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Statistically Convergent Fuzzy Sequence Spaces by Fuzzy Metric

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Abstract. In this article we study different properties of the statistically convergent and statistically null sequence classes of fuzzy real numbers with fuzzy metric, like completeness, solidness, sequence algebra, symmetricity and convergence free.

1. Introduction

The concept of fuzzy set, a set whose boundary is not sharp or precise has been introduced by L. A. Zadeh in 1965. It is the origin of new theory of uncertainty, distinct from the notion of probability. After the introduction of fuzzy sets, the scope for studies in different branches of pure and applied mathematics increased widely. The notion of fuzzy sets has successfully been applied in studying sequence spaces by Nanda [4], Nuary and Savas [5], Savas [7], Syau [9], Tripathy and Baruah [11], Tripathy and Dutta [12], Tripathy and Sarma ([13], [14]) and many others.

The notion of statistical convergence was introduced by Fast [1] and Schoenberg [8] independently. The potential of the introduced notion was realized in eighties by the workers on sequence spaces. Since than, a lot of work has been done on classical statistically convergent sequences. It is evidenced by the works of Fridy [2], alt [6], Tripathy [10], Tripathy and Sen [15] and many others. Though some work have been done on statistically convergent sequences of fuzzy real numbers under classical metric, but a very little work has been done on fuzzy metric. This motivated us to investigate statistically convergent sequence spaces of fuzzy real numbers by fuzzy metric.

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2. Definitions and Preliminaries

Definition 2.1. A *fuzzy real number* X is a fuzzy set on R, i.e. a mapping $X : R \to I(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.2. A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min\,(X(s), X(r))$, where $s < t < r$.

Definition 2.3. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

Definition 2.4. A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, is open in the usual topology of R for all $a \in I$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

Definition 2.5. The *α-level set* $[X]^\alpha$ of the fuzzy real number X, for $0 < \alpha \leq 1$, is defined by $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut. (The *strong α-cut* of the fuzzy real number X, for $0 \leq \alpha \leq 1$ is the set $\{t \in R : X(t) > \alpha\}$).

Let $X, Y \in R(I)$ and α-level sets be $[X]^\alpha = [a_1^\alpha,\ b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha,\ b_2^\alpha]$, $\alpha \in [0,1]$. Then the arithmetic operations on $R(I)$ in terms of α-level sets are defined as follows:

$$[X \oplus Y]^\alpha = [a_1^\alpha + a_2^\alpha,\ b_1^\alpha + b_2^\alpha],$$

$$[X \ominus Y]^\alpha = [a_1^\alpha - b_2^\alpha,\ b_1^\alpha - a_2^\alpha],$$

$$[X \otimes Y]^\alpha = \left[\min_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha,\ \max_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha\right]$$

$$\text{and} \quad [\bar{1} \div Y]^\alpha = \left[\frac{1}{b_2^\alpha},\ \frac{1}{a_2^\alpha}\right],\ 0 \notin Y.$$

For $X, Y \in R(I)$ consider a partial ordering $\leq$ (refer to Kaleva and Seikkala [3]) as

$$X \leq Y \text{ if and only if } a_1^\alpha \leq a_2^\alpha \text{ and } b_1^\alpha \leq b_2^\alpha\ , \text{ for all } \alpha \in (0,1],$$

where $[X]^\alpha = [a_1^\alpha,\ b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha,\ b_2^\alpha]$, $\alpha \in [0,1]$.

Definition 2.6. The *absolute value*, $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkala [3])

$$|X|(t) = \begin{cases} \max(X(t), X(-t)), & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

Definition 2.7. A fuzzy real number X is called *non-negative* if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Definition 2.8. A fuzzy real number sequence (X_k) is said to be *bounded* if $|X_k| \leq \mu$, for some $\mu \in R^*(I)$.

Definition 2.9. A subset E of N is said to have *natural density* $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \chi_E(k) \text{ exists,}$$

where $\chi_E(k)$ is the characteristic function of E. Clearly all finite subsets of N have zero natural density and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$.

Definition 2.10. A sequence (X_k) is said to be *statistically convergent* to L if for every $\varepsilon > 0$, $\delta(\{k \in N : |X_k - L| \geq \varepsilon\}) = 0$. We write $X_k \xrightarrow{stat} L$ or stat-lim $X_k = L$.

Definition 2.11. Let (X_k) and (Y_k) be two sequences, then we say that $X_k = Y_k$ for *almost all* k (in short *a.a.k.*) if $\delta(k \in N : X_k \neq Y_k) = 0$.

Definition 2.12. A class of sequences E^F is said to be *normal* (or *solid*) if $(Y_k) \in E^F$, whenever $|Y_k| \leq |X_k|$, for all $k \in N$ and $(X_k) \in E^F$.

Definition 2.13. Let $K = \{k_1 < k_2 < k_3 \ldots\} \subseteq N$ and E^F be a class of sequences. A *K-step set* of E^F is a class of sequences $\lambda_k^{E^F} = \{(X_{k_n}) \in w^F : (X_n) \in E^F\}$.

Definition 2.14. A *canonical pre-image of a sequence* $(X_{k_n}) \in \lambda_k^{E^F}$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n, & \text{for } n \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Definition 2.15. A *canonical pre-image of a step set* $\lambda_k^{E^F}$ is a set of canonical pre-images of all elements in $\lambda_k^{E^F}$, i.e. Y is in canonical pre-image $\lambda_k^{E^F}$ if and only if Y is canonical pre-image of some $X \in \lambda_k^{E^F}$.

Definition 2.16. A class of sequences E^F is said to be *monotone* if E^F contains the canonical pre-images of all its step sets.

From the above definitions we have the following well known Remark.

Remark 2.1. A class of sequences E^F is solid $\Rightarrow$ E^F is monotone.

Definition 2.17. A class of sequences E^F is is said to be *symmetric* if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where π is a permutation of N.

Definition 2.18. A class of sequences E^F is is said to be *sequence algebra* if $(X_k \otimes Y_k) \in E^F$, whenever $(X_k), (Y_k) \in E^F$.

Definition 2.19. A class of sequences E^F is is said to be *convergence free* if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

Fuzzy Metric Space:

Let d be a mapping from $R(I) \times R(I)$ into $R^*(I)$ and let the mappings $L, M : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $M(1,1) = 1$. Denote

$$[d(X,Y)]_\alpha = [\lambda_\alpha(X,Y), \rho_\alpha(X,Y)], \text{ for } X, Y \in R(I) \text{ and } 0 < \alpha \leq 1.$$

Definition 2.20. The quadruple $(R(I), d, L, M)$ is called a *fuzzy metric space* and d a *fuzzy metric*, if

(1) $d(X,Y) = \bar{0}$ if and only if $X = Y$,
(2) $d(X,Y) = d(Y,X)$ for all $X, Y \in X$,
(3) for all $X, Y, Z \in R(I)$,
  (i) $d(X,Y)(s+t) \geq L(d(X,Z)(s),\ d(Z,Y)(t))$ whenever $s \leq \lambda_1(X,Z)$, $t \leq \lambda_1(Z,Y)$ and $(s+t) \leq \lambda_1(X,Y)$,
  (ii) $d(X,Y)(s+t) \leq M(d(X,Z)(s),\ d(Z,Y)(t))$ whenever $s \geq \lambda_1(X,Z)$, $t \geq \lambda_1(Z,Y)$ and $(s+t) \geq \lambda_1(X,Y)$.

It is known (refer to Kaleva and Seikkala [3]) that in a fuzzy metric space (X, d, Min, Max) the triangle inequality (3) is equivalent to

$$d(X,Y) \leq d(X,Z) + d(Z,Y).$$

Let $\lambda : R(I) \times R(I) \to R^*(I)$ be such that $\lambda(X,Y) = \sup_{0<\alpha\leq 1} \lambda_\alpha(X,Y)$, where $\lambda_\alpha(X,Y) = \min\{| {a_1}^\alpha - {b_1}^\alpha |,\ | {a_2}^\alpha - {b_2}^\alpha |\}$, for α- cut of $X = [{a_1}^\alpha,\ {a_2}^\alpha]$ and α- cut of $Y = [{b_1}^\alpha,\ {b_2}^\alpha]$.

Similarly, let $\rho : R(I) \times R(I) \to R^*(I)$ be such that $\rho(X,Y) = \sup_{0<\alpha\leq 1} \rho_\alpha(X,Y)$, where $\rho_\alpha(X,Y) = \max\{| {a_1}^\alpha - {b_1}^\alpha |,\ | {a_2}^\alpha - {b_2}^\alpha |\}$, for α- cut of $X = [{a_1}^\alpha,\ {a_2}^\alpha]$ and α- cut of $Y = [{b_1}^\alpha,\ {b_2}^\alpha]$.

Throughout the paper we consider the fuzzy metric space with $L = Min$ and $M = Max$. Hence from Kaleva and Seikkala [3] it is clear that $(R(I), d, Min, Max)$ is a complete metric space.

With the concept of fuzzy metric, the following classes of sequences are defined.

$$\ell_\infty^F = \left\{ X = (X_k) \in w^F : \sup_k \lambda(X_k,\ \bar{0}) < \infty \text{ and} \sup_k \rho(X_k,\ \bar{0}) < \infty \right\}.$$

$$\ell_p^F = \left\{ X = (X_k) \in w^F : \sum_{k=1}^{\infty} \left\{ \lambda(X_k,\ \overline{0}) \right\}^p < \infty \text{ and } \sum_{k=1}^{\infty} \left\{ \rho(X_k,\ \overline{0}) \right\}^p < \infty \right\}.$$

$$c^F = \{X = (X_k) \in w^F : \lambda(X_k,\ L) \to \overline{0} \text{ and } \rho(X_k,\ L) \to \overline{0}, \text{ as } k \to \infty,$$

$$\text{for some } L \in R(I)\}.$$

$$c_0^F = \left\{ X = (X_k) \in w^F : \lambda(X_k,\ \overline{0}) \to \overline{0} \text{ and } \rho(X_k,\ \overline{0}) \to \overline{0}, \text{ as } k \to \infty \right\}.$$

$$\overline{c}^F = (\delta(\{k \in N : \lambda(X_k,\ T) \geq \epsilon\}) = 0 \text{ and } \delta(\{k \in N : \rho(X_k,\ T) \geq \epsilon\}) = 0),$$

$$\text{for some } T \in R(I).$$

$$\overline{c_0}^F = \left(\delta\left(\left\{ k \in N : \lambda(X_k,\ \overline{0}) \geq \epsilon \right\} \right) = 0 \text{ and } \delta\left(\left\{ k \in N : \rho(X_k,\ \overline{0}) \geq \epsilon \right\} \right) = 0 \right).$$

Throughout w^F, ℓ_∞^F, ℓ_p^F, c^F, c_0^F, $\overline{c}^F$, m^F and $\overline{c_0}^F$ denote the classes of *all*, *bounded*, *p-absolutely summable*, *convergent*, *null*, *statistically convergent*, *bounded statistically convergent* and *statistically null sequences* of fuzzy real numbers respectively.

3. Main Results

Theorem 3.1. *$m^F = \overline{c}^F \cap \ell_\infty^F$ is a closed subspace of the complete metric space ℓ_∞^F with the fuzzy metric d^* defined by*

$$[d^*(X, Y)]_\alpha = \left[\sup_k \lambda_\alpha(X_k,\ Y_k),\ \sup_k \rho_\alpha(X_k,\ Y_k) \right],$$

where $X = (X_k)$ and $Y = (Y_k)$ are in m^F and $0 < \alpha \leq 1$.

Proof. Since we are considering $(R(I), d, Min, Max)$ metric space, so it can be verified that d^* is a metric on ℓ_∞^F. Now we show that m^F is complete with respect to d^* . Let $(X^{(n)})$ is a Cauchy sequence in m^F . Then $(X^{(n)})$ be a Cauchy sequence in ℓ_∞^F . Since ℓ_∞^F is a complete metric space, so $X^{(n)} \to X$, as $n \to \infty$, say, in ℓ_∞^F . We show that

$$X \in m^F. \tag{1}$$

Since $X^{(n)} = (X_k^{(n)}) = (X_1^{(n)}, X_2^{(n)}, X_3^{(n)}, \ldots) \in m^F$, so for each $n \in N$ there exists $A_n \in R(I)$ such that

$$\text{stat-lim } X_k^{(n)} = A_n.$$

We prove the followings:

(i) $\lim_{n \to \infty} A_n = A$.

(ii) stat-lim $X_k = A$.

(i). Since $(X^{(n)})$ is a convergent sequence, so for a given $\overline{\varepsilon} > 0$, there exists such a $n_0 \in N$ that for each $m, n > n_0$ we have

$$d^*(X^{(m)}, X^{(n)}) < \frac{\overline{\varepsilon}}{3}.$$

i.e.,

$$\sup_k \lambda(X_k^{(m)}, X_k^{(n)}) < \frac{\varepsilon}{3} \text{ and } \sup_k \rho(X_k^{(m)}, X_k^{(n)}) < \frac{\varepsilon}{3}$$

(2) $$\Rightarrow \lambda(X_k^{(m)}, X_k^{(n)}) < \frac{\varepsilon}{3} \text{ and } \rho(X_k^{(m)}, X_k^{(n)}) < \frac{\varepsilon}{3}.$$

Again, since $X^{(n)} = (X_k^{(n)}) \in m^F$, so for a given $\varepsilon > 0$, we have

(3) $$\lambda(X_k^{(m)}, A_m) < \frac{\varepsilon}{3} \text{ and } \rho(X_k^{(m)}, A_m) < \frac{\varepsilon}{3}, \quad \text{for } a.a.k.$$

(4) $$\lambda(X_k^{(m)}, A_n) < \frac{\varepsilon}{3} \text{ and } \rho(X_k^{(m)}, A_n) < \frac{\varepsilon}{3}, \quad \text{for } a.a.k.$$

Now for each $m, n > n_0 \in N$ and from the inequalities (2), (3) and (4), we get

$$\lambda(A_m, A_n) \leq \lambda(A_m, X_k^{(m)}) + \lambda(X_k^{(m)}, X_k^{(n)}) + \lambda(X_k^{(n)}, A_n), \quad \text{for } a.a.k.$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and $$\rho(A_m, A_n) \leq \rho(A_m, X_k^{(m)}) + \rho(X_k^{(m)}, X_k^{(n)}) + \rho(X_k^{(n)}, A_n), \quad \text{for } a.a.k.$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus (A_n) is a Cauchy sequence in $R(I)$. Since $R(I)$ complete, so there exists such a number $A \in R(I)$ such that

$$\lim_{n\to\infty} A_n = A.$$

(ii). We have $X^{(n)} \to X$. For a given $\xi > 0$, there exists such a $q \in N$ that

$$\sup_k \lambda(X_k^{(q)}, X_k) < \frac{\xi}{3} \text{ and } \sup_k \rho(X_k^{(q)}, X_k) < \frac{\xi}{3}.$$

(5) $$\Rightarrow \lambda(X_k^{(q)}, X_k) < \frac{\xi}{3} \text{ and } \rho(X_k^{(q)}, X_k) < \frac{\xi}{3}, \text{ for each } k \in N.$$

The number q can be chosen in such a way that together with (5), we get

$$\lambda(A_q, A) < \frac{\xi}{3} \text{ and } \rho(A_q, A) < \frac{\xi}{3}.$$

Since, stat-lim $X_k^{(q)} = A_q$.

For a given $\xi > 0$,

$$\lambda(X_k^{(q)}, A_q) < \frac{\xi}{3} \text{ and } \rho(X_k^{(q)}, A_q) < \frac{\xi}{3}, \quad \text{for } a.a.k., \text{for each fixed } q.$$

Now,

$$\lambda(X_k, A) \leq \lambda(X_k, X_k^{(q)}) + \lambda(X_k^{(q)}, A_q) + \lambda(A_q, A), \quad \text{for } a.a.k., \text{for eachfixed } q.$$

$$< \tfrac{\xi}{3} + \tfrac{\xi}{3} + \tfrac{\xi}{3} = \xi.$$

$$\rho(X_k, A) \leq \rho(X_k, X_k^{(q)}) + \rho(X_k^{(q)}, A_q) + \rho(A_q, A), \quad \text{for } a.a.k., \text{for each fixed } q.$$

$$< \tfrac{\xi}{3} + \tfrac{\xi}{3} + \tfrac{\xi}{3} = \xi.$$

Hence stat-lim $X_k = A$. This proves the result.

Theorem 3.2. *The class of sequences* $\overline{c_0}^F$ *is solid and as such is monotone.*
Proof. Consider two sequences (X_k) and (Y_k) such that

$$|Y_k| \leq |X_k|, \text{ for all } k \in N \text{ and } (X_k) \in \overline{c_0}^F.$$

Then for a given $\varepsilon > 0$, we have

$$\{k \in N : \lambda(X_k,\ \overline{0}) \geq \varepsilon\} \supseteq \{k \in N : \lambda(Y_k,\ \overline{0}) \geq \varepsilon\}$$

and $$\{k \in N : \rho(X_k,\ \overline{0}) \geq \varepsilon\} \supseteq \{k \in N : \rho(Y_k,\ \overline{0}) \geq \varepsilon\}.$$

Since $(X_k) \in \overline{c_0}^F$, so $\delta\left(\{k \in N : \lambda(X_k,\ \overline{0}) \geq \varepsilon\}\right) = 0$
and $\delta\left(\{k \in N : \rho(X_k,\ \overline{0}) \geq \varepsilon\}\right) = 0.$

Hence $\delta\left(\{k \in N : \lambda(Y_k,\ \overline{0}) \geq \varepsilon\}\right) = 0$ and $\delta\left(\{k \in N : \rho(Y_k,\ \overline{0}) \geq \varepsilon\}\right) = 0$

Thus $(Y_k) \in \overline{c_0}^F$ and the class $\overline{c_0}^F$ is solid.

The class of sequences $\overline{c_0}^F$ is monotone follows from Remark 2.1.

Theorem 3.3. *The classes of sequences* $\overline{c}^F$ *and* m^F *are neither monotone nor solid.*

Proof. The result follows from the following example.

Example 3.1. Let us consider the sequence $(X_k) \in m^F$, defined as follows:

$$\text{For} \quad k = n^2, n \in N, \quad X_k(t) = \begin{cases} t-2, & \text{for } 2 \le t \le 3, \\ 4-t, & \text{for } 3 < t \le 4, \\ 0, & \text{otherwise} \end{cases}$$

$$\text{and for,} \quad k \ne n^2, n \in N, \quad X_k(t) = \begin{cases} 1 - k(t - 2^{-1}), & \text{for } 2^{-1} \le t \le 2^{-1} + k^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Now for $\alpha \in (0, 1]$ we get,

$$[X_k]^\alpha = \begin{cases} [2+\alpha, 4-\alpha], & \text{for } k = n^2, n \in N, \\ [2^{-1}, 2^{-1} + k^{-1}(1-\alpha)], & \text{otherwise.} \end{cases}$$

Clearly, $\lambda \in \left(X_k,\ \bar{\frac{1}{2}}\right) \le \varepsilon$ and $\rho \in \left(X_k,\ \bar{\frac{1}{2}}\right) \le \varepsilon$, for a. a. k. Thus $(X_k) \in \bar{c}^F$.

Let $J = \{k \in N : k = 2i, i \in N\}$ be a subset of N and let $\overline{\overline{(m^F)}}_J$ be the canonical pre-image of the J-step space $(m^F)_J$ of m^F, defined as follows:

$(Y_k) \in \overline{\overline{(m^F)}}_J$ is the canonical pre-image of $(X_k) \in m^F$ implies

$$Y_k = \begin{cases} X_k, & \text{for } k \in J, \\ \bar{0} & \text{for } k \notin J. \end{cases}$$

Now, for $\alpha \in (0, 1]$ we have,

$$[Y_k]^\alpha = \begin{cases} [4+\alpha, (4-\alpha)], & \text{for } k \in J \text{and } k = n^2, n \in N, \\ [2^{-1}, 2^{-1} + k^{-1}(1-\alpha)], & \text{for } k \in J \text{ and } k \ne n^2, \text{ for any } n \in N, \\ [0, 0], & k \notin J \end{cases}$$

For a given $\varepsilon > 0$, there is no definite point, say H such that $\lambda\,(X_k,\ H) \le \varepsilon$ and $\rho\,(X_k,\ H) \le \varepsilon)$, for *a.a.k*.

Thus $(Y_k) \notin \bar{c}^F (\supset m^F)$. Hence $\bar{c}^F$ and m^F are not monotone.

The classes $\bar{c}^F$ and m^F are not solid follows from the Remark 2.1.

Theorem 3.4. *The classes of sequences* $\bar{c}^F$, m^F *and* $\overline{c_0}^F$ *are not symmetric.*

Proof. The result follows from the following example.

Example 3.2. Consider the sequence $(X_k) \in Z$, for $Z = \bar{c}^F, m^F$ and $\overline{c_0}^F$ defined as follows:

$$\text{For}\quad k=n^2, n\in N,\quad X_k(t)=\begin{cases} t-2, & \text{for } 2\le t\le 3,\\ 4-t, & \text{for } 3<t\le 4,\\ 0, & \text{otherwise}\end{cases}$$

$$\text{and for,}\quad k\ne n^2, n\in N,\quad X_k(t)=\begin{cases} 1-2^{-1}kt, & \text{for } 0\le t\le 2^{-1}k,\\ 0, & \text{otherwise.}\end{cases}$$

Now for $\alpha\in(0,1]$ we have,

$$[X_k]^\alpha=\begin{cases}[2+\alpha,4-\alpha], & \text{for } k=n^2, n\in N,\\ [0,2k^{-1}(1-\alpha)], & \text{otherwise.}\end{cases}$$

Clearly, $(X_k)\in {\ell_\infty}^F$ and for a given $\varepsilon>0$, we have

$$\lambda\left(X_k,\overline{0}\right)\le\varepsilon\quad\text{and } \rho\left(X_k,\overline{0}\right)\le\varepsilon,\quad\text{for}\quad a.a.k.$$

Thus $(X_k)\in Z$, for $Z=\overline{c}^F, m^F$ and $\overline{c_0}^F$.

Let (Y_k) be a rearrangement of the sequence (X_k), defined as follows:

$$(Y_k)=(X_1,X_2,X_4,X_3,X_9,X_5,X_{16},X_6,X_{25},X_7\ \ldots\ ).$$

Then for $\alpha\in(0,1]$ we get,

$$[Y_k]^\alpha=\begin{cases}[2+\alpha,4-\alpha], & \text{for } k \text{ odd},\\ [0,2k^{-1}(1-\alpha)], & \text{for } k \text{ even.}\end{cases}$$

For a given $\varepsilon>0$, there is no definite point, say H such that $\lambda(X_k,H)\le\varepsilon$ and $\rho(X_k,H)\le\varepsilon)$, for $a.a.k.$

Thus $(Y_k)\notin Z$, for $Z=\overline{c}^F, m^F$ and $\overline{c_0}^F$.

Therefore the classes of sequences $\overline{c}^F, m^F$ and $\overline{c_0}^F$ are not symmetric.

Theorem 3.5. *The classes of sequences $\overline{c}^F, m^F$ and $\overline{c_0}^F$ are sequence algebra.*

Proof. We prove this result for the class $\overline{c_0}^F$, and for the other classes it can be proved by following.

Let $0<\varepsilon<1$ be given. Suppose $(X_k),(Y_k)\in\overline{c_0}^F$.

Then we have,

$$\left\{k\in N:\lambda(X_k\otimes Y_k,\overline{0})<\varepsilon\right\}\supseteq\left\{k\in N:\lambda(X_k,\overline{0})<\sqrt{\varepsilon}\right\}\cap\left\{k\in N:\lambda(Y_k,\overline{0})<\sqrt{\varepsilon}\right\}$$

and

$$\left\{k \in N : \rho(X_k \otimes Y_k, \overline{0}) < \varepsilon\right\} \supseteq \left\{k \in N : \rho(X_k, \overline{0}) < \sqrt{\varepsilon}\right\} \cap \left\{k \in N : \rho(Y_k, \overline{0}) < \sqrt{\varepsilon}\right\}$$

Since $\quad \delta\left(\left\{k \in N : \lambda(X_k,\ \overline{0}) < \sqrt{\varepsilon}\right\}\right) = 1, \quad \delta\left(\left\{k \in N : \lambda(X_k,\ \overline{0}) < \sqrt{\varepsilon}\right\}\right) = 1$

and $\quad \delta\left(\left\{k \in N : \rho(X_k,\ \overline{0}) < \sqrt{\varepsilon}\right\}\right) = 1, \quad \delta\left(\left\{k \in N : \rho(Y_k,\ \overline{0}) < \sqrt{\varepsilon}\right\}\right) = 1$

So, $\delta\left\{k \in N : \lambda(X_k \otimes Y_K,\ \overline{0}) < \varepsilon\right\} = 1$ and $\delta\left\{k \in N : \rho(X_k \otimes Y_K,\ \overline{0}) < \varepsilon\right\} = 1$

Thus $(X_k \otimes Y_k) \in \overline{c_0}^F$. Hence the class $\overline{c_0}^F$ is a sequence algebra.

Theorem 3.6. *The classes of sequences $\overline{c}^F, m^F$ and $\overline{c_0}^F$ are not convergence free.*

Proof. The result follows from the following example.

Example 3.3. Consider the sequence $(X_k) \in Z$, for $Z = \overline{c}^F, m^F$ and $\overline{c_0}^F$ defined as follows:

$$For \quad k = n^2,\ n \in N, \qquad X_k = \overline{0}$$

$$\text{and for} \quad k \neq n^2, n \in N, \quad X_k(t) = \begin{cases} 1 + 3^{-1}kt, & \text{for } -3k^{-1} \leq t \leq 0, \\ 1 - 3^{-1}kt, & \text{for } 0 < t \leq 3k^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $\alpha \in (0,\ 1]$ we have,

$$[X_k]^\alpha = \begin{cases} [0,\ 0], & \text{for } k = n^2,\ n \in N, \\ [3(\alpha - 1)k^{-1},\ 3(1 - \alpha)k^{-1}], & \text{otherwise.} \end{cases}$$

Hence, $(X_k) \in {\ell_\infty}^F$ and for a given $\varepsilon > 0$, we have

$$\lambda\left(X_k, \overline{0}\right) \leq \varepsilon \quad \text{and } \rho\left(X_k, \overline{0}\right) \leq \varepsilon, \quad \text{for } \ a.a.k.$$

Thus $(X_k) \in Z$, for $Z = \overline{c}^F, m^F$ and $\overline{c_0}^F$.

Let the sequence $(Y_k) \in Z$, be defined as follows:

$$For \quad k = n^2,\ n \in N, \qquad Y_k = \overline{0}$$

$$\text{and for} \quad k \neq n^2, n \in N, \quad Y_k(t) = \begin{cases} 1, & \text{for } k \leq t \leq k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $\alpha \in (0,\ 1]$ we have,

$$[Y_k]^\alpha = \begin{cases} [0,\ 0], & \text{for } k = n^2,\ n \in N, \\ [k,\ k + 1], & \text{otherwise.} \end{cases}$$

For a given $\varepsilon > 0$, there is no definite point, say H such that $\lambda(X_k, H) \leq \varepsilon$ and $\rho(X_k, H) \leq \varepsilon)$, for $a.a.k$.

Thus $(Y_k) \notin Z$, for $Z = \bar{c}^F, m^F$ and $\overline{c_0}^F$.

Hence the classes $\bar{c}^F, m^F$ and $\overline{c_0}^F$ are not convergence free.

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