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The Riesz basis property of discrete operators and application to a Euler–Bernoulli beam equation with boundary linear feedback control

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In this paper, we give an abstract condition of Riesz basis generation for discrete operators in Hilbert spaces, from which we show that the generalized eigenfunctions of a Euler–Bernoulli beam equation with boundary linear feedback control form a Riesz basis for the state Hilbert space. As an consequence, the asymptotic expression of eigenvalues together with exponential stability are readily presented.

Keywords: discrete operator; Riesz basis; spectrum-determined growth condition.

1. Introduction

For a linear vibration system, the property that the (generalized) eigenvectors of the system form a Riesz basis for the state Hilbert space is one of the most important features from both theoretical and practical points of view. The validity of this property will naturally result in the most significant problems such as spectrum-determined growth conditions and exponential stability of the system. But unfortunately, it is not easy to establish such a property, even for the most extensively studied systems, such as Euler–Bernoulli beam equations (Chen *et al.*, 1987). A typical simple example is the following Euler–Bernoulli beam equation with fixed conditions at the left-hand end and linear boundary feedback control at the free end:

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad 0 < x < 1, t > 0 \\ y(0,t) = y_x(0,t) = 0, \quad t \geqslant 0, \\ y_{xx}(1,t) = -k_1 y_{xt}(1,t), \; y_{xxx}(1,t) = k_2 y_t(1,t), \quad t \geqslant 0. \end{cases} \tag{1}$$

Although a simple asymptotic analysis technique can be used to obtain a very good estimation for high eigenmodes of the system, the proof of its exponential stability for a special case $k_1 > 0, k_2 = 0$ involves a very complicated estimation for resolvent but far more to the Riesz basis generation (Chen *et al.*, 1988). The Riesz basis generation for system (1) in the case of $k_1 = 0, k_2 > 0$ was recently examined by Conrad & Morgul

(1998) who proved that the generalized eigenfunctions of the system in this case form a Riesz basis for the state space for almost every $k_2 > 0$. The success of this result relies on an earlier result due to Bari (see, e.g. Gohberg & Krein, 1969) that if $\{\phi_n\}_1^\infty$ is a Riesz basis in a Hilbert space H and another ω-linearly independent sequence $\{\psi_n\}_1^\infty$ in H is quadratically close to $\{\phi_n\}_1^\infty$ in the sense that

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then $\{\psi_n\}_1^\infty$ is a Riesz basis itself for H. The reason for the result being available only for almost every $k_2 > 0$ in Conrad & Morgul (1998) is that there is difficulty in understanding the numbers of generalized eigenfunctions corresponding to 'low eigenmodes'. This severely limits the application of Bari's theorem, even if we clearly know the behaviour of 'high eigenmodes' and their algebraic multiplicities.

The purpose of this paper is to overcome this difficulty. In the next section, an abstract result on Riesz basis generation for discrete operators in Hilbert spaces is developed, which avoids the need to understand the 'low eigenmodes' as was done in many papers (Conrad & Morgul, 1998; Rao, 1997). Thus, the 'low eigenmodes' can be completely ignored under some 'almost' necessary and sufficient conditions. To demonstrate its applications, the result is easily used in Section 3 to show that a set of generalized eigenfunctions of system (1) form a Riesz basis for the state Hilbert space for any real k_1, k_2. As a consequence, the spectrum-determined growth condition holds for system (1). Our method not only gives the Riesz basis property of the system, but also an asymptotic expression of eigenvalues which implies trivially the exponential stability of the system; this greatly simplifies the known proof for the exponential stability and asymptotic distribution of the eigenvalues (Chen *et al.*, 1988).

2. Riesz basis property of discrete operators

Let us recall that for a closed linear operator A in a Hilbert space H, $(0 \neq) x \in H$ is called a generalized eigenvector of A, corresponding to an eigenvalue λ of A with finite algebraic multiplicity, if there is a positive integer n such that $(\lambda - A)^n x = 0$.

Let $\overline{\text{sp}}(A)$ be the closed subspace spanned by all generalized eigenvectors of A, which is usually called the root subspace of A.

LEMMA 1 Let $\{\phi_n\}_1^\infty$ be a Riesz basis for a Hilbert space H. Let $\{\psi_n\}_{N+1}^\infty$ be another sequence in H. If there is an $N \geqslant 0$ such that

$$\sum_{n=N+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then there exists a $M \geqslant N$ such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ is a Riesz basis for H.

Proof. Because $\{\phi_n\}_1^\infty$ is a Riesz basis, there is an invertible operator $T \in \mathcal{L}(H)$, the set of all linear bounded operators on H, such that

$$T\phi_n = e_n,$$

where $\{e_n\}_1^\infty$ is an orthonormal basis for H. By hypothesis, there is a $M \geqslant N$ such that

$$\theta^2 = \|T\|^2 \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 < 1.$$

Define operator Q as follows:

$$Qx = \sum_{n=1}^{M} a_n\phi_n + \sum_{n=M+1}^{\infty} a_n\psi_n, \text{ for } x = \sum_{n=1}^{\infty} a_n\phi_n \in H.$$

Then for any $x = \sum_{n=1}^{\infty} a_n\phi_n \in H$,

$$\begin{aligned}\|(I-Q)x\|^2 &= \left\| \sum_{n=M+1}^{\infty} a_n(\phi_n - \psi_n) \right\|^2 \\ &\leqslant \sum_{n=M+1}^{\infty} |a_n|^2 \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 \\ &\leqslant \|T\|^2 \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 \|x\|^2 = \theta^2\|x\|^2.\end{aligned}$$

This shows that $I - Q \in \mathcal{L}(\mathbf{H})$ and $\|I - Q\| \leqslant |\theta| < 1$. Therefore, both Q and Q^{-1} belong to $\mathcal{L}(H)$. Consequently, $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ is a Riesz basis for H because $Q\phi_n = \phi_n$ for $1 \leqslant n \leqslant M$ and $Q\phi_n = \psi_n$ as $n > M$.

THEOREM 1 Let A be a densely defined discrete operator in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for H. If there is an $N \geqslant 0$ with a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of A such that

$$\sum_{N+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then

(i) There are constant $M > N$ and generalized eigenvectors $\{\psi_{n0}\}_1^M$ of A such that $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H.

(ii) Let $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ correspond to eigenvalues $\{\sigma_n\}_1^\infty$ of A. Then $\sigma(A) = \{\sigma_n\}_1^\infty$, where σ_n is counted according to its algebraic multiplicity.

(iii) If there is an $M_0 > 0$ such that $\sigma_n \neq \sigma_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that all $\sigma_n, n > N_0$ are algebraically simple.

Proof. (ii) and (iii) are consequences of (i). Only proof of (i) is needed. By Lemma 1, there is an $M > N$ such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H; in particular, $\{\psi_n\}_{M+1}^\infty$ is ω-linearly independent.

Suppose that $\{\psi_n\}_{M+1}^\infty \cup \{\psi_\alpha\}$ is a 'maximal' subset of ω-linearly independent generalized eigenvectors of A, that is, $\{\psi_n\}_{M+1}^\infty \cup \{\psi_\alpha\}$ is a ω-linearly independent subset of generalized eigenvectors of A, and for any generalized eigenvector ψ of $\mathcal{A}$, the extended set $\{\psi_n\}_{M+1}^{\infty} \cup \{\psi_\alpha\} \cup \psi$ must be ω-linearly dependent. Therefore, $\{\psi_n\}_{M+1}^{\infty} \cup \{\psi_\alpha\}$ spans the root subspace $\overline{\text{sp}}(A)$ of A. By assumption and Bari's theorem, the number of $\{\psi_\alpha\}$ cannot exceed M. Let $\{\psi_\alpha\} = \{\psi_{n0}\}_1^L$, $L \leqslant M$. It follows from Theorem 3.2 in Rao (1997) that $\{\psi_{n0}\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ forms a Riesz basis for $\overline{\text{sp}}(A)$. We now show that $\overline{\text{sp}}(A) = H$.

Suppose, without loss of generality, that $0 \in \rho(A)$. Let $B = A^{-1}$. It is well-known that $\lambda \in \rho(A)$ if and only if $\lambda^{-1} \in \sigma(B)$ and the root subspace of A corresponding to λ is the same as the root subspace of B corresponding to λ^{-1} (see, e.g. Dunford & Schwartz, 1971, p. 2293). Hence $\{\psi_n\}_{N+1}^{\infty} \cup \{\psi_{n0}\}_1^M$ is also a 'maximal' set of ω-linearly independent generalized eigenvectors of B. However, our operator B now, as an assumption, is compact on H.

Suppose on the contrary: $\overline{\text{sp}}(A) \neq H$. Then by the orthogonal decomposition theorem, there is a subspace H_1, the orthogonal complement of H_2 in H, such that

$$H = H_1 \oplus \overline{\text{sp}}(A),\ H_1 = \overline{\text{sp}}(A)^{\perp},\ H_1 \neq \{0\}.$$

Because $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ is a Riesz basis for H, it follows from Bari's theorem that $\dim H_1 < \infty$.

Let $\mathbb{P}$ be the orthogonal projector from H to H_1. It follows from Lemma 4.2 in Gohberg & Krein (1969, p. 17) that $\mathbb{P}B\mathbb{P}$ is a Volterra operator and so is its adjoint $(\mathbb{P}B\mathbb{P})^* = \mathbb{P}B^*\mathbb{P}$, i.e. $\mathbb{P}B^*\mathbb{P}$ is compact and has no nonzero eigenvalues.

However, because $\overline{\text{sp}}(A)$ is an invariant subspace of B, H_1 is an invariant subspace of its adjoint B^*. Note that $\mathbb{P}B^*\mathbb{P}H_1 = B^*H_1$; therefore $B^*|_{H_1}$ (the operator B^* confined to H_1), has no nonzero eigenvalue. Because H_1 is finite dimensional, it must have $0 \in \sigma(B^*|_{H_1})$, i.e. there is a $x_0 \in H_1$, $x_0 \neq 0$ such that $B^*x_0 = 0$. This contradicts a well-known fact that $H = \overline{\mathcal{R}(B)} \oplus \mathcal{N}(B^*)$ and the assumption $\overline{\mathcal{R}(B)} = \overline{D(A)} = H$, where $\mathcal{R}(\bullet)$ and $\mathcal{N}(\bullet)$ denote respectively, the range and null subspace of operator $\bullet$. Hence $\overline{\text{sp}}(A) = H$.

Since a 'proper' subset of a basis cannot be a basis, it follows from Bari's theorem and assumption that $L = M$. This is (i). The proof is complete. □

REMARK 1 A is called a $[D]$-class operator in a Hilbert space H if there is an $N > 0$ with a sequence of generalized eigenvectors $\{\psi_n\}_1^{\infty}$ of A which form a Riesz basis for H such that

$$A\psi_n = \lambda_n\psi_n,\ \lambda_n \neq \lambda_m \text{ for } n, m > N,\ \lim_{n\to\infty} |\lambda_n| = \infty.$$

It is well known that a $[D]$-class operator must be a densely defined discrete operator. Therefore, our condition in Theorem 1 is 'almost' necessary and sufficient.

3. Application to a beam equation

Rewrite equation (1) as follows:

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad 0 < x < 1, \\ y(0,t) = y_x(0,t) = 0, \\ y_{xx}(1,t) = -k_1 y_{xt}(1,t),\ y_{xxx}(1,t) = k_2 y_t(1,t), \end{cases} \tag{2}$$

where k_i, $i = 1, 2$ are real. Let $H_E^2(0,1) = \{f(x) \in H^2(0,1) | f(0) = f'(0) = 0\}$, $\mathcal{H} = H_E^2(0,1) \times L^2(0,1)$ be the underlying state Hilbert space with the inner product induced norm:

$$\|(f,g)\|^2 = \int_0^1 [|f''(x)|^2 + |g(x)|^2]\,\mathrm{d}x.$$

Then (2) can be written as an evolutionary equation in $\mathcal{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \mathcal{A}Y(t), \tag{3}$$

where $Y(t) = (y, y_t)$ and the operator $\mathcal{A}$ is given by

$$\begin{cases} \mathcal{A}(f(x), g(x)) = (g(x), -f^{(4)}(x)), \\ D(\mathcal{A}) = \{(f,g) \in (H^4 \cap H_E^2) \times H_E^2 | f''(1) = -k_1 g'(1), f'''(1) = k_2 g(1)\}. \end{cases} \tag{4}$$

LEMMA 2

(i) For any real k_1, k_2, operator $\mathcal{A}$ defined by (4) is a densely defined discrete operator in $\mathcal{H}$; hence the spectrum $\sigma(\mathcal{A})$ consists of eigenvalues only.

(ii) For any $0 \neq \lambda = \mathrm{i}\tau^2 \in \sigma(\mathcal{A})$, there is only one associated (linearly independent) eigenfunction which takes the form $(f, \lambda f)$, where f is found to be

$$\begin{aligned} f(x) = {} & \tau[\operatorname{ch}\tau(1-x) - \cos\tau(1-x) + \operatorname{sh}\tau\sin\tau x \\ & + \operatorname{sh}\tau x\sin\tau - \operatorname{ch}\tau\cos\tau x + \operatorname{ch}\tau x\cos\tau] \\ & + k_2\mathrm{i}[-\operatorname{sh}\tau(1-x) - \sin\tau(1-x) + \operatorname{sh}\tau\cos\tau x \\ & + \operatorname{ch}\tau x\sin\tau - \operatorname{ch}\tau\sin\tau x - \operatorname{sh}\tau x\cos\tau], \end{aligned} \tag{5}$$

where ch stands for cosh, sh stands for sinh and τ satisfies the following characteristic equation:

$$\begin{aligned} & k_1\mathrm{i}\tau^2(\operatorname{sh}\tau\cos\tau + \operatorname{ch}\tau\sin\tau) \\ & + \tau[1 + \operatorname{ch}\tau\cos\tau - k_1k_2(1 - \operatorname{ch}\tau\cos\tau)] \\ & + k_2\mathrm{i}(\operatorname{ch}\tau\sin\tau - \operatorname{sh}\tau\cos\tau) = 0. \end{aligned} \tag{6}$$

In particular, each nonzero eigenvalue of $\mathcal{A}$ is geometrically simple.

Proof. We consider (ii) only because (i) is trivial. It is easily seen that $\lambda = \mathrm{i}\tau^2 \in \sigma(\mathcal{A})$ if and only if there is $f(x) \neq 0$ satisfying

$$\begin{cases} f^{(4)}(x) - \tau^4 f(x) = 0, \\ f(0) = f'(0) = 0, \\ f''(1) = -\mathrm{i}k_1\tau^2 f'(1),\ f'''(1) = \mathrm{i}k_2\tau^2 f(1), \end{cases} \tag{7}$$

and the associated eigenfunction is $(f, \lambda f)$. First, the general solution of

$$\begin{cases} f^{(4)}(x) - \tau^4 f(x) = 0, \\ f(0) = f'(0) = 0, \end{cases} \tag{8}$$

is of

$$f(x) = c_1(\operatorname{ch}\tau x - \cos\tau x) + c_2(\operatorname{sh}\tau x - \sin\tau x) \tag{9}$$

where c_1 and c_2 are constants. Secondly, by condition $f'''(1) = \mathrm{i}k_2\tau^2 f(1)$, we have

$$\begin{cases} c_1 = \tau(\operatorname{ch}\tau + \cos\tau) - k_2\mathrm{i}(\operatorname{sh}\tau - \sin\tau); \\ c_2 = -\tau(\operatorname{sh}\tau - \sin\tau) + k_2\mathrm{i}(\operatorname{ch}\tau - \cos\tau). \end{cases} \tag{10}$$

Substituting (10) into (9) yields (5). Finally, the last condition $f''(1) = -\mathrm{i}k_1\tau^2 f'(1)$ gives the characteristic equation (6).

LEMMA 3

(i) There is a family of eigenvalues $\{\lambda_n, \overline{\lambda_n}\}, \lambda_n = \mathrm{i}\tau_n^2$ of $\mathcal{A}$ with the following asymptotic expression:

$$\tau_n = m\pi + \mathcal{O}(n^{-1}). \tag{11}$$

More progressively,

$$\lambda_n = \begin{cases} -k_2 - \dfrac{1}{k_1} + \mathrm{i}(m\pi)^2 + \mathcal{O}(n^{-1}), & k_1 \neq 0, \\ -2k_2 + \mathrm{i}(m\pi)^2 + \mathcal{O}(n^{-1}), & k_1 = 0, \end{cases} \tag{12}$$

where n is a large positive integer and $m = n - 1/4$ if $k_1 \neq 0$ and $m = n - 1/2$ while $k_1 = 0$.

(ii) There is an eigenfunction $(f_n, \lambda_n f_n)$ of $\mathcal{A}$ corresponding to $\lambda_n = \mathrm{i}\tau_n^2$ such that

$$F_n(x) = \begin{pmatrix} \mathrm{e}^{-m\pi x} + \mathrm{e}^{m\pi(x-1)}[\sin m\pi + \cos m\pi] - [\sin m\pi x - \cos m\pi x] \\ \mathrm{i}\mathrm{e}^{-m\pi x} + \mathrm{i}\mathrm{e}^{m\pi(x-1)}[\sin m\pi + \cos m\pi] + \mathrm{i}[\sin m\pi x - \cos m\pi x] \end{pmatrix}^{\mathrm{T}} + \mathcal{O}(n^{-1}), \tag{13}$$

where

$$F_n(x) = \frac{2\tau_n^{-3}}{\mathrm{e}^{\tau_n}} \begin{pmatrix} f_n''(x) \\ \lambda_n f_n(x) \end{pmatrix}^{\mathrm{T}}. \tag{14}$$

$\lim_{n\to\infty} \|F_n\|^2_{L^2\times L^2} = 2$.

Proof. We distinguish two different cases.

CASE 1 $k_1 \neq 0$.

In this case, the characteristic equation (6) can be written asymptotically as

$$\begin{cases} \sin\tau + \cos\tau = \mathcal{O}(|\tau|^{-1}), \text{ or} \\ \sin\tau + \cos\tau = \frac{\mathrm{i}}{\tau}\left(k_2 + \frac{1}{k_1}\right)\cos\tau + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\tau}), \text{ as } \operatorname{Im}\tau \text{ bounded } \operatorname{Re}\tau \to \infty. \end{cases} \tag{15}$$

The first equation in (15) can be written as $\sin 2\tau = -1 + \mathcal{O}(|\tau|^{-2})$. Applying Rouche's theorem to this equation in a small neighbourhood of $m\pi = (n - 1/4)\pi$, where n is a large positive integer, we obtain a solution τ_n which is of the form (11). Substituting (11) into the second equation of (15), we have $2\mathcal{O}(n^{-1}) = \mathrm{i}/(m\pi)(k_2 + 1/k_1) + \mathcal{O}(n^{-2})$ for $\mathcal{O}(n^{-1})$ in the expression of τ_n in (11). Hence

$$\tau_n = m\pi + \mathrm{i}/(2m\pi)(k_2 + 1/k_1) + \mathcal{O}(n^{-2});$$

therefore, $\lambda_n = \mathrm{i}\tau_n^2 = -(k_2 + 1/k_1) + \mathrm{i}(m\pi)^2 + \mathcal{O}(n^{-1})$. This is (12) for $k_1 \neq 0$.

CASE 2 $k_1 = 0$.

In this case, the characteristic equation (6) can be written asymptotically as

$$\begin{cases} \cos\tau = \mathcal{O}(|\tau|^{-1}), \text{ or} \\ \cos\tau = \frac{\mathrm{i}}{\tau}k_2(\sin\tau - \cos\tau) + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\tau}), \text{ as } \operatorname{Im}\tau \text{ bounded } \operatorname{Re}\tau \to \infty. \end{cases} \tag{16}$$

By treatments similar to (15), we obtain (12) for the case of $k_1 = 0$.

Note that by (11), for any $y > 0$ and $0 \leqslant x \leqslant 1$,

$$\begin{cases} \mathrm{e}^{-\tau_n y} = \mathrm{e}^{-m\pi y} + \mathcal{O}(n^{-1}), \\ \sin\tau_n x = \sin m\pi x + \mathcal{O}(n^{-1}), \cos\tau_n x = \cos m\pi x + \mathcal{O}(n^{-1}). \end{cases} \tag{17}$$

Let $(f_n(x), \lambda_n f_n(x))$ be the eigenfunction of $\mathcal{A}$ corresponding to λ_n, where $f_n(x) = f(x)$ is defined by (5) with $\tau = \tau_n$. First, it follows from (5) that

$$\begin{aligned} \tau^{-2} f''(x) = {}& \tau[\operatorname{ch}\tau(1-x) + \cos\tau(1-x) - \operatorname{sh}\tau\sin\tau x + \operatorname{sh}\tau x\sin\tau \\ & + \operatorname{ch}\tau\cos\tau x + \operatorname{ch}\tau x\cos\tau] \\ & + k_2\mathrm{i}[-\operatorname{sh}\tau(1-x) + \sin\tau(1-x) - \operatorname{sh}\tau\cos\tau x + \operatorname{ch}\tau x\sin\tau \\ & + \operatorname{ch}\tau\sin\tau x - \operatorname{sh}\tau x\cos\tau]. \end{aligned}$$

Let $\tau = \tau_n$. Then, by (17),

$$\begin{aligned} 2\mathrm{e}^{-\tau}\tau^{-3} f''(x) &= \mathrm{e}^{-\tau x} - \sin\tau x + \mathrm{e}^{-\tau(1-x)}\sin\tau + \cos\tau x + \mathrm{e}^{-\tau(1-x)}\cos\tau + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\tau}) \\ &= \mathrm{e}^{-m\pi x} + \mathrm{e}^{m\pi(x-1)}[\sin m\pi + \cos m\pi] - [\sin m\pi x - \cos m\pi x] \\ &\quad + \mathcal{O}(n^{-1}). \end{aligned}$$

This is the estimate for the first component of F_n in (13). The estimate for f_n is similar; we omit the detail. The proof is complete. □

It should be noted that the asymptotic expression (12) is only valid for a 'family' of eigenvalues of $\mathcal{A}$ at the moment. However, we will show after verification of Riesz basis generation that (12) is really an asymptotic expression of eigenvalues of $\mathcal{A}$. This is one of the advantages of our approach in this paper.

To show that $\mathcal{A}$ satisfies the conditions of Theorem 1, we need a reference Riesz basis. In the following, we shall show how to find such a reference Riesz basis produced by generalized eigenvectors of a skew-adjoint, compact resolvent linear operator $\mathcal{A}_0$ in the space $\mathcal{H}$. The search for this $\mathcal{A}_0$ starts from equation (2) and characteristic equation (6). Let us show the process. The 'dominant term' in characteristic equation (6) can be obtained by letting $1/k_1 = k_2 = 0$ in the case of $k_1 \neq 0$ and by letting $k_1 = k_2 = 0$ in the case of $k_1 = 0$. Equation (2) will become the following two free systems:

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad 0 < x < 1, \\ y(0,t) = y_x(0,t) = 0, \, y_{xxx}(1,t) = y_{xt}(1,t) = 0, \end{cases} \tag{18}$$

in the case of $1/k_1 = k_2 = 0$ and

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad 0 < x < 1, \\ y(0,t) = y_x(0,t) = 0, \, y_{xx}(1,t) = y_{xxx}(1,t) = 0, \end{cases} \tag{19}$$

in the case of $k_1 = k_2 = 0$. The operator $\mathcal{A}_0$ is nothing but the system operator associated with systems (18) and (19):

$$\begin{cases} \mathcal{A}_0(f(x), g(x)) = (g(x), -f^{(4)}(x)), \\ \begin{cases} D(\mathcal{A}_0) = \{(f,g) \in (H^4 \cap H_E^2) \times H_E^2 | f'''(1) = 0, g'(1) = 0\}, k_1 \neq 0; \\ D(\mathcal{A}_0) = \{(f,g) \in (H^4 \cap H_E^2) \times H_E^2 | f''(1) = f'''(1) = 0, k_1 = 0. \end{cases} \end{cases} \tag{20}$$

It is easily shown that the operator $\mathcal{A}_0$ defined by (20) is skew-adjoint in the state space $\mathcal{H}$ with compact resolvent. Therefore, from general operator theory, all eigenvalues of $\mathcal{A}_0$ are located on the imaginary axis and there is a sequence of generalized eigenfunctions of $\mathcal{A}_0$ forming a Riesz basis for $\mathcal{H}$.

By letting $k_2 = 0$ in (5), we know that each nonzero eigenvalue $\mu = \mathrm{i}\omega^2$ of $\mathcal{A}_0$ is geometrically simple and hence algebraically simple. An eigenfunction $(\phi(x), \mu\phi(x))$ corresponding to $\mu = \mathrm{i}\omega^2$ can be obtained by letting $k_2 = 0$, $\tau = \omega$ in (5):

$$\begin{aligned} \phi(x) = \omega[&\mathrm{ch}\,\omega(1-x) - \cos\omega(1-x) + \mathrm{sh}\,\omega \sin\omega x \\ &+ \mathrm{sh}\,\omega x \sin\omega - \mathrm{ch}\,\omega \cos\omega x + \mathrm{ch}\,\omega x \cos\omega]. \end{aligned} \tag{21}$$

The characteristic equation for $(0 \neq)\mu = \mathrm{i}\omega^2 \in \sigma(\mathcal{A}_0)$ can be obtained by letting $1/k_1 = k_2 = 0$ in the case of $k_1 \neq 0$ and by letting $k_1 = k_2 = 0$ in the case of $k_1 = 0$, namely

$$\begin{cases} 1 + \mathrm{ch}\,\omega \cos\omega = 0, k_1 \neq 0, \\ \mathrm{sh}\,\omega \cos\omega + \mathrm{ch}\,\omega \sin\omega = 0, k_1 = 0. \end{cases} \tag{22}$$

Because eigenvalues of $\mathcal{A}_0$ lie on the imaginary axis, the nonzero eigenvalues of $\mathcal{A}_0$ consist of $\{\{\mu, \overline{\mu}\} | \mu = \mathrm{i}\omega^2, \omega$ is a positive solution of (22)$\}$. Asymptotically, we write (22) as

$$\begin{cases} \cos\omega = O(\mathrm{e}^{-\omega}), k_1 \neq 0, \\ \cos\omega + \sin\omega = O(\mathrm{e}^{-\omega}), k_1 = 0. \end{cases} \tag{23}$$

The positive solutions ω_n of (23) are of the form

$$\omega_n = m\pi + \mathcal{O}(\mathrm{e}^{-n}), \tag{24}$$

where $m = n - 1/4$ for $k_1 \neq 0$ and $m = n - 1/2$ for $k_1 = 0$ for large positive integer n.

It should be noted that the eigenvalues of $\mathcal{A}_0$ consist of all $\{\mu_n = \mathrm{i}\omega_n^2, \overline{\mu_n} = -\mathrm{i}\omega_n^2\}$ but a finite set, where ω_n is given by (24). This is unlike operator $\mathcal{A}$; (24) is indeed an asymptotic expression of eigenvalues of $\mathcal{A}_0$.

It is noted that in obtaining (13), only (11) is used which is certainly true for ω_n from (24). Therefore, the following results for operator $\mathcal{A}_0$ are valid.

LEMMA 4 Let $\mathcal{A}_0$ be defined as in (20). Then:

(i) $\mathcal{A}_0$ is skew-adjoint in the Hilbert space $\mathcal{H}$ with compact resolvent. Hence there is a sequence of generalized eigenfunctions of $\mathcal{A}_0$ forms a Riesz basis for $\mathcal{H}$.
(ii) All nonzero eigenvalues of $\mathcal{A}_0$ are algebraically simple.
(iii) The eigenfunction of $\mathcal{A}_0$ have the following asymptotic expression:

$$G_n(x) = \begin{pmatrix} \mathrm{e}^{-m\pi x} + \mathrm{e}^{m\pi(x-1)}[\sin m\pi + \cos m\pi] - [\sin m\pi x - \cos m\pi x] \\ \mathrm{i}\mathrm{e}^{-m\pi x} + \mathrm{i}\mathrm{e}^{m\pi(x-1)}[\sin m\pi + \cos m\pi] + \mathrm{i}[\sin m\pi x - \cos m\pi x] \end{pmatrix}^{\mathrm{T}} + \mathcal{O}(n^{-1}), \tag{25}$$

where

$$G_n(x) = \frac{2\omega_n^{-3}}{\mathrm{e}^{\omega_n}} \begin{pmatrix} \phi_n''(x) \\ \lambda_n \phi_n(x) \end{pmatrix}^{\mathrm{T}} \tag{26}$$

where $(\phi_n, \mu_n\phi_n)$ is an eigenfunction of $\mathcal{A}_0$ corresponding to eigenvalue $\mu_n = \mathrm{i}\omega_n^2$, where ω_n satisfies (24).

By Lemma 4, we may assume without loss of generality that a 'maximal' set of ω-linearly independent (generalized) eigenfunctions of $\mathcal{A}_0$ is

$$\{(\phi_n(x), \mu_n\phi_n(x))\}_1^\infty \cup \{\text{conjugates}\}, \tag{27}$$

which is a Riesz basis for $\mathcal{H}$ by (25). It follows from (13) and (25) that there is an $N > 0$ such that

$$\begin{aligned} &\sum_{n>N}^{\infty} \|2\tau_n^{-3}\mathrm{e}^{-\tau_n}(f_n, \lambda_n f_n) - 2\omega_n^{-3}\mathrm{e}^{-\omega_n}(\phi_n, \mu_n\phi_n)\|_{\mathcal{H}}^2 \\ &\quad = \sum_{n>N}^{\infty} \|F_n - G_n\|_{L^2(0,1)\times L^2(0,1)}^2 = \sum_{n>N}^{\infty} \mathcal{O}(n^{-2}) < \infty. \end{aligned} \tag{28}$$

The same thing is true for conjugates, i.e. operator $\mathcal{A}$ has a sequence of eigenfunctions which quadratically close to a Riesz basis in the sense of (28). Therefore, $\mathcal{A}$ satisfies all the conditions of Theorem 1. Consequently, we have obtained our main result in this section.

THEOREM 2 Let $\mathcal{A}$ be defined as in (4). Then

(i) There is a sequence of generalized eigenvectors of $\mathcal{A}$ which forms a Riesz basis for state space $\mathcal{H}$.
(ii) The eigenvalues of $\mathcal{A}$ have an asymptotic expression (12).

(iii) There is an $M > 0$ such that all λ_n in (12) are algebraically simple as $n > M$.

Therefore, $\mathcal{A}$ generates a C_0-group for any real k_1, k_2 and the spectrum-determined growth condition holds for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$. Moreover, $\mathcal{A}$ is a $[D]$-class operator as indicated in Remark 1.

COROLLARY 1 Let $\mathcal{A}$ be defined by (4). Then semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable if $k_1^2 + k_2^2 > 0$.

Proof. It is easily shown that under the condition $k_1^2 + k_2^2 > 0$, $\mathcal{A}$ is dissipative and there is no eigenvalue of $\mathcal{A}$ on the imaginary axis. By spectrum-determined conditions and asymptotic expression (12), $e^{\mathcal{A}t}$ is exponentially stable if and only if it is asymptotically stable. The result then follows from a well-known fact (see, e.g. Luo *et al.*, 1999) that a bounded C_0-semigroup generated by an operator with compact resolvent is asymptotically stable if and only if all eigenvalues of its generator are located on the open left half of the complex plane. □

REMARK 2 The exponential stability and asymptotic distribution of eigenvalues of system (2) were obtained by Chen *et al.* (1988) using every complicated methods. To show (12) is really an asymptotic expression of eigenvalues, it is not necessary to apply Theorem 1. Actually, by estimation (28) and Bari's theorem, the number of eigenvalues of $\mathcal{A}$ other than that given by (12) cannot exceed $2N$ (counted by algebraic multiplicity) because any finitely linear independent generalized eigenvector sequence of a discrete operator is also ω-linearly independent.

4. Concluding remarks

In this paper, we give an abstract condition on Riesz basis generation for the generalized eigenvectors of discrete operators in general Hilbert spaces. The condition neglects completely the 'low' eigenmodes and is almost necessary and sufficient for the Riesz basis property. By this general result, we prove in a very simple way that the usual Euler–Bernoulli beam equation with boundary linear feedback is a Riesz system. Meanwhile, the spectral analysis of the controlled system becomes very simple by examining only that of a reference conservative system, which in most cases is direct and simple. The approach demonstrated here has potential applications in the study of other types of perturbed system such as serially connected beam equations with joint linear feedback controls (Guo, 2001).

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