater than θ^* if her value lies outside this range. Bidding 0 is obviously dominated because she potentially looses the auction (depending on the tie-breaker rules) and gets a payoff of zero. Finally, bidding above θ^* is also dominated. That is because in the case where bidder 1's type is above θ^* and below bidder 1's deviant bid, bidder 2 wins the item at a price above his own value and cannot fully recover his payment from resale due to discounting; in other cases, submitting a bid above θ^* is no better than a positive bid below θ^*. Thus $b_2(\theta_2) = \bar{b}_2(\theta_2, \theta^*)$ is an optimal bid. ■

The structure of the equilibria is shown in Figure 1 and is described as follows. Both bidders bid their values if their types are above some cutoff value θ^*. Bidder-1 types below the cutoff abstain from the auction. Each bidder-2 type below the threshold $\tau(\theta^*)$ submits a bid $\bar{b}_2(\theta_2, \theta^*)$ that makes bidder 1 of type θ^* indifferent between value-bidding (and winning the initial auction at price $\bar{b}_2(\theta_2, \theta^*)$) and bidding zero (which means she purchases the could at price $T^*(\theta_2)$ in the resale market). Bidder-2 types above the threshold $\tau(\theta^*)$ submit a bid equal θ^*.[7]

The off-path beliefs specified in (17) and (18) are straightforward. If bidder 1 wins at a price in $[r, \bar{b}_2(0, \theta^*)) \cup (\bar{b}_2(\tau(\theta^*), \theta^*), \theta^*)$ she believes that bidder 2's value equals 0. Likewise, if bidder 2 wins at a price in $(r, \theta^*]$ she believes that bidder 1's value equals θ^*. The off-path beliefs for bidder 2 can be weakened (or changed) in two ways. The equilibrium would still be supported if, when bidder 2 wins at a price in $(r, \theta^*]$ her beliefs are the same as they are after observing a bid of 0, i.e., she believes that bidder 1's value is distributed $\hat{F}_{[0,\theta^*]}$. Then, a deviating bid that loses the initial auction results in the same resale price (an hence is not strictly improving). Alternatively, the off-path beliefs specified in (17) can be changed so that bidder 2 no longer chooses to offer the item for resale after winning at a price in $(r, \theta^*]$. Suppose, for instance that when bidder 2 wins at a price in $(r, \theta^*]$ she assigns probability 1 to bidder 1 having a value of zero. In this case, bidder 2 does not wait and offer the item for resale when she wins the initial auction, and hence bidder 1 is made worse off by deviating to a losing bid in $(r, \theta^*]$.[8]

Second-price auctions are often advertised as a means to allocate a good efficiently in markets with private values. A stark contrast would be a market where one of the buyers first owns the good and then acts as a monopoly seller; i.e., makes a take-it-or-leave-it offer to the other buyer. The equilibrium with $\theta^* = 1$ and $r = 0$ of Proposition 1 shows that such an extremely inefficient "bidder monopoly" may be produced as the equilibrium outcome of a second-price auction with resale, and all intermediate degrees of inefficiency are possible as well. This is expressed in the following corollary.

Corollary 1 *Let* $S^* = E[\max\{\theta_1, \theta_2\}]$ *denote the first-best surplus, and let* S_{M2} *denote the discounted expected surplus in a bidder-monopoly market*

[7]Note that because bidder 1 bids zero (or abstains) for any private use-value below θ^*, and since both bidders value-bid when their private use-values are above θ^*, it is immediate that initial seller revenue is reduced relative to the symmetric equilibrium for all $\theta^* > 0$.

[8]In this case, bidder 2's resale choice function $\rho(p_1, \theta_2) = YES$ if $p_1 = r$ and $\theta_2 \in [0, \tau(\theta^*)]$, and NO otherwise.

where buyer 2 first owns the good and then makes a take-it-or-leave-it offer to buyer 1. Then for all $S \in [S_{M2}, S^*]$ there exists an equilibrium of the second-price auction with resale and reserve price $r = 0$ such that the equilibrium surplus equals S.

A second corollary of Proposition 1 is that the initial seller may want to set a positive reserve price if she is efficiency-minded. More precisely, we are considering an extended game where the initial seller first announces a reserve price, and then the second-price auction with resale is played. The initial seller's payoff is the social surplus, which equals the expected consumption value of the good, depending on who consumes it.

By setting a positive reserve price, the initial seller may be able to prevent aggressive bidding and thus may be able to avoid an active resale market with an inefficient outcome. The inefficiency caused by the reserve price may be smaller than the inefficiency caused by aggressive bidding.

Corollary 2 *The extended game with reserve-price choice and an efficiency-minded initial seller has a perfect-Bayesian equilibrium where the initial seller chooses a positive reserve price.*

Proof. Suppose that given any reserve price $r < r_0$, the continuation equilibrium with $\theta^* = 1$ is played, and given any reserve price $r \geq r_0$, the continuation equilibrium with $\theta^* = 0$ is played. If $r_0 > 0$ is small, any reserve price $r < r_0$ will yield the surplus S_{M2}, and any reserve price $r \geq r_0$ will yield the surplus

$$S^* - E[\mathbf{1}_{\tilde{\theta}_1 < r,\ \tilde{\theta}_2 < r} \max\{\tilde{\theta}_1, \tilde{\theta}_2\}].$$

Hence, the reserve price $r = r_0$ is optimal. ■

4 Examples with Uniformly Distributed Values

Suppose there are two bidders whose values are determined independently and identically from the uniform distribution on $[0, 1]$. Assume that there are two periods: the initial auction and one resale market. The initial auction is a second-price auction with reserve $r < \delta_2(\frac{\theta^*}{2})^2$. In the resale auction, the period 1 winner makes a take-it-or-leave-it offer to the period 1 loser. We are considering an equilibrium where bidder 2 bids $b_2(\theta_2) > 0$ if $\theta_2 < \theta^*$ (and θ_2 otherwise), and bidder 1 bids zero if $\theta_1 < \theta^*$ (and θ_1 otherwise).

For any resale price p, the expected profit of bidder 2, who wins at price r, is

$$\pi_2(p, \theta_2) = (1 - \frac{p}{\theta^*})p + \frac{p}{\theta^*}\theta_2 - r \tag{19}$$

This is maximized at price

$$T^*(\theta_2) = \frac{\theta^* + \theta_2}{2} \tag{20}$$

Hence, bidder 2 waits and offers the good for resale if

$$\delta_2 \left[(1 - \frac{\theta^* + \theta_2}{2\theta^*})\frac{\theta^* + \theta_2}{2} + \frac{\theta^* + \theta_2}{2\theta^*}\theta_2\right] > \theta_2 \tag{21}$$

This is true for $\theta_2 \in [0, \theta^*]$ if and only if $\theta_2 < \tau(\theta^*) = \frac{\theta^*}{\delta_2}\left(2 - \delta_2 - 2\sqrt{1 - \delta_2}\right)$.

In order to make the bidding strategy $b_1(\theta_1) = 0$ optimal for bidder 1 when $\theta_1 \in [0, \theta^*]$ we require bidder 2's of type $\theta_2 \in [0, \tau(\theta^*)]$ to bid $\bar{b}_2(\theta_2, \theta^*)$ such that bidder 1 with type θ^* would be indifferent between winning the initial auction and waiting for resale (and types below θ^* would strictly prefer to wait):

$$\bar{b}_2(\theta_2, \theta^*) = \theta^* - \delta_1(\theta^* - \frac{\theta^* + \theta_2}{2}) \tag{22}$$

Finally, note that, from (20), $T^*(0) = \frac{\theta^*}{2}$. Hence off-path beliefs for bidder 1 regarding bidder 2's value apply when the initial auction price $p_1 \in [r, \theta^* - \delta_1(\frac{\theta^*}{2})) \cup (\theta^* - \frac{\delta_1\theta^*}{2} - \frac{\delta_1\theta^*}{2\delta_2}\left(2 - \delta_2 - 2\sqrt{1 - \delta_2}\right), \theta^*)$. Off-path beliefs for bidder 2 regarding bidder 1's value apply when the initial auction price $p_1 \in (0, \theta^*]$.

Case 1. $\theta^* = 1$, $\delta_1 = \delta_2 = 1$.

The key components of the equilibrium strategies are

$$T^*(\theta_2) = \frac{1 + \theta_2}{2} \tag{23}$$

$$\tau(\theta^*) = 1 \tag{24}$$

$$\bar{b}_2(\theta_2, \theta^*) = T^*(\theta_2). \tag{25}$$

Off-path beliefs for bidder 1 regarding bidder 2's value apply when the initial auction price $p_1 \in [0, .5)$. Off-path beliefs for bidder 2 regarding bidder 1's value apply when the initial auction price $p_1 \in (0, 1]$.

Case 2. $\theta^* = .6$, $\delta_1 = \delta_2 = .9$.

The key components of the equilibrium strategies are

$$T^*(\theta_2) = \frac{.6 + \theta_2}{2} \tag{26}$$

$$\tau(\theta^*) = .311\,7 \tag{27}$$

$$\bar{b}_2(\theta_2, \theta^*) = .33 + .45\theta_2. \tag{28}$$

Off-path beliefs for bidder 1 regarding bidder 2's value apply when the initial auction price $p_1 \in [0, .33) \cup (.47027, .6)$. Off-path beliefs for bidder 2 regarding bidder 1's value apply when the initial auction price $p_1 \in (0, .6)$.

5 Asymmetric Equilibrium with $n > 2$ bidders

Suppose there are $n > 2$ bidders whose values are determined independently and identically from the distribution F on $[0, 1]$. Assume that there are two periods: the initial auction and one resale market. The initial auction is a second-price auction with reserve $r < R^*$.[9] The resale auction is a second-price auction with an optimal reserve. Discounting is allowed. We are considering an equilibrium with $\theta^* = 1$ where bidder n bids $b_n(\theta_n) > 0$ and all other bidders bid zero.[10] The equilibrium conditions are obtained from Definition 1, which is easily extended to the case of $n > 2$ bidders.

In order to make the bidding strategy $b_j(\theta_j) = 0$ optimal for bidder $j \neq n$ we set $b_n(\theta_n) \equiv 1$ and we assume bidder n has posterior belief that any deviating bidder's value is 1 with probability 1. Hence bidder n's reserve in

[9] We now use R^* to denote the expected payoff of a zero-value seller who sets an optimal reserve in an $(n-1)$-bidder auction.

[10] This is also an equilibrium of the second-price auction without resale (See Blume and Heidhues, 2004). However, without resale, the equilibrium is in weakly dominated strategies and it does not exist for any positive reserve price.

the event that she sees a strictly positive (off-path) bid is 1. Hence deviations for any bidder $j \neq n$ are not profitable.

We also need to establish that bidder n would not deviate to an alternative bid. The only deviation that changes her payoff is to bid zero. The best case scenario for bidder n if she bids zero is to win the item in the resale market and pay zero. However, her expected payoff in the equilibrium is at least her value, since she can achieve an expected payoff at least as high as this by setting an optimal reserve as the reseller.

6 Concluding Remarks

In the equilibrium construction outlined in Proposition 1, bidder-1 types up to $T^*(0)$ are indifferent between abstaining and submitting a bid up to $T^*(0)$. This is because these types never get an acceptable resale offer anyway. To remove this indifference, one can argue that these types should at least lexicographically prefer to abstain rather than make the effort of participating in an auction where they expect to win with probability 0. The remaining indifference in the equilibrium construction is that of bidder-2 types up to θ^* between all bids in the range $[r, \theta^*]$. If the types in $[r, \theta^*]$ deviated to bid their values instead, the equilibrium would break down; in fact, it would become optimal for buyer-1 types in $[r, \theta^*]$ to bid their values, and the market would converge to the standard value-bidding equilibrium. In such a dynamic context, however, one can argue that buyer 2 may stick to her above-value equilibrium bid because she anticipates buyer 1's adaptive behavior, and because buyer 2's payoff is larger than in the standard bid-your-value equilibrium. In summary, there is reason to believe that our equilibria have good stability properties in dynamic contexts, provided one buyer is more forward-looking than the other.

Generalizing the equilibrium construction for $\theta^* < 1$ to second-price auctions with more than 2 bidders is nontrivial. The problem is that new equilibrium conditions come up. In particular, with multiple buyers it becomes difficult to verify that a buyer with a type who is supposed to abstain and wait for a resale offer has no incentive to increase her bid, win, and attempt resale herself. With 2 bidders, we know bidder 1 will never outbid bidder 2 because $\theta_2 < b_2(\theta_2)$. I.e., the maximum resale price bidder 1 could hope to achieve is less than the amount she pays. If there are $n > 2$ bidders (i.e., $n - 2$ other bidders bidding zero along with bidder 2) it is conceivable that one (or some) of them will have a value greater than $b_2(\theta_2)$ and hence bidder 1 might want to buy at $b_2(\theta_2)$ and attempt resale. This problem does not arise when $\theta^* = 1$.

Zheng (2000, Section 5.2) also constructs an equilibrium for a 2-bidder second-price-type auction where one bidder bids above her value and the other below. The main difference between the construction of Proposition 1 and that of Zheng, is that in Zheng's equilibrium the cutoff-type θ^* of buyer 1 is not indifferent between abstaining and bidding her value. Instead, abstaining is strictly optimal. This is only possible if $\theta^* = 1$; i.e., in the case where buyer 1 always abstains. Zheng's previous construction also immediately generalizes to markets with an arbitrary number of buyers.

If at least one of the two bidders does not discount future payoffs, the equilibria presented in Proposition 1 can be simplified. In particular, if we assume that bidder 2 does not discount future payoffs then bidder 2 will always offer the good for resale and hence $\tau(\theta^*)$ can be set equal to θ^*. This means there is no interval of types for bidder 2 that pool their bids and θ^*. If neither of the two bidders discounts future payoffs, the equilibria presented in Proposition 1 can be simplified even further. This is because bidder 2's bid in period 1 will be equal to her resale offer in period 2. Hence, bidder 1 is indifferent between winning and losing the initial auction. This makes it much easier to show that the proposed bidding strategies are optimal. A restatement and proof of Proposition 1 for the case of no discounting is provided in the Appendix.

7 Appendix

Proposition 2 *Fix $\theta^* \in (0,1]$. Suppose $\delta_1 = \delta_2 = 1$ and $r \leq R^*$. Then there exists a perfect Bayesian equilibrium of the second-price auction with resale such that for all $\theta_1, \theta_2 \in [0,1]$ and $p_1 \geq 0$,*

$$b_1(\theta_1) = \begin{cases} 0 & \textit{if } \theta_1 \in [0, \theta^*] \\ \theta_1 & \textit{if } \theta_1 \in (\theta^*, 1] \end{cases} \tag{29}$$

$$b_2(\theta_2) = T^*(\theta_2) \tag{30}$$

$$\rho_1(p_1, \theta_1) = NO \tag{31}$$

$$\rho_2(p_1, \theta_2) = YES \tag{32}$$

$$T_1(p_1, \theta_1) = \begin{cases} \theta_1 & \textit{if } p_1 \in [r, T^*(0)] \\ \max\{(T^*)^{-1}(p_1), \theta_1\} & \textit{if } p_1 \in (T^*(0), 1] \end{cases} \tag{33}$$

$$T_2(p_1,\theta_2) \;=\; \begin{cases} T^*(\theta_2) & \textit{if } p_1 \in [0,r] \\ \max\{\max\{\theta^*, b\}, \theta_2\} & \textit{if } p_1 \in (r,1] \end{cases} \tag{34}$$

The equilibrium outcome is supported by the following off-path beliefs

$$\forall \theta_1 \in [0,1]: \;\; \Pi_1(\theta_1 \mid p_1) = \mathbf{1}_{\theta_1 \geq \theta^*} \quad \textit{if } p_1 \in (r,\theta^*] \tag{35}$$

$$\forall \theta_2 \in [0,1]: \;\; \Pi_2(\theta_2 \mid p_1) = \mathbf{1}_{\theta_1 \geq 0} \quad \textit{if } p_1 \in [r, T^*(0)] \tag{36}$$

Proof. The bid function b_1 induces the posterior beliefs, for all $\theta_1 \in [0,1]$,

$$\Pi_1(\theta_1 \mid p_1) = \begin{cases} \hat{F}_{[0,\theta^*]}(\theta_1) & \text{if } p_1 = r \\ \mathbf{1}_{\theta_1 \geq p_1} & \text{if } p_1 \in (\theta^*, 1] \end{cases} \tag{37}$$

and off the equilibrium path we suppose the beliefs follow (17). Strict monotone comparative statics results (see Edlin and Shannon, 1998) show that the function $b_2 = T^*$ is strictly increasing. Therefore, for all $\theta_2 \in [0,1]$,

$$\Pi_2(\theta_2 \mid p_1) = \mathbf{1}_{\theta_1 \geq b_2^{-1}(p_1)} \quad \text{if } p_1 \in [T^*(0), 1], \tag{38}$$

and off the equilibrium path we suppose the beliefs follow (18). That the resale price function (15) satisfies (4) follows from (38) and (18). That (16) satisfies (4) follows from (37) and (17).

It remains to show (7). Consider $i = 1$. Let $\theta_1 \in [0,1]$ and $p_1 \geq 0$. Then,

$$\tilde{u}_1(b,\theta_1) = [\theta_1 - T^*(\theta_2)]^+ \;\; \text{if } p_1 = r. \tag{39}$$

Moreover,

$$\begin{aligned} \tilde{u}_1(b,\theta_1) \;=\; & \mathbf{1}_{b<T^*(\theta_2)} \; [\theta_1 - \max\{\theta^*, \tilde{\theta}_2\}]^+ \\ & + \mathbf{1}_{b>T^*(\theta_2)} \; (-T^*(\tilde{\theta}_2) + \max\{\theta_1, \tilde{\theta}_2\}) \;\; \text{if } b \in (r, \theta^*] \end{aligned} \tag{40}$$

and

$$\begin{aligned} \tilde{u}_1(b,\theta_1) \;=\; & \mathbf{1}_{\tilde{\theta}_2 > \theta^*} \; [\theta_1 - \tilde{\theta}_2]^+ \\ & + \mathbf{1}_{\tilde{\theta}_2 < \theta^*} \; (-T^*(\tilde{\theta}_2) + \max\{\theta_1, \tilde{\theta}_2\}) \;\; \text{if } b \geq \theta^*. \end{aligned} \tag{41}$$

Using (39), (40), and (41), it is straightforward to show that

$$\forall \theta_1 \in [0,1], b \geq 0: \;\; \tilde{u}_1(b_1(\theta_1), \theta_1) \geq \tilde{u}_1(b, \theta_1).$$

This implies (7) for $i = 1$.

It remains to show (7) for $i = 2$. Let $\theta_2 \in [0, 1]$ and $b \geq 0$. Then,

$$\begin{aligned} \tilde{u}_2(b, \theta_2) &= \mathbf{1}_{\tilde{\theta}_1 > \theta^*} [\theta_2 - \tilde{\theta}_1]^+ + \mathbf{1}_{T^*(\theta_2) \leq \tilde{\theta}_1 \leq \theta^*} (-r + T^*(\theta_2)) \\ &\quad + \mathbf{1}_{\tilde{\theta}_1 < T^*(\theta_2),\ \tilde{\theta}_1 \leq \theta^*} (-r + \theta_2) \quad \text{if } b \geq T^*(0). \end{aligned} \tag{42}$$

Hence, a deviation from the bid $b_2(\theta_2) \geq T^*(0)$ to any bid in the range $[T^*(0), \infty)$ is not profitable. If buyer 2 submits a bid $b \in (r, T^*(0))$ (or $b = r$ and $r > 0$), her payoff may differ from (42) because if $\tilde{\theta}_1 > \theta^*$ buyer 1's resale offer depends on her off-path posterior beliefs about buyer 2. But a deviation to b cannot be profitable because buyer 1 will not re-offer the good at a price below $\tilde{\theta}_1$. If $b = 0$ and $r = 0$, buyer 2's payoff may be even smaller because buyer 2 may lose the tie with buyer 1.

Any abstaining bid $b < r$ yields the payoff

$$\tilde{u}_2(b, \theta_2) = \mathbf{1}_{\tilde{\theta}_1 > \theta^*} [\theta_2 - \tilde{\theta}_1]_+.$$

Hence, by (42) the bid $b_2(\theta_2) \geq T^*(0)$ is at least as good as abstaining if and only if

$$E[\mathbf{1}_{T^*(\theta_2) \leq \tilde{\theta}_1}\ T^*(\theta_2) + \mathbf{1}_{\tilde{\theta}_1 < T^*(\theta_2)} \theta_2 \mid \tilde{\theta}_1 \leq \theta^*] \geq r.$$

Because the left-hand side is bounded below by R^*, optimality of bid $b_2(\theta_2)$ follows from the assumption $r \leq R^*$. ■

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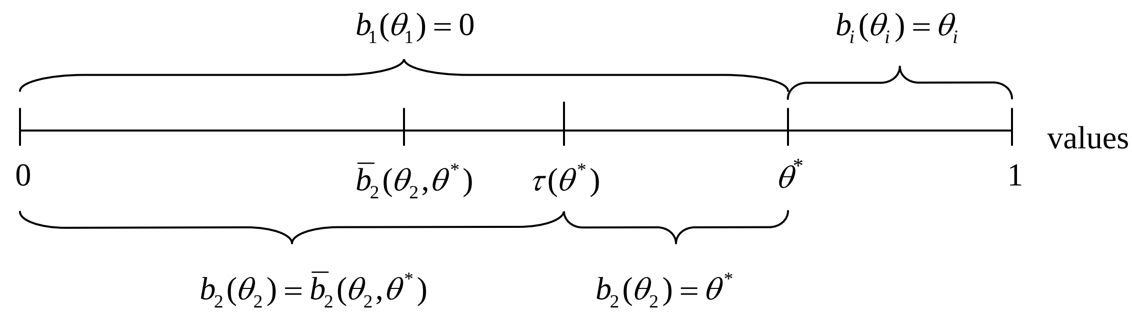
$b_1(\theta_1) = 0$
$b_i(\theta_i) = \theta_i$
values
0
$\bar{b}_2(\theta_2, \theta^*)$
$\tau(\theta^*)$
θ^*
1
$b_2(\theta_2) = \bar{b}_2(\theta_2, \theta^*)$
$b_2(\theta_2) = \theta^*$

Figure 1