Centrality in directed social networks. A game theoretic approach

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A R T I C L E  I N F O

JEL classification: C71

Keywords:
Social networks
Game theory
Centrality
Shapley value

A B S T R A C T

In this paper we define a family of centrality measures for directed social networks from a game theoretical point of view. We follow the line started in our previous paper (Gómez et al., 2003) and, besides the definition, we obtain a characterization of the measures and an additive decomposition in three submmands that can be interpreted in terms of emission, betweenness and reception centrality components.

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1. Introduction

A social network is a set of nodes representing people, groups, organizations, enterprises, etc., that are connected by links showing relations or flows between them. Network analysis studies the implications of the restrictions of different actors in their communications and then in their opportunities of relation. The fewer constraints an actor faces, the more opportunities he/she will have, and thus he will be in a more favorable position to bargain in exchanges and to intermediate in the bargains of others that need him, increasing his influence. Then, among other goals, network analysis tries to obtain indices, as objective as possible, to measure hypothetical or indirectly observable variables such as influence, opportunities, and better position.

Social networks analysts consider the closely related concepts of centrality and power as fundamental properties of individuals, that inform us about aspects as who is who in the network, who is a leader, who is an intermediary, who is almost isolated, who is central, who is peripheral. Under the network approach it is assumed that this power is inherently relational. Social networks researchers have developed several centrality measures. Degree, Closeness and Betweenness centralities are without doubt the three most popular ones.

Degree centrality (Shaw, 1954; Nieminen, 1974) focuses on the level of communication activity, identifying the centrality of a node with its degree.

Closeness centrality (Beauchamp, 1965; Sabidussi, 1966) considers the sum of the geodesic distances between a given actor and the remaining as a decentrality measure in the sense that the lower this sum is, the greater the centrality. Closeness centrality is, then, a measure of independence in the communications, in the relations or in the bargaining, and thus, it measures the possibility to communicate with many others depending on a minimum number of intermediaries.

Betweenness centrality (Bavelas, 1948; Freeman, 1977) emphasizes the value of the communication control: the possibility to intermediate in the relation of others. Here, all possible geodesic paths between pairs of nodes are considered. The centrality of each actor is the number of such paths in which it lies.

Stephenson and Zelen (1989) abandon the geodesic path as structural element in the definition of centrality, to introduce a measure based on the concept of information as it is used in the theory of statistical estimation. The defined measure uses a weighted combination of all paths between pairs of nodes, the weight of each path depending on the information contained in it.

Bonacich (1972, 1987) suggests another concept of centrality. He proposes to measure the centrality of different nodes using the eigenvector associated with the largest characteristic eigenvalue of the adjacent matrix. The ranking of web sites as they appear in the web search engine Google was created from this measure by Brin and Page (1998).

All previous approaches assume that the direct relation between two nodes (whenever it exists) is symmetrical. Nevertheless it is easy to find situations in which the connections are directed, having an specific sense: for example in the case of the network of citations in scientific papers or in the walks across the pages in the www. It seems, then, to be relevant to define measures of centrality (or to adapt the already existing ones) for these special situations that can be considered, in fact, more general than the not directed ones. Contributions in this direction can be found in White and Borgatti (1994); Borgatti (2005), that generalize the Freeman’s geodesic measures for betweenness in undirected graphs, Tutzauer (2007), who uses the entropy as a measure of centrality in networks characterized by path-transfer flow, and Pollner et al. (2008),
that introduces an algorithm to calculate the centrality for cohesive subgroups in directed networks.

In this paper we propose a family of centrality measures for directed graphs using a game-theoretical point of view. The seminal work in applying game theory to the topic of centrality for nodes in graphs is Grofman and Owen (1982). They used the framework of games with restrictions in the communication introduced by Myerson (1977, 1980). In Gómez et al. (2003), we extend previous ideas to obtain a new family of centrality measures with some appealing properties and the corresponding calculation methods. Other contributions are closely related with the problem of centrality but ignore it focusing only in the definition and properties, including characterizations, of allocation rules for games with restrictions in the cooperation, these restrictions being given by graphs or digraphs. An excellent survey of the work on this topic can be found in Slikker and van den Nouweland (2001). Other recent relevant contributions are Amer et al. (2007), that define a family of measures for a concept they call accessibility in oriented networks, van den Brink and Borm (2002) for a special type of digraphs representing competitions, González-Arangüena et al. (2008), where the classical Myerson value is generalized to games with restrictions in the communication given by digraphs, van den Brink and Gilles (2000), van den Brink et al. (2008) and Hendrickx et al. (2009). Kim and Jun (2008) present some different types of connectivity in directed networks and associated characterizations of allocations rules.

The approach we present here assumes that actors in a directed network are simultaneously players in a TU game (an n-person cooperative game in characteristic function form, or a model for a cooperative situation in which actors can transfer their utilities). This game models the social interests among the actors. For instance: the number of e-mails among the members of a scientific community, the number of marriages among the individuals of a group of families, or the possibility of obtaining majorities by the parties in a given parliament, to name only a few. The restrictions in the communication generated by the digraph modify such game transforming it in a generalized TU one: the digraph restricted game. In these generalized TU games, as introduced by Nowak and Radzik (1994), the worth of a coalition depends not only on its members but also on the order in which they incorporate to that coalition. The centrality of each actor is then measured as the variation of his outcome from the game without restrictions to the digraph-restricted one. We will use as point solution for players in a TU game the Shapley value (Shapley, 1953) and, in generalized TU games, a parametric family of indices that include those characterized by Nowak and Radzik (1994); Sánchez and Bergantinos (1997). Therefore a family of measures for each digraph is obtained, each member of this family corresponding to a particular election of the a priori social interests (the game) and of the fixed index. Implicitly we are assuming that the centrality of each individual, even when the relations are fixed, depends on the interests that motivate the interaction among actors. So, the links an actor has give him a different relational power depending on the social interests in the group.

This approach is closely related with the ones in Gómez et al. (2003) and in Amer et al. (2007), especially in the technical framework. Moreover the introduced measures are characterized. This characterization is based on two properties: component efficiency and α-directed fairness. The consideration of arcs (directed links) as units of relation, instead of the classical links of a graph, introduces an element of asymmetry in the bilateral relations and, as a consequence, a possible different bargaining power for both incident nodes. This is the meaning of the α-directed fairness, a closely related property with the α-hierarchical payoff property in Slikker et al. (2005).

The centrality should possibly be conceived as a vector magnitude instead of a scalar one. With this idea in mind, we obtain a decomposition of each measure in three summands that can be viewed as components or factors in the directions of emission, reception (closedness ones) and betweenness. The defined centrality will be then the modulus (using the city-block distance) of this three-dimensional vector.

The remaining of the paper is organized as follows: Section 2 contains the notation and some preliminary concepts. In Section 3 the definition and some properties (including a characterization) of the proposed family of centrality measures are given. Section 4 is devoted to the aforementioned decomposition. Final conclusions appear in Section 5. Appendix A collect the proofs of all the results in the paper.

2. Preliminaries

2.1. Games and generalized games

In many social networks, the formation of coalitions is a process in which not only the members of the coalitions are important but also the order in which they appear. Taking this idea into account, Nowak and Radzik (1994) introduced the concept of game in generalized characteristic function form.

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. For each $S \subseteq 2^N \setminus \{\varnothing\}$, let us denote by $\Pi(S)$ the set of all permutations or ordered coalitions of the players in $S$ and, for notational convenience, $\Pi(\varnothing) = \{\varnothing\}$. We will denote $\Omega(N) = \{T \in \Pi(S) | S \subseteq N\}$ the set of all ordered coalitions with players in $N$. Given an ordered coalition $T \in \Omega(N)$, there exists $S \subseteq N$ such that $T \in \Pi(S)$. We will denote $H(T) = S$ for the set of players in the ordered coalition $T$, and $r = |H(T)|$.

A game in generalized characteristic function form is a pair $(N, v)$, $N$ being the players set and $v$ a real function (the generalized characteristic function) defined on $\Omega(N)$, and satisfying $v(\varnothing) = 0$. For each $S \subseteq N$, and for every ordered coalition $T \in \Pi(S)$, $v(T)$ represents the social or economic possibilities of the players in $S$ if the coalition is formed following the order given by $T$. When there is no ambiguity with respect to the set of players $N$, we will identify the (generalized) game $(N, v)$ with its (generalized) characteristic function $v$.

We will denote by $\mathcal{G}^n$ the set of all generalized cooperative games with players set $N$. $\mathcal{G}^n$ is a vector space with dimension $|\Omega(N)| - 1$. Let us denote $\mathcal{G}^n$ the subspace of $\mathcal{G}^n$ consisting of all games for which $v(T) = v(R)$ if $H(T) = H(R)$ holds. $\mathcal{G}^n$ will be identified with the $2^n - 1$ dimensional space of cooperative games in characteristic function form or Transferable Utility games (TU-games). Intuitively, for games in $\mathcal{G}^n$, the order in which the coalitions are formed is irrelevant.

Each ordered coalition $T = (i_1, \ldots, i_r) \in \Omega(N)$ establishes a strict linear order $\prec_T$ in $H(T)$, defined as follows. For all $i, j \in H(T)$, $i \prec_T j$ (i precedes $j$ in $T$) if and only if there exists $k, l \in \{1, \ldots, t\}$, $k < l$, such that $i = i_k, j = i_l$.

Based on this strict linear order, we define an inclusion relation in $\Omega(N)$ in this way: for $A, B \in \Omega(N)$ we will say that $A$ is included in $B$ (and we will denote $A \subset B$) if $H(A) \subset H(B)$ and, for all $i, j \in H(A)$ such that $i \prec_T j$, $i \prec_B j$ holds. Given an ordered coalition $T \in \Omega(N)$ and $i \in H(T)$ we will denote $i^T$ for the position of player $i$ in coalition $T$.

In this paper, an usual basis of $\mathcal{G}^n$, the generalized unanimity basis, consisting of the (generalized) unanimity games $(w_T)_{T \in \Omega(N)}$, will often be used. For any $T \in \Omega(N) \setminus \{\varnothing\}$, the generalized characteristic function $w_T$ is defined as follows:

\[
\begin{align*}
    w_T(R) &= \begin{cases} 
    1 & \text{if } T \subset R \\
    0 & \text{otherwise.}
    \end{cases}
\end{align*}
\]

for all $R \in \Omega(N)$.
The family \(|u_S|, S \neq N|N\) defined:

\[ u_S = \sum_{T \in \Omega(S)} w_T, \quad \text{for each } S \in 2^N \setminus \{\emptyset\}, \]

is a basis of \(G_N\), that we will identify with the unanimity basis of the classical TU games. For a given \(v \in G_N\), \(|\Delta^u(T)|, T \in \Omega(N)\) is the set of the generalized unanimity coefficients of \(v\) (the coordinates of \(v\) in the generalized unanimity basis). Sánchez and Bergantiños (1997) proved that, for all \(T \in \Omega(N)\setminus \emptyset\):

\[ \Delta^u_T = \sum_{R \in T, R \neq T} \Delta^u_T(R) = \sum_{R \in T} (-1)^{\#T-R} v(R). \]

Given \(v \in G_N\) and \(S \subset N\), \(S \neq \emptyset\), \(|\Delta^u_T(S)| = \Delta^u(S)\) holds for all \(T \in \Omega(S), \{\Delta^u_T(S)\} \neq \emptyset \subset N\) being the coordinates of \(v\) in the basis \(|u_S|, S \neq \emptyset \subset N\). These coefficients are known as the Harasanyi divindends (Harsanyi, 1963). The coefficients \(\Delta^u_T(S)\) (and the \(\Delta^u(S)\) too) can be interpreted as the synergy earnings of cooperation for the coalition that were not yet obtained by its proper subcoalitions.

A game \(v \in G_N\) is \(0\)-normalized if \(v(|i|) = \Delta_T(|i|) = 0\) for all \(i \in N\). A game in \(G_N\) is totally positive (Vasilev, 1975, 1981) if all its Harasanyi dividends are nonnegative, and it is almost positive if previous condition is satisfied for dividends of coalitions with at least two members. \(AP_N\) will denote the subset of all almost-positive games in \(G_N\). Obviously, a \(0\)-normalized game \(v\) is almost-positive if and only if it is totally positive. The generalization of these definitions to \(G_N\) is straightforward.

Recently, totally positive games have gained popularity as in many applications of cooperative game theory to economic allocation problems, such as auction games, river games and queuing games, the game is totally positive (see, for example, van den Brink et al., 2009). Moreover, totally positive games have appealing properties. First, for these games the Harasanyi set equals the core (Vasilev, 1981; Derks et al., 2000; Vasilev and van der Laan, 2002) and, second, the set of almost-positive games describes the whole class of cooperative games from the lattice point of view (Llerena and Rafels, 2006). As a consequence, the core of a game can be expressed as the intersection of the cores of almost-positive games.

In this paper, we will use other classes of games in \(G_N\): A game \(v \in G_N\) is superadditive if for all coalitions \(S, T \subset N\) with \(S \cap T = \emptyset\), \(v(S \cup T) \geq v(S) + v(T)\) holds; it is convex if for all coalitions \(S, T \subset N\), \(v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\) holds and it is symmetric if for every coalition \(S \subset N\), \(v(S)\) depends only on the cardinality of \(S\) and, thus, there is some \(f : N \rightarrow \mathbb{R}\) such that \(v(S) = f(|S|)\). The subspace of \(G_N\) formed by symmetric games will be denoted \(S^N\), \(S^N \subset G_N\) will be the subspace of games that are also symmetric and \(0\)-normalized.

In their seminal paper on games in generalized characteristic function form, Nowak and Radzik (1994) define and characterize a value \(\psi_N^R\) for these games. For each \(v \in G_N\) and all \(i \in N\) this value is given by:

\[ \psi^R_i(N, v) = \sum_{S \in N \setminus \{i\}} \sum_{(T, 1, (i_1, i_2, \ldots, i_t), \emptyset) \in \Omega(S)} \frac{(n-t-1)!}{n!} (v(i_1, i_2, \ldots, i_t, i) - v(T)). \]

An alternative expression for this value (based on the generalized unanimity coefficients of \(v\)) is:

\[ \psi^R_i(N, v) = \sum_{T \in \Omega(N), i \in T} \frac{\Delta^u_T}{|T|}. \]

Later, Sánchez and Bergantiños (1997) define and study another value for games in \(G_N\), differing from the former in the dummy player axiom and needing also a symmetry axiom. This value can be obtained from the two alternative equivalent expressions:

\[ \psi^R_i(N, v) = \sum_{T \in \Omega(N), i \in T} \frac{\Delta^u_T}{|T|}. \]

In this paper we use a parametric family of functions defined on \(G_N\), \(\psi^R_{\alpha}(T)\), \(\alpha \in [0, 1]\). Each one of them can be considered as a particular point solution. Given \(\alpha \in [0, 1]\), \(\psi^R_{\alpha}\) is defined, for each generalized TU-game \((N, v) \in G_N\) and all \(i \in N\) by:

\[ \psi^R_{\alpha}(N, v) = \sum_{T \in \Omega(N), i \in T} \frac{\Delta^u_T}{|T|}. \]

The defined family includes the values \(\psi^R_{0}\) and \(\psi^R_{1}\). In particular, \(\psi^R_{0} = \psi^R (0)\) (it is implicit in (1) the abuse of notation \(0^0 = 1\)) and \(\psi^R_{1} = \psi^R_{1}\).

The idea behind this family of values is that, in a unanimity game \(v\), we share the reward among the non-dummy players, neither equally (as Sánchez and Bergantiños does), nor all for the last one (as Nowak and Radzik does), but proportionally to the player position in \(T\).

All these values, when applied to games in \(G_N\), coincide with the Shapley value for classical TU games:

\[ \phi_i(N, v) = \sum_{S \in N \setminus \{i\}} \frac{(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S)), \quad \text{for all } i \in N. \]

An alternative expression for the Shapley value is:

\[ \phi_i(N, v) = \sum_{S \in N \setminus \{i\}} \frac{\Delta^u(S)}{|S|}, \quad \text{for all } i \in N. \]

2.2. Graphs and directed graphs

A directed graph or digraph is a pair \((N, d)\), \(N = \{1, 2, \ldots, n\}\) be a set of nodes and \(d\) a subset of the collection of all ordered pairs \((i, j)\), \(i \neq j\), of elements of \(N\). Each pair \((i, j)\) in \(d\) is called an arc. In the following, if there is no ambiguity with respect to \(N\), we will refer to the digraph \((N, d)\) as \(d\). We will denote \(D_N^d\) for the set of all possible digraphs with nodes set \(N\).

Given a digraph \((N, d)\), if \((i, j) \in d\), we will say that \(i\) is directly connected with \(j\). Obviously, if \(i\) is directly connected with \(j\), the reverse is not necessarily true. If \(i\) is not directly connected with \(j\) in the digraph, it may still be possible to connect them, provided that there are other nodes through which we can do so. We will say that \(i\) is connected with \(j\) in the digraph \((N, d)\) if there is a directed path connecting them, i.e., there exists an ordered sequence of nodes in \(N\), \((i_1, i_2, \ldots, i_t)\), such that \(i_1 = i, i_t = j\) and \((i_l, i_{l+1}) \in d\) for all \(l \in \{1, 2, \ldots, t - 1\}\).

We will say that an ordered set \(T = (i_1, i_2, \ldots, i_t) \in \Omega(N)\) is connected in the digraph \((N, d)\) if, for all \(l = 1, \ldots, t - 1, i_l\) is directly connected with \(i_{l+1}\) in the digraph \((N, d)\).
Given \((N, d)\), two nodes \(i, j \in N\) are in the same component if it exists a sequence of nodes in \(N\), \(i = i_1, i_2, \ldots, i_s = j\) such that for \(l = 1, 2, \ldots, s - 1\), \(i_l \in d\) or \(i_{l+1} \in d\).

So, given a digraph \((N, d) \in D^N\) we will establish a partition of \(N\) in components. We will denote by \(N/d\) the set of all the components of the directed graph \((N, d)\).

Let us observe that, given a component \(C \in N/d\), any \(T \in \Omega(C)\) can be a not connected ordered set in \(N\).

Then the connection concept for components we use is clearly weaker than the one used for ordered sets. We will say that the digraph \(d \in D^N\) is (weakly) connected if \(|N/d| = 1\).

In the following we will use, for short, \((N, d^+)\) instead of \((N, d \setminus \{(i, j)\})\) for the digraph obtained when removing the arc \((i, j)\) from \((N, d)\).

### 2.3. Digraph communication situations

A digraph communication situation (or directed communication situation\(^2\)) is a triplet \((N, v, d)\) where \((N, v)\) is a TU-game in \(G^N\) and \((N, d)\) is a digraph in \(D^N\). We will denote \(D_N\) for the family of all directed communication situations with nodes-players set \(N\).

The subset of \(D_N\) corresponding to directed communication situations in which the game is a symmetric one will be denoted \(D_{CS}^N\), whereas \(D_{NS}\) will correspond to directed communication situations in which the game is simultaneously almost positive and symmetric. The subsets of \(D_{CS}^N\) and \(D_{ND}^N\) formed by directed communication situations in which the game is 0-normalized will be denoted \(D_{CS}^N\) and \(D_{ND}^N\), respectively.

### 3. A family of centrality measures for digraph communication situations

#### 3.1. Definition of the centrality measures

In order to define centrality measures for digraphs using a game theoretical approach, let us first consider a digraph communication situation \((N, u, d) \in D_N\), \(S \subseteq N\), with \(u\) a unanimity game. The restrictions in the communication modeled by a digraph affect the outcome of several coalitions and thus a new game arises to take into account these constraints. This is the classical approach to study the problem of games with restrictions in the communications (Myerson, 1977, 1980).

Given the directed communication situation \((N, u, d)\), the digraph-restricted game \((N, u^d)\) can be interpreted as the game to connect \(S\) in \(d\), and in our approach, we propose to consider all the connected ordered coalitions \(T \in \Pi(S)\). This leads to define the game \((N, u^d)\) as the generalized TU one with (generalized) characteristic function:

\[
u^d = \sum_{T \in \Pi(S) \in d} w_T.
\]

We will extend the previous definition to \(D_N\) by linearity and thus, given \((N, v, d) \in D_N\), we define the digraph-restricted game \((N, v^d) \in G^N\) as the one with generalized characteristic function:

\[
u^d = \sum_{\sigma \in \Omega(S) \in d} \Delta(\sigma) v^d.
\]

In the next proposition we will give an expression for \(v^d\) in terms of \(v\). Its proof is a direct consequence of a Sánchez and Bergantiños (1997)'s result, and so it is omitted.

**Proposition.** Given \((N, v, d) \in D_N\), the generalized characteristic function of the digraph restricted game \((N, v^d)\) is given by:

\[
u^d(T) = \sum_{\sigma \in \Omega(S) \in d} \lambda^d(\sigma) v(S),
\]

with \(\lambda^d(\sigma) = \sum_{R \in K \in T, K \in \sigma} (-1)^{k+r}.

Given \((N, v, d) \in D_N\) and \(\alpha \in [0, 1]\), we can consider the difference:

\[
K^d(\alpha)(N, v, d) = \Psi^\alpha(N, v^d) - \psi(N, v),
\]

as a measure of the centrality of node \(i \in N\). As it is obvious from the previous expression, we are assuming that the centrality of a given node can be measured as the difference in its allocation when the restrictions in the communication given by the digraph are taken into account and the corresponding allocation when the restrictions do not exist.

Interpreting the Shapley value and \(\Psi^\alpha\), for all \(\alpha \in [0, 1]\), as indices of power in \(G^N\) and \(G^d\), respectively, the defined centrality measure for a given node can be viewed as the variation in its power due to its position in the digraph.

In order to avoid a priori differences among players given by different status in the original game, we propose to use symmetric games. As a consequence, the term \(\psi(N, v)\) will be equal for all players and then, removing it, only a shift transformation will be produced. Another shift transformation (that will permit us to associate null centrality to isolated nodes) is obtained when replacing each symmetric game by its 0-normalized version. Then, when defining the centrality, we will restrict ourselves to symmetric and 0-normalized games.

From now on, we will use this alternative definition of centrality.

**Definition.** Given \((N, v, d) \in D_N\) and \(\alpha \in [0, 1]\), the centrality of node \(i \in N\) with respect to game \(v\) (the centrality of node \(i\), for short) is defined:

\[
k^d(\alpha)(N, v, d) = \Psi^\alpha(N, v^d).
\]

In the previous definition, \(\alpha \in [0, 1]\) evaluates the asymmetry of the arcs impact and can be viewed as a discount factor of the importance of the initiator node versus the receiver one. Of course, choosing a value for \(\alpha\) is a critical aspect of the defined family of measures.

This value can be obtained from empirical information, from the relative importance to be initiator in a relation... According to the proposed centrality measure, being receiver is preferred to being initiator. This is so when, for instance, the relation is a citation of a scientific paper, or when the initiator is a subordinate of the receiver as in games with permission structures (van den Brink, 1997). Nevertheless, social networks are usually seen as the paths through which agents transmit information to their neighbors. With this interpretation, being the first in the chain of information transmission will be always preferred. Obviously, if we prefer this last interpretation, similar results to those proposed here can be obtained replacing \(\alpha\) with \(1/\alpha\).

**Remark.** As a consequence, the special case \(\alpha = 1\) can be interpreted as the one in which both incident nodes in an arc play symmetrical roles. Then, it is natural to assume that switching these roles for all players, the respective centralities are not affected, i.e.: \(k(\alpha)(N, v, d) = k(\alpha)(N, v, d)\), \(\tilde{d}\) being the digraph defined \((\tilde{d}, i, j) \in N, (i, j) \in d\).

The proof of this statement is trivial.

---

\(^2\) The model of a directed communication situation \((N, v, d)\) with \((N, v)\) a TU-game and \((N, d)\) a digraph appears, as a game with a permission structure, in Gilles et al. (1992), and as a game under precedence constraints in Faigle and Kern (1993). This denomination is also used in Slikker and van den Nouweland (2001), but with a different meaning.
3.2. Definition of properties

In a sociological context it is assumed that adding an arc between two given nodes increase (or at least do not decrease) their centralities. In the case of allocation rules for communication situations, this property is known as stability.

**Definition 4.** Given \( A \subset \mathcal{DCS}_0^N \), a function \( \xi : A \to \mathbb{R}^n \) is stable if, for all \((N, v, d) \in A \) and \((i, j) \in d'\):

\[
\xi_i(N, v, d) \geq \xi_j(N, v, d'), \quad \text{for } i, j.
\]

Unfortunately, in order to meet this property, we must restrict ourselves to use games in a subset of \( G^N \). The next example illustrates this point.

**Example 5.** Consider the directed communication situation \((N, v, d)\) with \( N = \{1, 2, 3, 4, 5, 6, 7\}\), \( v \) the symmetric and super-additive game \( v(S) = \sum_{k=2}^{7} \binom{s}{k} (-1)^k \), and \( d = \{(2, 3), (2, 4), (2, 5), (2, 6), (2, 7)\} \). Consider also the directed communication situation \((N, v, d')\) with \( d' = d \cup \{(1, 2)\} \), as in Fig. 1.

As \( v = \sum_{S \subseteq N, |S| = 2} (-1)^{|S|} \mu_S \), and using the definition of the restricted game:

\[
v^d = v(1,2) + v(2,3) + v(2,4) + v(2,5) + v(2,6) + v(2,7),
\]

\[
v^{d'} = v(1,2) + v(2,3) + v(2,4) + v(2,5) + v(2,6) + v(2,7) - v(2,1,3) - v(2,1,4) + v(2,1,5) - v(2,1,6) + v(2,1,7)
\]

and thus, for all \( \alpha \in [0, 1] \),

\[
\kappa^v_0(N, v, d) = 0, \quad \text{and} \quad \kappa^v_1(N, v, d') = \frac{\alpha}{2(1 + \alpha)} - \frac{5\alpha^2}{3(1 + \alpha + \alpha^2)}.
\]

It is easy to see that, if \( \alpha > \frac{\sqrt{2} - 1}{2} \), then \( \kappa^v_0(N, v, d) > \kappa^v_1(N, v, d') \).

The previous example points out an (a priori) undesirable aspect of the proposed measures, that vanishes if we restrict ourselves to the family (cone) \( A\mathcal{DCS}_0^N \subset G^N \).

A symmetrical, 0-normalized, and almost-positive game \( v \in G^N \) can be written: \( v = \sum_{k=2}^{n} \mu_k v_k \), with \( \mu_k \geq 0 \) and \( v_k = \sum_{t \in \mathcal{T}} t u_T \), and thus, for all \( S \subset N \):

\[
v_k(S) = \begin{cases} \lfloor k \rfloor \binom{s}{k} \quad &k \leq s \leq n \quad \text{for all } k = 2, \ldots, n. \quad \text{(2)} \\ 0 &k = s + 1, \ldots, n. \end{cases}
\]

This is so because, for a 0-normalized game \( v \), \( \Delta_0((i)) = 0 \) for all \( i \in N \); for a symmetric game, \( \Delta_0(S) \) depends only on \( s \), and for an almost positive game, \( \Delta_0(S) \geq 0 \) for all \( S \subset N \).

They have an intuitive interpretation as communication games in which the interests of players focus on the number of ordered groups that can be found or in the diffusion of information or in sending messages among players. In this sense, we consider them as appropriated games to measuring social interests and thus as a started point to measure centrality.

If we consider the characteristic function of a game \( v \in S^N_0 \) as a real function \( f, v \in A\mathcal{DCS}_0^N \) holds if the first \( n \) derivatives of \( f \) are nonnegative, presenting \( f \), at least, an exponential increase rate.

In the next subsection we will prove that, when choosing almost positive games, the defined measures satisfy stability.

In order to compare centralities of nodes belonging to different digraphs, it would be interesting to know the total centrality in a given digraph, i.e., the amount that we must allocate among the different actors in the network.

In the next example, we motivate the convenience of imposing the so called efficiency in connected digraphs property to our centrality measures.

**Example 6.** Consider the directed communication situation \((N, v, d)\) with \( N = \{1, 2, 3\} \), \( v = v_2 \) (i.e.: \( v(S) = s(s - 1) \) for all \( S \subset N \) with \( s \geq 1 \), and \( d = \{(1, 2), (2, 3)\} \)).

As it is said, \( v_2(S) \) can be interpreted as the number of potential directed information transmission channels between (ordered) pairs of actors in \( S \); that is, the number of different ordered pairs of elements in \( S \).

With the restrictions in the communication imposed by the digraph \( d \), the restricted game results \( v^d = 2v_2(1,2) + v_2(3,2) \) and thus, actors have only 2 communication channels between (ordered) pairs of elements in \( S \).

As it is said, \( v_2(S) \) can be interpreted as the number of potential directed information transmission channels between (ordered) pairs of actors in \( S \); that is, the number of different ordered pairs of elements in \( S \).

**Definition 7.** Given \( A \subset \mathcal{DCS}_0^N \), a function \( \xi : A \to \mathbb{R}^n \) is efficient in connected digraphs if, for all \((N, v, d) \in A \) with \((N, d)\) a connected digraph:

\[
\sum_{i \in N} \xi_i(N, v, d) = \sum_{T \in \Pi(N)} \frac{v^d(T)}{n!}.
\]

Another question of interest is the impact of removing a directed relation between two individuals has in their centralities. As we will prove in the next subsection, the defined family of measures satisfies the \( \alpha \)-directed fairness property, covering the possibility of an asymmetrical impact. This property states that, assuming the change in the centrality as a result of the removal of an existing arc can be different for the initiator and the receiver, it holds that the ratio between these two differences (if they are non-null) is the same for all directed communication relations. So, the initiator of a severed directed communication relation experiences a lesser (or equal, if \( \alpha = 1 \)) influence on his centrality than the receiver does. Of course, changing \( \alpha \) to \( 1/\alpha \) a similar interpretation, in which initiator and receiver exchange their roles, can be obtained.

**Definition 8.** Given \( \alpha \in [0, 1] \) and \( A \subset \mathcal{DCS}_0^N \), a function \( \xi : A \to \mathbb{R}^n \) satisfies the \( \alpha \)-directed fairness property if, for all \((N, v, d) \in A \) and
i, j ∈ N such that (i, j) ∈ d:

$$\xi_i(N, v, d) - \xi_i(N, v, d^0) = \alpha[\xi_i(N, v, d) - \xi_i(N, v, d^0)].$$

There exists a relation between $\alpha$-directed fairness and fairness for games with a permission structure. For example, taking $\alpha = 1$, the 1-directed fairness property is related to disjunctive fairness in van den Brink (1997).

3.3. Results

This subsection includes three results, whose proofs appear in Appendix A: the first one showing that the proposed measures satisfy stability, second one showing that it is the unique measure satisfying (under the suitable conditions) component efficiency and $\alpha$-directed fairness properties, and the last one deals with the particular case $\alpha = 0$.

Proposition 9. For each $\alpha \in [0, 1], \kappa^\alpha : \mathcal{D}_{AC}^N \rightarrow \mathbb{R}^n$ is stable.

Theorem 10. For each $\alpha \in [0, 1], \kappa^\alpha : \mathcal{D}_{AC}^N \rightarrow \mathbb{R}^n$ is the unique function defined on $\mathcal{D}_{AC}^N$ satisfying component efficiency and $\alpha$-directed fairness.

The next example proves that the two properties in the statement of Theorem 10 are not sufficient to guarantee the uniqueness of $\kappa^0$.

Example 11. Given $(N, v, d) \in \mathcal{D}_{AC}^N$, let $N(d) = \{C_1, \ldots, C_k\}$. Consider the digraphs $(C_k, d^k)$, $k = 1, 2, \ldots, k$, where, for each $i \in C_k$, $d^k = (i, j)$, $j \neq i$. In this case, $i$ is the out-star with hub at $i$ and satellites the remaining nodes in $C_k$.

Let us define as follows a function $\xi$ on $\mathcal{D}_{AC}^N$. Given $(N, v, d) \in \mathcal{D}_{AC}^N$, for each $i \in N$, let $C_k$ be the component of $(N, d)$ to which $i$ belongs.

$$\xi_i(N, v, d) = \kappa_i^0(N, v, d) + \sum_{d^k \in \delta} [\xi_i(N, v, d^k) - \kappa_i^0(N, v, d^k)],$$

where:

$$\xi_i(N, v, d^k) = \begin{cases} 0 & \text{if } i = i_k \\ b_i & \text{if } i \neq i_k, \end{cases}$$

with $b_i \in \mathbb{R}$, and $\sum_{i \neq i_k} b_i = \sum_{T \in \Pi(C_k)} \nu(T) / c_k!$.

Let us recall $\kappa_i^0(N, v, d) = v(\{i\}) = 0$ for the hub of an out-star. Because of the 0-directed fairness, we can sequentially remove all the arcs without change of the $i$-values and thus, $\kappa_i^0(N, v, d) = \kappa_i^0(N, v, d^0)$. Then, by component efficiency, $\kappa_i^0(N, v, d) = v(\{i\}) = 0$. Choosing the values $b_i, i \neq i_k$ in (3) in an appropriate way (this is always possible when there exists $k$ with $c_k \geq 3$), we have that $\xi$ differs from $\kappa^0$.

Let us now prove that this new function, $\xi$, as $\kappa^0$ does, satisfies component efficiency and 0-directed fairness.

The efficiency is given by the fact that:

$$\sum_{i \in C_k} \xi_i(N, v, d) = \sum_{i \in C_k} \kappa_i^0(N, v, d) + \sum_{i \in C_k} \sum_{d^k \in \delta} [\xi_i(N, v, d^k) - \kappa_i^0(N, v, d^k)]$$

$$= \sum_{i \in C_k} \kappa_i^0(N, v, d) + \sum_{d^k \in \delta} \sum_{i \in C_k} \xi_i(N, v, d^k) - \kappa_i^0(N, v, d^k)$$

$$= \sum_{T \in \Pi(C_k)} \nu(T) / c_k!.$$
Example 15. Let us consider the directed communication situation \( (N, v, d) \), with \( N = \{1, 2, 3\} \), \( v = v_3 \) and \( d = \{(1, 2), (2, 3)\} \). In this case, only one ordered communication is feasible (given the digraph): \( (1, 2, 3) \). If \( \alpha = 1 \), the three nodes have equal contribution in this communication, and thus:

\[
\kappa^1(N, v, d) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)
\]

Nevertheless the role of node 1 is to initiate the communication, node 2 intermediates and node 3 receives the communication.

**Definition 16.** Given \( (N, v, d) \in \mathcal{D}^N_{r=0} \), we define, for each \( \alpha \in [0, 1] \), the emission component of the centrality \( \kappa^\alpha \) of node \( i \) as:

\[
e^\alpha_v(N, v, d) = \frac{\sum_{S \subseteq N} \Delta_v(S) \sum_{T \in \Pi(S) \cap \mathcal{C}_d(S), k(T) = 1} \psi^\alpha_v(N, w_T)}{\sum_{S \subseteq N} \Delta_v(S) \sum_{T \in \Pi(S) \cap \mathcal{C}_d(S), k(T) = 1} \sum_{k=0}^{t-1} \alpha^k}.
\]

**Definition 17.** Given \( (N, v, d) \in \mathcal{D}^N_{r=0} \), we define, for each \( \alpha \in [0, 1] \), the betweenness component of the centrality \( \kappa^\alpha \) of node \( i \) as:

\[
b^\alpha_v(N, v, d) = \frac{\sum_{S \subseteq N} \Delta_v(S) \sum_{T \in \Pi(S) \cap \mathcal{C}_d(S), 1 < k(T) < t} \psi^\alpha_v(N, w_T)}{\sum_{S \subseteq N} \Delta_v(S) \sum_{T \in \Pi(S) \cap \mathcal{C}_d(S), 1 < k(T) < t} \sum_{k=0}^{t-1} \alpha^k}.
\]

**Definition 18.** Given \( (N, v, d) \in \mathcal{D}^N_{r=0} \) and \( \alpha \in [0, 1] \), the reception component of the centrality \( \kappa^\alpha \) of node \( i \) is given by:

\[
r^\alpha_v(N, v, d) = \frac{\sum_{S \subseteq N} \Delta_v(S) \sum_{T \in \Pi(S) \cap \mathcal{C}_d(S), k(T) = t} \psi^\alpha_v(N, w_T)}{\sum_{S \subseteq N} \Delta_v(S) \sum_{T \in \Pi(S) \cap \mathcal{C}_d(S), k(T) = t} \sum_{k=0}^{t-1} \alpha^k}.
\]

Note that, in the case \( \alpha = 0 \), \( \kappa^0(N, v, d) = \rho^0(N, v, d) \) and then, the decomposition is inessential.

The defined components are obtained from the value assigned to \( i \) by its position in the ordered connected sets. The first position permits to initiate the communication, the intermediate positions are clearly associated with betweenness and finally, to be at the end of each connected chain transforms a node in a receiver one. These ideas justify the chosen notation and its interpretation.

The next result establishes that these centrality components give us an additive decomposition of \( \kappa^\alpha \).

**Proposition 19.** For each \( (N, v, d) \in \mathcal{D}^N_{r=0} \) and \( \alpha \in [0, 1] \) we have,

\[
\kappa^\alpha(N, v, d) = e^\alpha(N, v, d) + b^\alpha(N, v, d) + r^\alpha(N, v, d).
\]

Let us turn now our attention to some intuitive properties of these components. The following propositions, whose proofs appear in Appendix A, refer to three particular networks (in-star, out-star and oriented chain) and show us the proper behavior of the components vs. the defined centrality measures.

**Proposition 20.** Given \( (N, v, d^S) \in \mathcal{D}^N_{r=0} \) and \( d^S = \{(i, 1), i \in N \setminus \{1\}\} \), for all \( (N, d) \in \mathcal{D}^N \) and \( \alpha \in [0, 1] \):

\[
\rho^\alpha_v(N, v, d) \leq \frac{\rho^\alpha_v(N, v, d)}{\kappa^\alpha_v(N, v, d)}, \quad i = 1, 2, \ldots, n.
\]

Symmetrically we obtain the next result for emission centrality and out-stars.

**Proposition 21.** Given \( (N, v, d^S) \in \mathcal{D}^N_{r=0} \) and \( d^S = \{(1, i), i \in N \setminus \{1\}\} \), for all \( (N, d) \in \mathcal{D}^N \) and \( \alpha \in [0, 1] \):

\[
e^\alpha_v(N, v, d) \leq \frac{e^\alpha_v(N, v, d)}{\kappa^\alpha_v(N, v, d)}, \quad i = 1, 2, \ldots, n.
\]

Finally, we will prove that, in an oriented chain, for almost positive and symmetrical games, the emission centrality is maximal at the initial node, whereas reception centrality is maximal at the last node. Moreover, betweenness centrality increases from the first node to the median one.

**Proposition 22.** Given \( (N, v, d) \in \mathcal{D}^N_{r=0} \) and \( d = \{(1, 2), (2, 3), \ldots, (n-1, n)\} \) (odd) an oriented chain, for all \( \alpha \in [0, 1] \):

\[
e^\alpha_v(N, v, d) \leq \rho^\alpha_v(N, v, d), \quad i = 1, 2, \ldots, n.
\]

**Corollary 23.** Given \( (N, v, d) \in \mathcal{D}^N_{r=0} \) with \( d = \{(1, 2), (2, 3), \ldots, (n-1, n)\} \) an oriented chain (odd), we have:

\[
e^1_v(N, v, d) \geq \rho^1_v(N, v, d), \quad i = 1, 2, \ldots, n.
\]

5. Conclusions

This paper proposes a new instrument to measure nodes centrality in directed networks, following our previous work on the undirected case (Gómez et al., 2003). Actors in the network are simultaneously players in a cooperative game. This game represents the interests that motivate the interactions among actors, whereas a directed network imposes restrictions in the cooperation. Given the (directed) network and the game, a new (generalized) restricted game is obtained.

In the literature there are many centrality measures, but they usually assume the links are undirected, i.e., both agents connected by a link equally benefit from its existence. In the present paper, we consider the directed case.

In our proposal, the centrality of nodes is measured as the difference between:

(i) the Shapley value (this is assumed to be the node’s outcome when no digraph is binding the interaction among agents), and

(ii) a new value for generalized TU games belonging to a parametric family that contains the two existing ones defined by Nowak and Radzik (1994); Sánchez and Bergantiños (1997). This new index is interpreted as the node’s outcome considering the restrictions imposed by the directed network. In particular, this index weights the specific ranking among agents derived from the directed network structure.
The paper discusses several properties of these measures. For each \( \alpha \in [0, 1] \), the corresponding measure is characterized in terms of efficiency and \( \alpha \)-directed fairness properties. In the case \( \alpha = 0 \), a symmetry axiom is also needed.

This characterization must be interpreted in a double sense: first, it fixes the range of variation of our measure and second, it highlights the fact that the initiator and the receiver do not necessarily play symmetrical roles in the type of analyzed relations.

The defined centrality perhaps can be thought as a vector measure instead of a scalar one. With this idea in mind, we additively decompose the centrality of each node in three submands that can be seen as components or factors of the centrality in the dimensions of emission, reception and betweenness. Each one of the proposed measures is, then, the modulus, using the city-block distance, of the three-dimensional vector given by its components.

Acknowledgements

This research has been supported by the “Plan Nacional de I+D+i” of the Spanish Government, under the project MTM2008-06778-C02-02/MTM. The authors would like to thank two anonymous referees for their helpful comments.

Appendix A.

Proof of the Proposition 3.2. Consider first the unanimity games \((N, u_s), S \subseteq N, s \geq 2\). Obviously \((N, u_s) \in \mathcal{AP}^N \cap \mathcal{S}_d^N\). Then, for all \(i \in N\) and all \(\alpha \in [0, 1]\),

\[
\kappa_\alpha^i(N, u_s, d) = \sum_{T \in \Pi(S) \cap c_d^N} \Psi_\alpha^i(N, w_T) = \kappa_\alpha^i(N, u_s, d^i),
\]

the inequality holding because \(c_d^N \subset \mathcal{S}_d^N\) and \(\Psi_\alpha^i(N, w_T) \geq 0\). The result for a more general directed communication situation \((N, v, d) \in \mathcal{DC}_{\mathcal{AP}^N}\) follows using the linearity of the measure and the fact that all dividends are nonnegative. As a straightforward consequence, we obtain the stability of the defined measures. \(\square\)

Proof of the Theorem 3.1. Let us first prove that \(\kappa_\alpha\) satisfies component efficiency. For each \(\alpha \in [0, 1]\), \(\sum_{i \in N} \kappa_\alpha^i(N, v, d) = \sum_{i \in N} \Psi_\alpha^i(N, v, d^i)\). So, we need to prove the average efficiency of \(\Psi_\alpha\). In fact, for all \((N, w) \in \mathcal{C}_d^N:\)

\[
\sum_{i \in N} \Psi_\alpha^i(N, w) = \sum_{i \in N} \sum_{T \in \Omega(N)} \Delta_\alpha^i(N, w_T) = \sum_{i \in N} \sum_{T \in \Omega(N)} \Delta_\alpha^i(N, w_T) = \sum_{T \in \Omega(N)} \frac{1}{|T|} \sum_{i \in N} (-1)^{|T|-r} w(R).
\]

Moreover, taking \(l = t - r:\)

\[
\sum_{\theta \neq R \subseteq T} \frac{(-1)^{t-r}}{l!} = \sum_{l=0}^{n-r} \binom{r+l-1}{l} \frac{(-1)^{t-l}}{(r+l)!} = \frac{1}{l!} \sum_{l=0}^{n-r} \binom{n-r}{l} (-1)^{l} = \begin{cases} 1 \quad \text{if } r = n \\ 0 \quad \text{otherwise} \end{cases}
\]

and therefore:

\[
\sum_{i \in N} \Psi_\alpha^i(N, w) = \sum_{R \in \Pi(N)} w(R).
\]

And, taking \(w = \nu^d\), component efficiency of \(\kappa_\alpha\) is obtained. With the aim to prove the \(\alpha\)-directed fairness property, let us first consider, for each \(S \subseteq N\) with \(s \geq 2\), the unanimity game \((N, u_s)\). Then,

\[
u_d^i - \nu_d^{h_i} = \sum_{T \in \Pi(S) \cap c_d^N} \Psi_\alpha^i(N, w_T),
\]

and therefore:

\[
k_\alpha^i(N, u_s, d) - k_\alpha^i(N, u_s, d^h) = \sum_{T \in \Pi(S) \cap c_d^N} \alpha^{t-(|T|)} \sum_{l=0}^{n-r} \binom{r+l-1}{l} (-1)^{l} = \alpha \sum_{T \in \Pi(S) \cap c_d^N} \alpha^{t-(|T|)+1} \sum_{l=0}^{n-r} \binom{r+l-1}{l} (-1)^{l} = \alpha \kappa_\alpha^i(N, u_s, d) - k_\alpha^i(N, u_s, d^h).
\]

As \(\kappa_\alpha\) is linear in \(\nu\), it also satisfies the \(\alpha\)-directed fairness property.

Conversely, suppose now \(\alpha \in [0, 1]\) and \(\xi^\alpha: \mathcal{DC}_{\mathcal{AP}^N} \to \mathbb{R}^n\) is a function satisfying these two properties. We will prove, by induction on \(|d|\), the number of arcs in \((N, d)\), that \(\xi^\alpha(N, v, d) = k_\alpha^\alpha(N, v, d)\) for all \((N, v, d) \in \mathcal{DC}_{\mathcal{AP}^N}\) and all \(\alpha \in [0, 1]\).

If \(|d|=0, N/d = \{1\}, \ldots, \{n\}\) holds and thus, by component efficiency, \(\xi^\alpha(N, v, d) = v(\{i\}) = 0 = k_\alpha^\alpha(N, v, d)\) for all \(i \in N\).

Suppose, then, that \(\xi^\alpha(N, v, d) = k_\alpha^\alpha(N, v, d)\) for all \((N, v, d) \in \mathcal{DC}_{\mathcal{AP}^N}\) with \(|d|\leq m\), and consider \((N, v, d) \in \mathcal{DC}_{\mathcal{AP}^N}\) with \(|d| = m\). For all \((h, k) \in d\), using the induction hypothesis and the fact that both functions \(\xi^\alpha\) and \(\kappa_\alpha\) satisfy \(\alpha\)-directed fairness, we have:

\[
\xi^\alpha_h(N, v, d) - \alpha \xi^\alpha_k(N, v, d) = \xi^\alpha_h(N, v, d^{hk}) - \alpha \xi^\alpha_k(N, v, d^{hk}) = k_\alpha^\alpha_h(N, v, d^{hk}) - \alpha k_\alpha^\alpha_k(N, v, d^{hk}) = k_\alpha^\alpha_h(N, v, d) - \alpha k_\alpha^\alpha_k(N, v, d),
\]

and thus:

\[
\xi^\alpha_h(N, v, d) - k_\alpha^\alpha_h(N, v, d) = \alpha [\xi^\alpha_k(N, v, d) - k_\alpha^\alpha_k(N, v, d)].
\]

Consider \(i \in N\) and let \(C \subseteq N/d\) be the component to which \(i\) belongs. If \(|C| = 1\), it is trivial by component efficiency that \(\xi^\alpha_j(N, v, d) = k_\alpha^\alpha(N, v, d)\) for \(\alpha \in [0, 1]\). Suppose now there is \(j \in C, j \neq i\). Then, it exists a sequence \(i_0 = i, i_1, \ldots, i_{l+1} = i, j\) such that for \(t = 0, 1, \ldots, l+1, (i_t, i_{t+1}) \in d\) or \((i_t, j) \in d\), or both possibilities.

Let us define \(r_t(i, j) = \begin{cases} 1 \quad \text{if } (i_t, i_{t+1}) \in d \\ -1 \quad \text{otherwise} \end{cases}\).
Obviously, $r_i(i, j)$ depends on the considered sequence $i_0 = i_1$, \ldots, $i_l = j$, but this fact will be ignored in the notation. Then, repeatedly using (7):

$$
\xi_i^0(N, v, d) = \sum_{l=0}^{\infty} r_i(l, j)\xi_i^0(N, v, d) - \kappa_i^0(N, v, d) = \alpha \xi_i^0(N, v, d) - \kappa_i^0(N, v, d).
$$

Therefore,

$$
\sum_{j \in C} [\xi_i^0(N, v, d) - \kappa_i^0(N, v, d)] = \left(1 + \sum_{j \in C, j \neq i} \alpha \right) \sum_{k=0}^{\infty} \xi_i^0(N, v, d) - \kappa_i^0(N, v, d).
$$

But using the component efficiency, the left hand side in (8) equals zero. As $1 + \sum_{j \in C, j \neq i} \alpha \neq 0$ for all $\alpha \in [0, 1]$, we conclude that, for all $i \in N$:

$$
\xi_i^0(N, v, d) = \kappa_i^0(N, v, d),
$$

which completes the proof. \qed

**Proof of the Proposition 3.3.** It is obvious that $\kappa^0$ satisfies symmetry. Following the steps in the previous proof it is easy to see that $\kappa^0$ satisfies 0-directed fairness and component efficiency. Consider $\xi : T^{C_{\mathcal{N}^0}} \rightarrow R^n$, satisfying these three properties. We will prove, by backward induction on the cardinality of $d$, that $\xi(N, v, d) = \kappa_0(N, v, d)$ for all $(N, v) \in T^{C_{\mathcal{N}^0}}$.

If $d = K_0$ is the $n$-nodes complete digraph then, by symmetry and efficiency, $\xi(N, v, K_0) = v(N)/n = \kappa_0(N, v, K_0), i = 1, \ldots, n$.

Suppose $\xi$ coincides with $\kappa_0$ for all $(N, v, d)$ with $|d| \geq k$ and consider $(N, v, d)$ with $|d| = k - 1$. Then, for $i \in N$ and any $(i, j) \in K_n \setminus d$:

$$
\xi_i(N, v, d \cup \{(i, j)\}) - \xi_i(N, v, d) = 0,
$$

as $\xi$ satisfies 0-directed fairness. Therefore:

$$
\xi_i(N, v, d) = \xi_i(N, v, d \cup \{(i, j)\}) = \kappa_0(N, v, d \cup \{(i, j)\}.
$$

last equality holding due to the induction hypothesis. As $\kappa_0$ satisfies 0-directed fairness, $\kappa_0(N, v, d \cup \{(i, j)\}) = \kappa_0(N, v, d)$ and thus, $\xi_i(N, v, d) = \kappa_0(N, v, d)$.

\qed

**Proof of the Proposition 4.1.** If $(N, v, d) \in T^{C_{\mathcal{S}^0}}$, $\alpha \in [0, 1]$ and $i \in N$:

$$
\kappa_i^0(N, v, d) = \Delta_i(S) \sum_{T \in T^{S \cap C_{\mathcal{S}^0}}} \Psi_i^0(N, w_T)
$$

\begin{align*}
&= \Delta_i(S) \sum_{T \in T^{S \cap C_{\mathcal{S}^0}}} \Psi_i^0(N, w_T) + \left(\sum_{T \in T^{S \cap C_{\mathcal{S}^0}}} \Psi_i^0(N, w_T) - \sum_{T \in T^{S \cap C_{\mathcal{S}^0}}} \Psi_i^0(N, w_T)\right) + \sum_{T \in T^{S \cap C_{\mathcal{S}^0}}} \Psi_i^0(N, w_T) - \kappa_i^0(N, v, d) + \psi_i^0(N, v, d) + \rho_i^0(N, v, d)\end{align*}

and thus the result is proved. \qed

**Proofs of the Propositions 4.2 and 4.3.** It is obvious that for each $(N, v) \in T^{C_{\mathcal{N}^0}} \cap S_{\mathcal{N}^0}$, $\rho_i^w(N, v, d^3) = \kappa_i^w(N, v, d^3)$ and the left hand side of the inequality in the statement of the Proposition 4.2 is equal to 1. As the game $v$ is almost positive, $\rho_i^w(N, v, d) \leq \kappa_i^w(N, v, d)$ for all $i \in N$ and the result is proved.

Symmetrically we obtain the corresponding result for emission centrality. \qed

**Proof of the Proposition 4.4.** In order to simplify the proof, we will suppose an odd number of nodes in the chain. As for every $(N, v) \in T^{C_{\mathcal{N}^0}}$, $v$ is a linear combination of $(v_2, \ldots, v_n)$ with non-negative scalars, it is sufficient to prove the results for each $v_k$, $k = 2, \ldots, n$.

First, (4) holds because, for $k = 2, 3, \ldots, n$ and $\alpha \in [0, 1]$:

$$
e^0_i(N, v_k, d) = \begin{cases} \alpha^{k-1} & \text{for } i = 1, 2, \ldots, n - k + 1 \\ \sum_{r=0}^{k-1} \alpha^r & \text{for } i = n - k, \ldots, n. \end{cases}
$$

Analogously, (5) is satisfied as, for $\alpha \in [0, 1]$:

$$
\rho_i^w(N, v_k, d) = \begin{cases} \frac{1}{k-1} & i = k, \ldots, n \\ \sum_{r=0}^{k-1} \alpha^r & i = 1, 2, \ldots, k - 1. \end{cases}
$$

Finally, in order to prove (6), consider $\alpha \in [0, 1]$ and $i \in N$, $i \leq m$.

An ordered coalition of size $k$, with $3 \leq k \leq n$, containing $i$, connected in $(N,d)$ and where $i$ has a non extreme position is:

$$T(i, r, k) = (i - r + 1, i - r, \ldots, i - 1, i, i + 1, \ldots, i + k - r),$$

with max $(2, i + k - n) \leq r \leq \min \{i, k - 1\}$. Then, for $3 \leq k \leq n$,

$$\rho_i^w(N, v_k, d) = \sum_{r=\max\{2, i + k - n\}}^{\min\{i, k - 1\}} k!\psi_i^w(N, w_{T(i, r, k)}) + \sum_{r=\max\{2, i + k - n\}}^{\min\{i, k - 1\}} k!\psi_i^w(N, w_{T(i, r, k)}) + \sum_{r=\max\{2, i + k - n\}}^{\min\{i, k - 1\}} k!\psi_i^w(N, w_{T(i, r, k)}) \leq \sum_{r=\max\{2, i + k - n\}}^{\min\{i, k - 1\}} k!\psi_i^w(N, w_{T(i, r, k)}) + \sum_{r=\max\{2, i + k - n\}}^{\min\{i, k - 1\}} k!\psi_i^w(N, w_{T(i, r, k)})$$

last inequality holding because, for $r \leq i + k - m - 1$, we have $m \leq i + k - r - 1$ and thus, $m \in T(i, r, k)$ and

$$\psi_i^w(N, w_{T(i, r, k)}) = \sum_{l=0}^{\min\{i, k - 1\}} \alpha^{i-r} \sum_{l=0}^{\min\{i, k - 1\}} \alpha^{i-r} \leq \sum_{l=0}^{\min\{i, k - 1\}} \alpha^{i-r} \sum_{l=0}^{\min\{i, k - 1\}} \alpha^{i-r}$$

as $m(T(i, r, k)) \geq r$. 

\qed
On the other hand, (because m is the median node), for each
\( r = i + k - m, \ldots, \min \{ i, k - 1 \}, \)
\[
\psi^r_i(N, w_{T(m,r,k)}) = \sum_{k=0}^{\bar{\alpha}^k-r} \sum_{l=0}^{\bar{\alpha}^l} \psi^l_i(N, w_{T(m,r,k)}).
\]
and it exists \( T(m, r, k) \) such that \( \psi^m_i(N, w_{T(m,r,k)}) = \psi^r_i(N, w_{T(m,r,k)}). \)
Thus,
\[
\beta^m_i(N, v_k, d) \leq \sum_{r=\max(2,i+k-n)}^{\min\{i,k-1\}} k! \psi^m_i(N, w_{T(m,r,k)}) \leq \beta^m_i(N, v_k, d),
\]
this last inequality holding because \( \beta^m_i(N, v_k, d) \) contains at least all the terms in both summands in the left hand side, but possibly additional nonnegative terms. □

References