Delsarte clique graphs

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Abstract

In this paper, we consider the class of Delsarte clique graphs, i.e. the class of distance-regular graphs with the property that each edge lies in a constant number of Delsarte cliques. There are many examples of Delsarte clique graphs such as the Hamming graphs, the Johnson graphs and the Grassmann graphs.

Our main result is that, under mild conditions, for given $s \geq 2$ there are finitely many Delsarte clique graphs which contain Delsarte cliques with size $s + 1$. Further we classify the Delsarte clique graphs with small $s$.

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1. Introduction

Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the eigenvalues of $\Gamma$. If $C$ is a clique of $\Gamma$ then $C$ has at most $(1 - \frac{k}{\theta_D})$ vertices ([1, Proposition 4.4.6]). A clique $C$ containing $(1 - \frac{k}{\theta_D})$ vertices is called a Delsarte clique of $\Gamma$. It is known that each Delsarte clique $C$ of $\Gamma$ is a maximal clique and a completely...
regular code in the sense of [1, p. 345]. Moreover the parameters of a Delsarte clique, as a completely regular code, depend not on the particular Delsarte clique but only on the intersection numbers of $\Gamma$.

In this paper, we consider the class of Delsarte clique graphs, i.e. the class of distance-regular graphs with the property that each edge lies in a constant number of Delsarte cliques (for a more precise definition, see Definition 1.1). There are many examples of Delsarte regular graphs with the property that each edgelies in a constant number of Delsarte cliques and each clique of size $s_C$ contains a full Delsarte set $C$. Hence any vertex of $\Gamma$ lies in exactly $\frac{k s_C}{s_C - 1}$ Delsarte cliques of $\mathcal{C}$.

Remark 1.2. (i) Let $(\Gamma, C)$ be a Delsarte pair with parameters $(k, s_C, n_C)$, where $k \geq 2$, $s_C \geq 1$ and $n_C \geq 1$ are integers. Then $C$ is called a Delsarte set of $\Gamma$ with parameters $(s_C, n_C)$. A Delsarte set $C$ is called full if $C = \{C \mid C$ is a clique of size $s_C + 1$ in $\Gamma\}$.

(b) A non-complete graph $\Gamma$ with valency $k$ is called a Delsarte clique graph with parameters $(k, s, n)$ if $\Gamma'$ contains a full Delsarte set $C$ with parameters $(s, n) = (s_C, n_C)$, where $k \geq 2$, $s_C \geq 1$ and $n_C \geq 1$ are integers.

Remark 1.2. (i) Let $(\Gamma, C)$ be a Delsarte pair with parameters $(k, s_C, n_C)$, then by [1, Proposition 4.4.6(i)], $-\frac{k}{s_C}$ is the smallest eigenvalue of $\Gamma$ and each clique of size $s_C + 1$ in $\Gamma$ is a Delsarte clique. Moreover any vertex of $\Gamma'$ lies in exactly $\frac{k n_C}{s_C}$ Delsarte cliques of $\mathcal{C}$.

(ii) If $(\Gamma, C)$ and $(\Gamma', C')$ are two Delsarte pairs with parameters $(k, s_C, n_C)$ and $(k, s_C', n_C')$, respectively, then $s_C = s_C'$ must hold as $\theta_D = -\frac{k}{s_C} = -\frac{k}{s_C'}$. Hence in this paper, for any Delsarte pair $(\Gamma, C)$ on a given distance-regular graph $\Gamma'$, write $(k, s, n_C) := (k, s_C, n_C)$.

(iii) Let $(\Gamma, C)$ be a Delsarte pair and $\Gamma''$ be a distance-regular graph with the same intersection numbers as $\Gamma$. Then $\Gamma''$ does not have to contain a Delsarte set $C'$. Examples for this case are Doob graphs [1, Section 9.2], the three Chang graphs [1, Section 3.11] and the distance-regular graphs recently found by Van Dam and Koolen [4].

Note that a Delsarte pair $(\Gamma, C)$ with $s = 1$ is exactly the same as a pair of a bipartite distance-regular graph and the set of edges $C$. We call a Delsarte set or a Delsarte pair thick if $s$ is at least 2.

A graph $\Gamma$ is called coconnected if the complement of $\Gamma$ is connected. Note that distance-regular graphs which are not coconnected are exactly the complete multipartite graphs.
Our first main result is that for a thick Delsarte pair \((\Gamma, C)\) with parameters \((k, s, n_C)\), either \(\Gamma\) is not coconnected or the valency \(k\) of \(\Gamma\) is bounded by a function of \(s\) and the diameter \(D\).

**Theorem 1.3.** Let \(\Gamma\) be a distance-regular graph with valency \(k\) and diameter \(D \geq 2\), containing a thick Delsarte set \(C\) with parameters \((s, n_C)\). If \(\Gamma\) is coconnected then

\[
k < s + s\frac{4D}{h}
\]

holds where \(h := |\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}|\).

**Theorem 1.3** is a generalization of a result by Hiraki and Koolen. In [6, Theorem 1], they showed that if \(\Gamma\) is a distance-regular graph of order \((s, t)\) (i.e. \(\Gamma\) is locally the disjoint union of \(t + 1\) cliques of size \(s\)) with an eigenvalue \(-t - 1\) then

\[
t < s\frac{4D}{h} - 1
\]

holds, and hence

\[
k = (t + 1)s < s + s\frac{4D}{h}.
\]

Note that if \(\Gamma\) is a distance-regular graph of order \((s, t)\) with an eigenvalue \(-t - 1\), then \(\Gamma\) is a Delsarte clique graph with parameters \(((t + 1)s, s, 1)\).

In the rest of the paper we will concentrate on a thick Delsarte pair \((\Gamma, C)\) such that \(\Gamma\) is coconnected and has an induced subgraph \(K_{2,1,1}\). Note that a distance-regular graph \(\Gamma\) has no induced subgraph \(K_{2,1,1}\) if and only if \(\Gamma\) is of order \((s, t)\) with \(s = a_1 + 1\) and \(t = \frac{b_1}{s}\).

We will show that for given \(s \geq 2\) there are finitely many coconnected distance-regular graphs \(\Gamma\) which have an induced subgraph \(K_{2,1,1}\) and a family \(C\) of Delsarte cliques of size \(s + 1\) such that \((\Gamma, C)\) is a Delsarte pair. We will show this by showing that the diameter and valency of \(\Gamma\) is bounded above by \(s\).

**Theorem 1.4.** Let \(\Gamma\) be a non-complete coconnected distance-regular graph containing a Delsarte clique of size \(s + 1 \geq 3\) and an induced subgraph \(K_{2,1,1}\). If \(\Gamma\) contains a Delsarte set, then the following hold:

(i) \(D \leq s\);

(ii) \(k < \left(\frac{s}{\lfloor \frac{s}{2} \rfloor}\right)^4\).

In particular, for fixed \(s \geq 2\) there are finitely many coconnected distance-regular graphs \(\Gamma\) which have an induced subgraph \(K_{2,1,1}\) and contain a Delsarte set \(C\) with parameters \((s, n_C)\).

**Remark 1.5.** (i) The diameter bound \(D \leq s\) is tight. For instance, the Johnson graphs \(J(2s, s)\) are Delsarte clique graphs with parameters \((s^2, s, 2)\) and diameter \(D = s\).

(ii) Although in **Theorem 1.4** we showed that \(k\) is bounded by a function of \(s\), we think this bound is far from sharp. We think the bound should be a polynomial in \(s\).

This paper is organized as follows. In Section 2, we set up the notation of distance-regular graphs. In Section 3 we give constructions and examples of Delsarte sets and Delsarte clique graphs. In Section 4, we introduce and calculate parameters for a Delsarte
pair, and then prove Theorem 1.3. In Section 5, we use the notions of the fourth section to illustrate combinatorial properties for a distance-regular graph having a Delsarte set. We then prove Theorem 1.4. In Section 6, we classify distance-regular graphs containing a Delsarte set with \( s = 2, 3 \).

2. Preliminaries and definitions

Suppose that \( \Gamma \) is a finite connected graph with vertex set \( V(\Gamma) \). The distance between any two vertices \( x \) and \( y \) of \( \Gamma \), \( d(x, y) \), is defined as the length of any shortest path between \( x \) and \( y \), and the diameter \( D \) of \( \Gamma \) to be the largest distance between any pair of vertices in \( V(\Gamma) \). For \( x \in V(\Gamma) \) and \( i \leq D \), let \( \Gamma_i(x) \) denote the set of vertices in \( V(\Gamma) \) that are at distance \( i \) from \( x \) and put \( \Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset \). The graph \( \Gamma \) is called distance-regular if there are integers \( b_i, c_i \) \((0 \leq i \leq D)\) so that for any two vertices \( x \) and \( y \) in \( V(\Gamma) \) at distance \( i \), there are precisely \( c_i \) neighbors of \( y \) in \( \Gamma_{i-1}(x) \) and \( b_i \) neighbors of \( y \) in \( \Gamma_{i+1}(x) \). Clearly such a graph is regular with valency \( k := b_0 \). The numbers \( c_i, b_i, \) and \( a_i \), where

\[
a_i := k - b_i - c_i \quad (i = 0, \ldots, D),
\]

are called the intersection numbers of \( \Gamma \), and the array

\[
\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}
\]

is called the intersection array of \( \Gamma \). Now, suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 2 \), diameter \( D \geq 2 \) and intersection array \( \iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\} \). Let \( \theta_0 > \theta_1 > \cdots > \theta_D \) be the eigenvalues of \( \Gamma \). The standard sequence \((u_i(\theta))_{0 \leq i \leq D}\) corresponding to an eigenvalue \( \theta \) of \( \Gamma \) is a sequence satisfying the following recurrence relation:

\[
u_0(\theta) = 1, \quad u_1(\theta) = \theta/k, \\
c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) = \theta u_i(\theta) \quad (1 \leq i \leq D).
\]

Then the multiplicity of the eigenvalue is given by

\[
m(\theta) = \frac{|V(\Gamma)|}{D} \sum_{i=0}^D k_i u_i^2(\theta)
\]

where \( k_i := |\Gamma_i(x)| \) for a vertex \( x \in V(\Gamma) \). For more background information about distance-regular graphs we refer the reader to [1].

3. Constructions and examples

For a non-complete connected graph \( \Omega \), we denote by \( \Omega_2 \) the graph whose vertices are those of \( \Omega \) and whose edges are the 2-subsets of vertices at distance 2 in \( \Omega \). In particular, if \( \Omega \) is connected and bipartite, then \( \Omega_2 \) has exactly two components, say \( \Omega^+ \) and \( \Omega^- \). The graphs \( \Omega^+ \) and \( \Omega^- \) are called the halved graphs of \( \Omega \).
The following proposition shows that for a halved graph of a bipartite distance-regular graph with even diameter there exists a natural Delsarte pair.

**Proposition 3.1.** Let $\Omega$ be a bipartite distance-regular graph with valency $k \geq 2$ and even diameter $D \geq 4$. Let $\Gamma := \Omega^+$ and let $C := \{\Omega_1(w) \mid w \in V(\Omega^-)\}$. Then $(\Gamma, C)$ is a Delsarte pair with parameters $\left(\frac{k(k-1)}{c_2(\Omega)}, k-1, c_2(\Omega)\right)$.

**Proof.** The valency $k(\Gamma)$ of $\Gamma$ equals $\frac{k(k-1)}{c_2(\Gamma)}$, and all elements of $C$ are cliques of size $k$. As $\Omega$ has even diameter and is bipartite, it has $0$ as eigenvalue, which implies that the smallest eigenvalue of $\Gamma$ equals $-\frac{k}{c_2(\Gamma)}$. It follows that a clique of size $k$ is a Delsarte clique in $\Gamma$. Let $x, y \in V(\Gamma)$ at distance $1$ in $\Gamma$. This means that in $\Omega$ they are at distance $2$ and there are $c_2(\Omega)$ common neighbors in $V(\Omega^-)$ of $x$ and $y$ in $\Omega$. This shows the proposition. \qed

We will call a Delsarte pair $(\Gamma, C)$ graphic if the bipartite graph $\Omega$ with the vertex set $V(\Omega) = V(\Gamma) \cup C$ and the edge set $\{(x, C) \mid x \in V(\Gamma), C \in C$ and $x \in C\}$ is a bipartite distance-regular graph.

Recall that a graph is called edge-transitive if its automorphism group acts transitively on the set of edges; cf. [1, p. 435]. The next proposition gives a sufficient condition for a graph to be a Delsarte clique graph.

**Proposition 3.2.** If $\Gamma$ is an edge-transitive distance-regular graph with diameter $D \geq 2$ containing a Delsarte clique, then $\Gamma$ is a Delsarte clique graph.

**Proof.** As each edge lies in the same number of Delsarte cliques from the property of edge-transitivity, the result follows immediately. \qed

As a consequence we have the following:

**Proposition 3.3.** Let $\Gamma$ be a halved graph of a bipartite distance-regular graph with valency $k \geq 2$ and even diameter $D \geq 4$. If $\Gamma$ is edge-transitive then $\Gamma$ is a Delsarte clique graph.

**Proposition 3.4.** Let $\tilde{\Gamma}$ be a distance-regular antipodal cover with even diameter $D \geq 4$, and let $\Gamma_{\tilde{\Gamma}}$ be its folded graph. Let $C$ be a set of cliques in $\Gamma$ and let $\tilde{C} := \{\tilde{C} \subseteq V(\tilde{\Gamma}) \mid \tilde{C}$ is a clique of $\tilde{\Gamma}$ and folds to an element of $C$ in $\Gamma\}$. Then $(\Gamma, C)$ is a Delsarte pair with parameters $(k, s, n_C)$ if and only if $(\tilde{\Gamma}, \tilde{C})$ is a Delsarte pair with parameters $(k, s, n_{\tilde{C}})$.

**Proof.** By [1, Proposition 4.2.3] the graphs $\Gamma$ and $\tilde{\Gamma}$ have the same smallest eigenvalue and hence any clique in $\Gamma$ is a Delsarte clique of $\Gamma$ if and only if every clique in $\tilde{C}$ is a Delsarte clique of $\tilde{\Gamma}$. It is straightforward to see that each edge in $\Gamma$ lies in $n_C$ edges if and only if each edge of $\tilde{\Gamma}$ lies in $n_{\tilde{C}} = n_C$ edges. \qed

As an application we see that the folded Johnson graphs $\overline{J}(4m, 2m)$ are Delsarte clique graphs when $m \geq 2$. 

3.1. Examples

Now we list some Delsarte clique graphs. In the list below, \( \left[ \frac{\ell}{m} \right] \) means the \( q \)-ary Gaussian binomial coefficient:

\[
\left[ \frac{\ell}{m} \right] = \left[ \frac{\ell}{m} \right]_q = \frac{(q^\ell - 1) \cdots (q^{\ell-m+1} - 1)}{(q^m - 1) \cdots (q - 1)}.
\]

1. Any bipartite distance-regular graph with valency \( k \) is a Delsarte clique graph with parameters \((k, s, n) = (k, 1, 1)\).

2. A complete multipartite graph \( K_{(s+1)\times \frac{k}{s}} \) is a Delsarte clique graph with parameters \((k, s, n) = (k, s, (\frac{k}{s})^{s-1})\).

3. The complement of square grid \( \ell \times \ell \) is a Delsarte clique graph with parameters \((k, s, n) = ((\ell - 1)^2, (\ell - 1), (\ell - 2)!)) \) for \( \ell \geq 4 \).

4. The complement of the triangular graph \( T(2\ell) \) is a Delsarte clique graph with parameters \((k, s, n) = (\ell - 1)(2\ell - 3), \ell - 1, \prod_{i=3}^{\ell}(2\ell - 2i + 1)) \) for \( \ell \geq 4 \).

5. The Johnson graph \( J(m, e) \) with \( m \geq 2e \) is a Delsarte clique graph with parameters (cf. [1, Section 9.1])

\[
(k, s, n) = \begin{cases} 
(e(m - e), e, 2) & \text{if } m = 2e, \\
(e(m - e), m - e, 1) & \text{if } m > 2e.
\end{cases}
\]

6. The folded Johnson graph \( \overline{J}(4m, 2m) \) with \( m \geq 2 \) is a Delsarte clique graph with \((k, s, n) = (4m^2, 2m, 2) \) (cf. Proposition 3.4).

7. The Grassmann graph \( J_q(m, e) \) with \( m \geq 2e \) is a Delsarte clique graph with parameters

\[
(k, s, n) = \begin{cases} 
\left( q \left[ \begin{array}{c} e \\ 1 \end{array} \right]^2, q \left[ \begin{array}{c} e \\ 1 \end{array} \right], 2 \right) & \text{if } m = 2e \\
\left( q \left[ \begin{array}{c} e \\ 1 \end{array} \right], q \left[ \begin{array}{c} m - e \\ 1 \end{array} \right], q \left[ \begin{array}{c} m - e \\ 1 \end{array} \right], 1 \right) & \text{if } m > 2e
\end{cases}
\]

(cf. [1, Section 9.3]).

8. A regular near \( 2D \)-gon is a Delsarte clique graph with parameters \((k, s, n) = (k, a_1 + 1, 1) \) (cf. [1, Section 6.4]). In particular, the Hamming graphs \( H(D, q) \) and the dual polar graphs (with diameter \( D \) over \( GF(q) \)) are Delsarte clique graphs with respective parameters \((q - 1)D, q - 1, 1\), (cf. [1, Section 9.2]) and \( \left( q^e \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], q^e, 1 \right) \), with \( e \in \{0, 1, 2, \frac{1}{2}, \frac{3}{2}\} \) depending on the type of form (cf. [1, Theorem 9.4.3]).

9. The halved \( 2m \)-cube for \( m \geq 3 \), denoted by \( D_{2m, 2m}(1) \), is a Delsarte clique graph with parameters \((k, s, n) = (2m^2 - m, 2m - 1, 2) \) as they are locally \( J(2m, 2) \).

10. The folded halved \( 4m \)-cube with \( m \geq 2 \) is a Delsarte clique graph with \( (k, s, n) = (8m^2 - 2m, 4m - 1, 2) \) (cf. Proposition 3.4).

11. The half dual polar graph \( D_{2m, 2m}(q) \) with \( m \geq 3 \) is a Delsarte clique graph with parameters

\[
(k, s, n) = \left( q \left[ \begin{array}{c} m \\ 2 \end{array} \right], q \left[ \begin{array}{c} 2m \\ 1 \end{array} \right] - 1, q + 1 \right)
\]

(cf. [2, Theorem 3.5], [1, Theorem 9.4.8]).
12. The Ustimenko graphs $Ust_{2m}(q)$ with $m \geq 3$ (the distance 1 or 2 graph of the dual polar graph $Sp(2(2m - 1), q)$) is a Delsarte clique graph with parameters

$$(k, s, n) = \left(\left[\frac{m}{2}\right], \left[\frac{2m}{1}\right] - 1, q + 1\right)$$

(cf. [2, Theorem 3.7], [1, Theorem 9.4.10]). If $q$ is odd, then $Ust_{2m}(q)$ is not edge-transitive, and hence it is not isomorphic to $D_{2m, 2m}(q)$, although they have the same parameters.

13. The Patterson graph is a Delsarte clique graph with parameters $(k, s, n) = (280, 10, 4)$ (cf. [1, Section 13.7]).

14. The halved Foster graph with intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$ is a Delsarte clique graph with parameters $(6, 2, 1)$ (cf. [1, Section 13.2A]).

15. The $3.O_6^+(3)$-graph, an antipodal 3-cover with diameter 4 on 378 vertices with intersection array $\{45, 32, 12, 1; 1, 6, 32, 45\}$, is a Delsarte clique graph with parameters $(45, 5, 3)$ (cf. [1, Section 13.2 C]).

16. The distance 2 graph of the Gosset graph is a Delsarte clique graph with parameters $(27, 3, 5)$.

17. The distance 2 graph of the halved 6-cube is a Delsarte clique graph with parameters $(15, 3, 3)$.

If $m \geq 3$ then for the cases of the halved $2m$-cubes, the folded halved $4m$-cubes, the half dual polar graphs $D_{2m, 2m}(q)$ and the Ustimenko graphs $Ust_{2m}(q)$, the full set of Delsarte cliques is also graphic.

Sometimes, although we know a graph is a Delsarte clique graph as it is edge-transitive, it is not known what the corresponding $n$ is. An example of this is provided by the complement of the Berlekamp–Van Lint–Seidel graph. Also in this case there exists a graphic set of Delsarte cliques, but it is not the full set of Delsarte cliques (see [3]).

Finally, we would like to give an example of a distance-regular graph which is not a Delsarte clique graph, but contains a Delsarte set. That is, let $\Gamma$ be the complement of the Shrikhande graph. It has intersection array $\{9, 4; 1, 6\}$ and the smallest eigenvalue $-3$. Hence cliques of size 4 in $\Gamma$ are Delsarte cliques. There are 72 edges in $\Gamma$, and 48 of them are contained in exactly 1 Delsarte clique, and 24 of them contain exactly 2 Delsarte cliques, So $\Gamma$ is not a Delsarte clique graph. There are 16 Delsarte cliques in $\Gamma$. Exactly 4 of them contain only edges of the second type and they are mutually disjoint. Let $C$ be the set of the remaining 12 cliques. Then each edge of $\Gamma$ lies in exactly one clique of $C$ and it is a (unique) Delsarte set with parameters $(3, 1)$.

4. Delsarte pairs

In this section, we calculate the parameters for any Delsarte pair $(\Gamma', C)$, and then show Theorem 1.3 which states that valency $k$ is bounded in terms of the size of a clique in $C$ for a coconnected thick Delsarte pair $(\Gamma', C)$.

Let $\Gamma'$ be a distance-regular graph of diameter $D \geq 2$. Let $C$ be a Delsarte clique of $\Gamma'$, and define $C_i := \{x \in V(\Gamma') \mid d(x, C) = i\}$ for each $1 \leq i \leq \rho(C)$ where $\rho(C) := \max\{i \mid C_i \neq \emptyset\}$ and $C_0 = C$. Then by [5, Section 13.7], $\rho := \rho(C) = D - 1$
holds and there exist nonnegative integers $\gamma_i(C), \alpha_i(C)$ and $\beta_i(C)$ ($i = 0, 1, \ldots, D - 1$) for the number of neighbors of $x \in C_i$ in $C_{i-1}, C_i$ and $C_{i+1}$, respectively, where $\beta_0(C) = \gamma_0(C) = 0$. For $0 \leq i \leq D - 1$ and a vertex $x \in C_i$, define

$$
\psi_i(x, C) := |\{y \in C \mid d(x, y) = i\}|.
$$

By [5, Section 11.7], the numbers $\psi_i(x, C)$ ($0 \leq i \leq D - 1$) depend not on the pair $(x, C)$ but only on the distance $i = d(x, C)$. Hence let

$$
\psi_i := \psi_i(x, C) \quad (0 \leq i \leq D - 1).
$$

By an easy induction argument, one can see that the numbers $\alpha_i(C), \beta_i(C), \gamma_i(C)$ are not depending on the particular Delsarte clique $C$ and hence we define $\alpha_i := \alpha_i(C), \beta_i := \beta_i(C), \gamma_i := \gamma_i(C)$.

If we say a graph $\Gamma$ has a Delsarte set with $\psi_1 = p$ for some positive integer $p$, then we mean that each Delsarte clique in $\Gamma$ has $\psi_1 = p$.

Now let $(\Gamma, C)$ be a Delsarte pair with parameters $(k, s, n_C)$. For $x, y \in V(\Gamma)$ with $d(x, y) = i$ ($1 \leq i \leq D$) define $\tau_i(x, y; C)$ as the number of a clique $C$ containing $x$ with $d(y, C) = i - 1$.

**Lemma 4.1.** The number $\tau_i(x, y; C)$ ($1 \leq i \leq D$) depends not on the specific pair $(x, y)$ but only on the distance $i = d(x, y)$ and $C$.

**Proof.** Let $x, y \in V(\Gamma)$ with $d(x, y) = i$ ($1 \leq i \leq D$). By counting the number of pairs $(z, C)$, where $z \in \Gamma_{i-1}(y) \cap \Gamma_1(x)$ and $C$ is a clique containing $x$ and $z$ in $C$, in two ways, we find $c_i n_C = \psi_{i-1}\tau_i(x, y; C)$. It follows that

$$
\tau_i(x, y; C) = \frac{c_i n_C}{\psi_{i-1}}, \quad (3)
$$

and hence the number $\tau_i(x, y; C)$ depends not on the pair $(x, y)$ but only on the distance $d(x, y) = i$ and $n_C$.

By the previous lemma, we may put, for $1 \leq i \leq D$,

$$
\tau_i(C) := \tau_i(x, y; C)
$$

where $(x, y)$ is any pair of vertices with $d(x, y) = i$ and $(\Gamma, C)$ is a Delsarte pair.

Now we will derive a generalization of [6, Lemma 4] for distance-regular graphs which contain a Delsarte set.

**Proposition 4.2.** Let $\Gamma$ be a non-complete distance-regular graph with valency $k$ containing a Delsarte set $C$ with parameters $(s, n_C)$. Let $m$ be the multiplicity of the smallest eigenvalue $\frac{k}{s}$ of $\Gamma$. Then the following hold for $\Gamma$:

(i) $c_j = \frac{\tau_j(C)}{n_C} \psi_{j-1}$ $(1 \leq j \leq D)$ and $b_j = \left(\frac{k}{s} - \frac{\tau_j(C)}{n_C}\right) (s + 1 - \psi_j - 1) (1 \leq j \leq D - 1)$.

(ii) Let $u_j := u_j \left(-\frac{k}{s}\right)$ for $0 \leq j \leq D$. Then

$$
u_j = \frac{-\psi_{j-1}}{s + 1 - \psi_{j-1}} u_{j-1}, \quad (4)$$

where $u_j$ and $\psi_j$ are polynomials of $D$-th degree in $\tau_j(C)$.
In particular
\[ u^2_j \geq s^{-2j}. \]  
(5)

(iii) \( m \leq s^{2D} \). Moreover the equality holds if and only if \( s = 1 \).

**Proof.** (i) Let \( x, y \) be two vertices at distance \( j \). By counting the number of pairs \((z, C)\), where \( z \in \Gamma(x) \cap \Gamma(y) \) and \( C \) a clique containing \( x \) and \( z \) in \( C \), in two ways, we obtain the equation
\[ b_j n_C = (s + 1 - \psi_j) \left( \frac{kn_C}{s} - \tau_j(C) \right). \]
This implies the formula for \( b_j \) and by (3) the formula for \( c_j \) holds.

(ii) We prove Eq. (4) by induction on \( j \). As \( u_0 = 1, u_1 = -\frac{1}{s} \) and \( \psi_0 = 1 \), the result holds for the case \( j = 1 \). Let \( 1 \leq j \leq D - 1 \) and assume that
\[ (s + 1 - \psi_{j-1})u_j = -\psi_{j-1}u_{j-1}. \]
Then the following holds:
\[ b_j u_{j+1} = \left( -\frac{k}{s} - a_j \right) u_j - c_j u_{j-1} \]
\[ = \left\{ -\frac{k}{s} - a_j + \frac{c_j(s + 1 - \psi_{j-1})}{\psi_{j-1}} \right\} u_j \]
\[ = -\psi_j \left( \frac{k}{s} - \frac{\tau_j(C)}{n_C} \right) u_j. \]
Hence Eq. (4) holds for \( j + 1 \) by (i). This shows that Eq. (4) holds for all \( 1 \leq j \leq D \). Eq. (5) follows immediately from (4).

(iii) This follows immediately from
\[ m = \frac{|V(\Gamma)|}{\sum_{j=0}^{D} k_j u^2_j} \leq \frac{|V(\Gamma)|}{\left( \frac{1}{s} \right)^{2D} \sum_{j=0}^{D} k_j} = s^{2D} \]
and the inequality is attained if and only if \( s = 1 \). \( \Box \)

**Proof of Theorem 1.3.** If \( \Gamma \) does not contain an induced subgraph \( K_{2,1,1} \), then \( \Gamma \) is a distance-regular graph of order \((s, \frac{k}{s} - 1)\) with the smallest eigenvalue \(-\frac{k}{s}\) and \( s > 1 \). Hence by [6, Theorem 1] we obtain
\[ \frac{k}{s} - 1 < s^{\frac{4D-1}{2}} \]
and the result follows in this case. So we may assume that \( \Gamma \) has an induced subgraph \( K_{2,1,1} \). Then \( c_2 \geq 2 \) and \( h = 1 \). Let \( m \) be the multiplicity of an eigenvalue \(-\frac{k}{s}\). Note that by [1, Proposition 4.4.8], \( m > 2 \) holds as \( s \geq 2 \) and \( D \geq 2 \). Since \( \Gamma \) is coconnected, we have \( k \leq \frac{1}{2}(m-1)(m+2) \) by [1, Theorem 5.3.2]. The theorem now follows in this case by Proposition 4.2(iii). This completes the proof. \( \Box \)
5. Combinatorial properties

In this section we look at the combinatorial properties of a distance-regular graph $\Gamma$ having a Delsarte set. First we will look at what we can derive when $\Gamma$ has a Delsarte clique.

**Lemma 5.1.** Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a Delsarte clique $C$ of size $s + 1$. Then the following hold:

(i) \[ s(a_1 - s + 1) = (k - s)(\psi_1 - 1). \]

In particular, $\psi_1 = 1$ if and only if $C$ has size $a_1 + 2$.

(ii) $c_2 = \psi_1$ if and only if $c_2 = 1$.

(iii) If $c_2 > 1$ then $\Gamma$ contains quadrangles.

**Proof.** (i) Let $x \in C$. This follows by counting the number of edges between $C \setminus \{x\}$ and $\Gamma_1(x) \setminus C$.

(ii) $(\Rightarrow)$: We assume $c_2 = \psi_1$. Let $x \in C$ and $y \in \Gamma_1(x) \setminus C$. Define $U := \Gamma_1(y) \cap C$. We prove that $(\Gamma_1(x) \cap \Gamma_1(y)) \cup \{x, y\}$ is a clique of size $a_1 + 2$. Suppose there exist $z \neq z' \in (\Gamma_1(x) \cap \Gamma_1(y))$ which are not adjacent. We may assume that $z \not\in U$. Then $\Gamma_1(z) \cap C = U$, as otherwise there exists $w \in (\Gamma_1(z) \cap C) \setminus U$, and thus $\{z\} \cup U \subseteq \Gamma_1(w) \cap \Gamma_1(y)$, which contradicts $c_2 = \psi_1$. This implies that $z' \not\in U$ and $\Gamma_1(z) \cap C = U$, as before. Then $\{y\} \cup U \subseteq \Gamma_1(z) \cap \Gamma_1(z')$ which contradicts $c_2 = \psi_1$.

It follows that $(\Gamma_1(x) \cap \Gamma_1(y)) \cup \{x, y\}$ is a clique of size $a_1 + 2$, and hence it must be a Delsarte clique. This implies that $c_2 = \psi_1 = 1$ from (i).

$(\Leftarrow)$: If $c_2 = 1$ then $\psi_1 \leq 1$; hence $\psi_1 = 1$.

(iii) Let $x \in C$ and let $y \in C_1$ at distance 2 from $x$. By (ii) there exists a common neighbor $z$ of $x$ and $y$ which is not in $C$. As $z$ is adjacent to $x$ there exists a vertex $w$ in $C$ which is connected to $y$ but not to $z$. Now the induced subgraph on $\{x, w, y, z\}$ forms an induced quadrangle.

**Lemma 5.2.** Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a Delsarte clique $C$ of size $s + 1$. Then the following hold:

(i) Let $i$ and $j$ be positive integers such that $i + j \leq D - 1$. Then $\psi_i + \psi_j \leq s + 1$.

(ii) If $\psi_1 > \frac{s + 1}{2}$ then $D = 2$.

(iii) Suppose $\psi_1 > 1$. Then for all $x \in C$ the local graph $\Delta(x)$ is connected and its second largest eigenvalue equals $s - \psi_1$. Moreover if $D = 2$ then $\Gamma$ has eigenvalues $k, s - \psi_1, -\frac{k}{2}$.

**Proof.** (i) Let $x \neq y \in C$. Take $z \in \Gamma_i(x) \cap \Gamma_{i+1}(y)$ and $z' \in \Gamma_j(y) \cap \Gamma_{i+j+1}(z)$. Then $C \cap \Gamma_i(z)$ and $C \cap \Gamma_j(z')$ are disjoint. Hence we have

$\psi_i + \psi_j = |C \cap \Gamma_i(z)| + |C \cap \Gamma_j(z')| \leq |C| = s + 1$.

(ii) Assume $D \geq 3$. Then we have $2\psi_1 \leq s + 1$, by putting $i = j = 1$ in (i). This is a contradiction.
(iii) Let $\Delta(x)$ be the subgraph induced on $\Gamma_1(x)$. Then the partition $\{C \setminus \{x\}, \Delta(x) \setminus C\}$ of $\Delta(x)$ is equitable with the quotient matrix

$$Q = \begin{pmatrix} s - 1 & a_1 - s + 1 \\ \psi_1 - 1 & a_1 - \psi_1 + 1 \end{pmatrix}.$$  

As $Q$ has eigenvalues $a_1$ and $s - \psi_1$, it follows that $\Delta(x)$ has an eigenvalue $s - \psi_1$. Lemma 5.1(i) implies that $sb_1 = s(k - 1 - a_1) = (k - s)(s - \psi_1 + 1)$. By [1, Theorem 4.4.3],

$$-1 - \frac{b_1}{\theta_D + 1} = -1 + \frac{sb_1}{k - s} = s - \psi_1$$

is an upper bound for the second largest eigenvalue of $\Delta(x)$. Since $a_1 \geq s - 1 > s - \psi_1$, the second largest eigenvalue of $\Delta(x)$ equals $s - \psi_1$ and $\Delta(x)$ is connected.

Moreover if $D = 2$, then the partition $\{C, V(\Gamma) \setminus C\}$ of $V(\Gamma)$ is equitable with the quotient matrix

$$Q' = \begin{pmatrix} s & k - s \\ \psi_1 & k - \psi_1 \end{pmatrix},$$

and $Q'$ has eigenvalues $k, s - \psi_1$. This completes the proof. \qed

For the definition and properties of a distance-regular antipodal cover and its folded graph, we refer the reader to [1, Section 4.2, p. 438]. Now let us return to Delsarte pairs.

**Theorem 5.3.** Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a thick Delsarte set $C$ with parameters $(s, n_C)$. Then the following hold:

(i) $\psi_1 = 1$ holds if and only if $\Gamma$ is of order $(s, \frac{k}{s} - 1)$ with the smallest eigenvalue $-\frac{k}{s}$.
(ii) If $\psi_1 \geq 2$ then for all $x \in V(\Gamma)$ the local graph $\Delta(x)$ is connected and its second largest eigenvalue equals $s - \psi_1$.
(iii) If $c_2 \geq 2$ then $\Gamma$ contains quadrangles and $\tau_2(C) \geq 2n_C$ holds. In particular $c_2 \geq 2\psi_1$ holds.
(iv) If $\psi_1 + \psi_{D-i-1} = s + 1$ holds for all $i = 0, 1, \ldots, D - 1$, then $\Gamma$ is an antipodal cover. Moreover, if $D$ is odd then $\Gamma$ is an antipodal $2$-cover.

**Proof.** (i), (ii) These follow from Lemmas 5.1(i) and 5.2(iii).

(iii) As $\Gamma$ contains quadrangles from Lemma 5.1(iii), consider the induced quadrangle on $\{x, y, u, w\}$ with $d(x, y) = d(u, w) = 2$. Then $\tau_2(C) = \tau_2(x, y; C) \geq 2n_C$ holds by considering the Delsarte cliques of $C$ containing the edge $(x, u)$ and the Delsarte cliques of $C$ containing the edge $(x, w)$. From the equation $c_2 = \tau_2(C)\psi_1/n_C$, the result follows.

(iv) We find $u_D := u_D(\theta_D) = (-1)^D$ by Proposition 4.2 and our assumption. Then [1, Proposition 4.4.7] implies that $\Gamma$ must be antipodal as $\Gamma$ is not bipartite. Moreover if $D$ is odd then we have $u_D = -1$. Let $x$ and $y$ be vertices of $\Gamma$ at distance $D$. Then $(\bar{x}, \bar{y}) = u_D = -1$ and thus $\bar{x} = -\bar{y}$, where $\bar{\cdot}$ denote the standard representation corresponding to $\theta_D$. (See [1, Proposition 4.4.11].) If there exists $z \in \Gamma_D(x) \cap \Gamma_D(y)$ then $\bar{x} = -\bar{z} = -\bar{y}$ which is a contradiction. Hence $\Gamma_D(x) \cap \Gamma_D(y) = \emptyset$ and $\Gamma$ must be an antipodal $2$-cover. \qed
Now we will show that for any Delsarte pair with $\psi_1 \geq 2$, the sequence $(\psi_i)_i$ is strictly increasing. As a consequence, we obtain a diameter bound in terms of the size of a Delsarte clique. First we need the following lemma.

**Lemma 5.4.** Let $\Gamma$ be a distance-regular graph with diameter $D \geq 2$. Let $C$ and $C'$ be distinct Delsarte cliques in $\Gamma$ such that $C \cap C' \neq \emptyset$ holds. Then one of the following holds.

(i) $|C \cap C'| = \psi_1$.
(ii) $|C \cap C'| \leq \psi_i - \psi_{i-1}$ for all $i = 2, \ldots, D - 1$.

**Proof.** Let $s + 1$ be the size of Delsarte cliques. Let $T := C \cap C'$ and $t := |T| > 0$. Since $C \neq C'$, there exists $x \in C' \setminus C$ and

$$1 \leq t = |T| \leq |C \cap \Gamma_1(x)| = \psi_1.$$ 
Suppose $t < \psi_1$; we show that (ii) holds.

For each $i = 2, \ldots, D - 1$, take $y \in \Gamma_{i-1}(x) \cap C_i$. Then $y \in C'_{i-1}$. Set $M := C \cap \Gamma_i(y)$ and $N := C \setminus M$. Then $|M| = \psi_i$, $|N| = s + 1 - \psi_i$ and $T \subseteq M$. Set $P := C' \cap \Gamma_{i-1}(y)$ and $Q := C' \setminus (P \cup T)$. Then $P \cap T = \emptyset$, $|P| = \psi_{i-1}$ and $|Q| = s + 1 - \psi_{i-1} - t$. Note that there are no edges between $N$ and $P$ as $N \subseteq \Gamma_{i+1}(y)$ and $P \subseteq \Gamma_{i-1}(y)$. By counting the number of edges between $N$ and $Q$ we have

$$|N|(|\psi_1 - t) = \sum_{u \in N} |Q \cap \Gamma_1(u)| = \sum_{w \in Q} |N \cap \Gamma_1(w)| \leq |Q|(|\psi_1 - t).$$

Since $t < \psi_1$, we have $s + 1 - \psi_i = |N| \leq |Q| = s + 1 - \psi_{i-1} - t$. This shows that (ii) holds. This completes the proof. □

**Theorem 5.5.** Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a thick Delsarte set $C$ with parameters $(s, n_C)$. Suppose $\psi_1 > 1$. Then

$$1 < \psi_1 < \cdots < \psi_{D-1}.$$ 

In particular, $D \leq s$.

**Proof.** Suppose $\psi_{i-1} = \psi_i$ for some $2 \leq i \leq D - 1$. Then Lemma 5.4 implies that any two intersecting Delsarte cliques have exactly $\psi_1$ common vertices. Let $C$ be a Delsarte clique in $C$ and $x \in C_1$. Set $W := C \cap \Gamma_1(x)$. As $|W| = \psi_1 \geq 2$, there are two vertices $w \neq w' \in W$. Let $C^{(1)}, \ldots, C^{(n_C)}$ be all the Delsarte cliques in $C$ containing the edge $(x, w)$. Since $|C \cap C^{(j)} = \psi_1$ holds, $C \cap C^{(j)} = C \cap \Gamma_1(x)$ and thus $w, w' \in C \cap C^{(j)}$ for all $1 \leq j \leq n_C$. Thus there are $n_C + 1$ Delsarte cliques $C^{(1)}, \ldots, C^{(n_C)}$, $C \in \mathcal{C}$ containing the edge $(w, w')$. This is a contradiction. Therefore $1 < \psi_1 < \cdots < \psi_{D-1}$. Moreover $j \leq \psi_{j-1}$ ($2 \leq j \leq D$) holds by induction on $j$. Hence $D \leq \psi_{D-1} \leq s$ holds. This shows the theorem. □

**Corollary 5.6.** Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a thick Delsarte set $C$ with parameters $(s, n_C)$. Suppose $\psi_1 > 1$ and $D = s$. Then $\psi_i = i + 1$ holds for all $i = 0, 1, \ldots, D - 1$ and $\Gamma$ is an antipodal cover. Moreover, if $D$ is odd then $\Gamma$ is an antipodal 2-cover.
Proof. The proof of the above theorem implies that \( D = s \) holds if and only if \( \psi_i = i + 1 \) holds for all \( 0 \leq i \leq D - 1 \). The rest of the assertion follows immediately from Theorem 5.3(iv). \( \square \)

Proof of Theorem 1.4. (i) This follows immediately from Theorems 5.3 and 5.5.

(ii) By Theorem 5.5, we have \( \psi_i < \psi_{i+1} \). Let \( u_j = u_j( -\frac{k}{s}) \). By induction on \( j \), it follows that for each \( 0 \leq j \leq D - 1 \) we have

\[
|u_j| \geq \frac{1 \cdot 2 \cdots \lfloor \frac{s}{2} \rfloor}{s \cdot (s - 1) \cdots (s + 1 - \lfloor \frac{s}{2} \rfloor)} = \left( \frac{s}{\lfloor \frac{s}{2} \rfloor} \right)^{-1}.
\]

Let \( m \) be the multiplicity of an eigenvalue \( -\frac{k}{s} \) of \( \Gamma \). As \( s \geq 2 \) and \( D \geq 2 \), \( m > 2 \) holds by [1, Proposition 4.4.8]. As \( \Gamma \) is coconnected, we have \( k \leq \frac{1}{s}(m - 1)(m + 2) \) by [1, Theorem 5.3.2]. Substituting Eq. (6) in Biggs’ Formula (2) for \( m \), the result follows. This completes the proof. \( \square \)

6. Classifications

In this section we will classify distance-regular graphs containing a Delsarte set with \( \psi_1 = s \geq 2 \) and \( \psi_1 = s - 1 \geq 2 \), respectively. As an application we will classify distance-regular graphs containing a Delsarte set with small \( s \).

6.1. \( \psi_1 = s, s - 1 \)

First we will look at the case \( \psi_1 = s \).

Proposition 6.1. Let \( \Gamma \) be a distance-regular graph with valency \( k \) and diameter \( D \geq 2 \), containing a Delsarte clique \( C \) with size \( s + 1 \geq 3 \). Then \( \psi_1 = s \) holds if and only if \( \Gamma \) is a complete multipartite graph \( K_{(s+1) \times \frac{k}{s}} \).

Proof. Suppose \( \psi_1 = s \). Then \( D = 2 \) holds from Lemma 5.2(ii) as \( s \geq 2 \). It follows, by Lemma 5.2(iii), that \( \Gamma \) has eigenvalues \( k, 0, -\frac{k}{s} \), and hence \( \Gamma \) is a complete multipartite graph \( K_{(s+1) \times \frac{k}{s}} \). Clearly if \( \Gamma \) is complete multipartite, then \( \psi_1 = s \). This shows the proposition. \( \square \)

Now let us consider the case \( \psi_1 = s - 1 \). Checking the examples in Section 3 we see that the following graphs have a Delsarte set with \( \psi_1 = s - 1 \geq 2 \): the distance 2 graph of the halved 6-cube, the distance 2 graph of the Gosset graph, the Johnson graphs \( J(5, 2) \) and \( J(6, 3) \), the complements of the square grid \( \ell \times \ell \) and the complements of triangular graphs \( T(2\ell) \) for integral \( \ell \geq 4 \), and the complement of the Shrikhande graph.

Now we will show that these graphs are all the distance-regular graphs having a Delsarte set with \( \psi_1 = s - 1 \geq 2 \).

Theorem 6.2. Let \( \Gamma \) be a non-complete distance-regular graph with valency \( k \), containing a Delsarte clique of size \( s + 1 \) with \( \psi_1 = s - 1 \geq 2 \). Then \( \Gamma \) contains a Delsarte set if and only if \( \Gamma \) is one of the following graphs:

(i) the complement of the triangular graph \( T(2\ell) \) for \( \ell \geq 4 \);
(ii) the complement of square grid \( \ell \times \ell \) for \( \ell \geq 4 \);
(iii) the complement of the Shrikhande graph;
(iv) the Johnson graph $J(5, 2)$;
(v) the Johnson graph $J(6, 3)$;
(vi) the distance 2 graph of halved 6-cube;
(vii) the distance 2 graph of the Gosset graph.

To prove the theorem we need the following result shown by Seidel [8]; cf. [1, Theorem 3.12.4(i)].

**Proposition 6.3.** Let $\Gamma$ be a strongly regular graph with the smallest eigenvalue $-2$; then $\Gamma$ is either a complete multipartite graph $K_{\ell \times 2}$ ($\ell \geq 2$), a triangular graph $T(\ell)$ ($\ell \geq 5$), a square grid $\ell \times \ell$ ($\ell \geq 3$), or one of the graphs of Petersen, Clebsch, Schläfli, Shrikhande, or Chang.

**Proof of Theorem 6.2.** Suppose $\Gamma$ contains a Delsarte set. Let us first assume $D = 2$. By Lemma 5.2(iii), the second largest eigenvalue of $\Gamma$ is $1$ as $\psi_1 = s - 1$. Then the complement of $\Gamma$ is a strongly regular graph with the smallest eigenvalue $-2$. Hence $\Gamma$ is the complement of one of the graphs shown in Proposition 6.3. It is easy to see that the complement of the Clebsch graph, i.e. the folded 5-cube, and the complements of the triangular graphs $T(2\ell + 1)$ do not contain any Delsarte clique. The complement of the square grid $3 \times 3$, the complement of the triangular graph $T(6)$ and the complement of the Schläfli graph are the generalized quadrangle $GQ(2, 1)$, $GQ(2, 2)$ and $GQ(2, 4)$, respectively, and hence $\psi_1 = 1$ in these cases, which do not satisfy our assumption. It can be checked that the complements of the three Chang graphs do not contain any Delsarte sets. And the complement of the Petersen graph is the Johnson graph $J(5, 2)$. Hence $\Gamma$ is one of the graphs (i)–(iv) in this case.

Now let us assume $D \geq 3$. Lemma 5.2(ii) shows that $2 \leq s - 1 \leq \frac{3\ell+1}{4}$ holds and hence $s = 3$. Then $\Gamma$ is an antipodal 2-cover of diameter 3 and valency $k = \frac{3}{2}(a_1 - 1)$, by Lemma 5.1(i), Theorem 5.5 and Corollary 5.6. Hence $\Gamma$ is a Taylor graph with intersection array $\{3a_1 - 3, 2a_1 - 4, 1; 1, 2a_1 - 4, 3a_1 - 3\}$ and the distance 2 graph $\tilde{\Gamma}$ of $\Gamma$ is also a Taylor graph with intersection array $\{3a_1 - 3, a_1, 1; 1, a_1, 3a_1 - 3\}$ ([1, p. 431]). It follows from [1, Corollary 1.15.3] that $a_1 \in \{4, 6, 10\}$. Therefore $\tilde{\Gamma}$ is the Johnson graph $J(6, 3)$, the halved 6-cube and the Gosset graph for $a_1 = 4, 6$ and 10, respectively. Note that $\Gamma$ is also the distance 2 graph of $\tilde{\Gamma}$ and the distance 2 graph of the Johnson graph $J(6, 3)$ is itself. Hence $\Gamma$ is one of the graphs (v), (vi), (vii) in this case.

The theorem follows from the fact that the seven families of graphs all contain a Delsarte set.

**Remark 6.4.** We also could have used the classification of the distance-regular graphs with an eigenvalue $-1 - \frac{b_1}{2}$ by Koolen [7, Theorem 1] for Theorem 6.2 as $-\frac{k}{3} = -1 - \frac{b_1}{2}$ is the smallest eigenvalue of $\tilde{\Gamma}$ by Lemma 5.1(i). But Koolen did miss three cases, namely the three Taylor graphs with $(k, c_2) \in \{(9, 4), (15, 8), (27, 16)\}$. In the following theorem we correct his result. His proof is essentially correct. At the end of the proof in [7] he obtained eight possible pairs for $(k, a_1)$ and diameter $D \leq 3$. It is easy to show that the number of vertices is bounded by 900. He checked the tables in [1], but he forgot to check the antipodal graphs of diameter 3. The corrected version for [7, Theorem 1] is the following:
Theorem 6.5. Let $\Gamma$ be a distance-regular graph with $a_1 \geq 2$. Then $\Gamma$ has an eigenvalue $-1 - \frac{b_1}{2}$ if and only if one of the following holds:

(i) $\Gamma$ is a complete graph.
(ii) $\Gamma$ is a connected strongly regular graph with the second largest eigenvalue 1.
(iii) $\Gamma$ is the Conway–Smith graph with $\iota(\Gamma) = \{10, 6, 4, 1; 1, 2, 6, 10\}$.
(iv) $\Gamma$ is the Johnson graph $J(6, 3)$ with $\iota(\Gamma) = \{9, 4, 1; 1, 4, 9\}$.
(v) $\Gamma$ is the distance-2 graph of the halved 6-cube with $\iota(\Gamma) = \{15, 8, 1; 1, 8, 15\}$.
(vi) $\Gamma$ is the distance-2 graph of the Gosset graph with $\iota(\Gamma) = \{27, 16, 1; 1, 16, 27\}$.

6.2. $s = 2, 3$

In this subsection, we will classify Delsarte clique graphs with $s = 2, 3$.

Proposition 6.6. Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a Delsarte set $C$ with parameters $(2, nC)$. Then $\Gamma$ is either a graph of order $(2, k^2 - 1)$ with the smallest eigenvalue $-k$, or the complete multipartite graph $K_{3 \times \frac{k}{2}}$.

Proof. Note that $1 \leq \psi_1 \leq s = 2$. By Theorem 5.3(i) and Proposition 6.1 the result follows. □

Theorem 6.7. Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D \geq 2$, containing a Delsarte set $C$ with parameters $(3, nC)$. Then $\Gamma$ is one of the following graphs:

(i) distance-regular graphs of order $(3, k^3 - 1)$ with the smallest eigenvalue $-\frac{k}{3}$;
(ii) the complete multipartite graph $K_{4 \times \ell}$ for $\ell \geq 2$;
(iii) the complement of the triangular graph $T(8)$;
(iv) the complement of the square grid $4 \times 4$;
(v) the complement of the Shrikhande graph;
(vi) the Johnson graph $J(5, 2)$;
(vii) the Johnson graph $J(6, 3)$;
(viii) the distance 2 graph of the halved 6-cube;
(ix) the distance 2 graph of the Gosset graph.

Proof. Note that $1 \leq \psi_1 \leq s = 3$. Then by Theorem 5.3(i), Proposition 6.1 and Theorem 6.2 the result follows. □

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