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# WAVE KERNELS RELATED TO SECOND-ORDER OPERATORS

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**Abstract**
*The wave kernel for a class of second-order subelliptic operators is explicitly computed. This class contains degenerate elliptic and hypo-elliptic operators (such as the Heisenberg Laplacian and the Grušin operator). Three approaches are used to compute the kernels and to determine their behavior near the singular set. The formulas are applied to study propagation of the singularities. The results are expressed in terms of the real values of a complex function extending the Carnot-Caratheodory distance, and the geodesics of the associated sub-Riemannian geometry play a crucial role in the analysis.*

**Contents**



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## 1. Introduction

We study in this paper properties of fundamental solutions of wave equations associated with several subelliptic second-order self-adjoint operators $L$. We give an explicit expression for the Grušin operator, the Heisenberg Laplacian, and the harmonic oscillator.

Recall that the general solution of the wave equation

$$\frac{\partial^2 w_L}{\partial t^2} = -L w_L,$$
$$w_L(0) = f,$$
$$\frac{\partial w_L(0)}{\partial t} = g, \tag{1.1}$$

has the formal expression

$$w_L(t) = \frac{\sin(tL^{1/2})}{L^{1/2}} g + \cos(tL^{1/2}) f. \tag{1.2}$$

It suffices to compute the fundamental solution $w_L$ that satisfies equation (1.1) and $w_L(0) = 0$, $\partial w_L(0)/\partial t = \delta_0$, where $\delta_0$ denotes the Dirac distribution at the origin zero, that is,

$$w_L(t) = \frac{\sin(L^{1/2}t)}{L^{1/2}}, \tag{1.3}$$

or to compute $\partial w_L/\partial t = \cos(tL^{1/2})$—the solution of (1.1) where $w_L(0) = \delta_0$, $\partial w_L(0)/\partial t = 0$.

We consider the wave kernel for a number of second-order operators. We obtain in some cases an explicit representation for the kernel and some information about propagation of singularities.

The wave kernel for the standard wave equation in the general $n$-dimensional Euclidean space $\mathbb{R}^n$ was computed first by J. Hadamard (see R. Courant and D. Hilbert [5]), who considered more generally the case where $L$ is elliptic. The kernel is of the form $\delta^{(p)}(t^2 - |x|^2)$ for the usual (Euclidean) Laplacian. In the general elliptic case, the solution is represented as a sum of terms, starting from the less regular to the more regular (see [5]). Once again the leading term is proportional to $\delta^{(p)}(t^2 - |x|^2)$, where now $|x|$ is the distance between $x$ and the origin in the Riemannian metric associated with the second-order elliptic operator.

The wave equation for a subelliptic $L$ has been considered in [10]. The finite speed of propagation was established in [12]. The first systematic analysis of the wave kernel for the Heisenberg Laplacian was performed in [13], where the "light cone" was described and propagation of singularities discussed. The computations are somewhat complicated. In [15] the broad features of the propagation of singularities for the Heisenberg Laplacian are also described using a different method based on deforming the path of integration in the complex plane. The geometry of the light cone is rather different from the standard Euclidean case.

It turns out that finite speed of propagation is associated with the so-called Carnot-Caratheodory distance defined as the sub-Riemannian length of a minimizing geodesic (see [4], [14]); the formula for the wave kernel, and the full light cone, involve all geodesics. Sub-Riemannian geometry differs substantially (even locally) from usual Riemannian geometry. On the other hand, a complex-valued function $f$, appearing in the integral representation for the heat kernel on Heisenberg group (see [6]), was shown in [1] to satisfy a Hamilton-Jacobi equation with the symbol of the Heisenberg Laplacian as Hamiltonian. Critical points and critical values of this function $f$ (extended analytically to the complex plane) correspond to sub-Riemannian geodesics and their lengths, respectively. A curve on which the function $f$ is real is constructed in [15] and in [1]. Our formulas involve integration along this curve. Moreover, adding a time-dependent term to $f$, we obtain a complex phase satisfying a Hamilton-Jacobi equation with the symbol of the wave operator as Hamiltonian.

Observe that in both [13] and [15] propagation of singularities is studied without actually computing the wave kernel. In [13] the kernel is given as a limit of expressions containing integrals (or an infinite series); one could presumably get a closed form with extra effort. No attempt at calculating the kernel is made in [15]; the appearance of fractional powers in [15, (8.7)] makes explicit computations difficult. One of the main purposes of the present paper is to obtain a more explicit formula for the Heisenberg wave kernel. Known properties of singularities (such as propagation) are then easily obtained. Moreover, the relationship between the sub-Riemannian geometry and complex integration formulas (such as in [6] and [1]) is put into context.

Explicit formulas for model operators (such as the wave kernels for the Heisenberg Laplacian or the Grušin operator), while interesting in their own right, may also offer new insights into the problem and may serve as principal terms in approximations for more general cases.

Three methods are applied in this paper for explicit computation of the wave kernels. The first involves separation of variables, summation of series containing Hermite polynomials, and deformation of integration path in the complex plane. This approach is utilized in Section 2 for solving (1.1) if $-L/2$ is the Grušin operator in $\mathbb{R}^2$,

$$-\frac{L}{2} = \frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2}\Big), \tag{1.4}$$

and in Section 3 for the case where $-L/2$ is the Harmonic oscillator in $\mathbb{R}^1$,

$$-\frac{L}{2} = \frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} - x^2\Big). \tag{1.5}$$

As an additional illustration, we solve in Section 3 the Klein-Gordon equation. The integral representing the wave kernel for the Grušin operator may be evaluated explicitly by the residue theorem (Proposition 4), and all properties (geometry of the light cone, behavior near this cone, band structure) may be read off the resulting (rather explicit) formula. On the other hand, the formula for the harmonic oscillator (Proposition 7) involves an integral over a path where the phase is purely imaginary, and it seems that the integral may not be easily evaluated. To the best of our knowledge, wave kernels for the Grušin operator and for the harmonic oscillator were not calculated before, nor was the propagation of singularities for the Grušin operator studied in detail.

The second method involves inversion of the transmutation formula (Proposition 9; see [8]) and deforming the integration path used in the integral formula for the heat kernel from the real axis to a path in the complex plane where the exponent is real. This approach is described in Section 4 and applied for the case where $-L/2$ is the Heisenberg Laplacian $\Delta_H$ defined on $\mathbb{R}^{2n+1} = \{(x_1, \dots, x_{2n}, x_0)\}$ by

$$\Delta_H = \frac{1}{2}\bigg(\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j-1}} + 2\alpha_j x_{2j}\frac{\partial}{\partial x_0}\Big)^2 + \frac{1}{2}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j}} - 2\alpha_j x_{2j-1}\frac{\partial}{\partial x_0}\Big)^2\bigg), \tag{1.6}$$

where $\alpha_1, \dots, \alpha_n$ are positive constants. Throughout most of the paper we consider the isotropic case in which all the $\alpha_j$'s are equal to a constant $\alpha$ (see, e.g., (4.11), (5.15)). (We comment on the extension to the general anisotropic case at various points in the paper.) The expression for the wave kernel using a complex contour is given in Theorem 2 (formula (4.14)). One may rewrite the formula using integration over real intervals (Theorem 3). The role of the geodesics emerges clearly, as the integration is performed over intervals of the form $(d_{2k-1}^2, d_{2k}^2)$, where $d_j$ is the length of the $j$th geodesic. One may obtain a closed form (not involving integrations) if $x = (x_1, \dots, x_{2n}) = 0$ (Theorem 4). The leading singularities of the wave kernel are calculated (and are compatible with the results of [13]). Observe that in Section 4 we deal with the kernel of $\cos(tL^{1/2})$, unlike the rest of the paper where $\sin(tL^{1/2})/L^{1/2}$ is treated.

The third method is based on an analytic continuation of the Green function of the operator $-L + \partial^2/\partial y^2$ and uses an idea due to M. Taylor [15]. This method is described in Section 5 and applied to the Heisenberg Laplacian, as well to the case where $-L$ is a degenerate elliptic operator of the type studied in [2]. This class contains the Grušin operator and the Baouendi-Goulaouic operator as special cases. While the results for the Heisenberg Laplacian parallel those of Section 4, new phenomena occur for degenerate operators—if certain dimensions are odd (as is the case for the Grušin operator), the integral representing the kernel is computable (by the residue theorem or otherwise) and yields a simple expression for the kernel. The results coincide with those of Section 2.

There is of course a certain amount of redundancy in rederiving the same results by different methods. We feel, however, that each method has its advantages. Thus separation of variables is directly applicable to the harmonic oscillator; using the heat kernel, we may compute $\cos(tL^{1/2})$ directly; and analytic continuation of the Green function enables a straightforward calculation for degenerate elliptic operators without prior computation of the associated heat kernel. Moreover, one should not forget that separation of variables underlies the computation of heat kernels in [15], as well as that of the Green kernels in [3].

An entirely different method for computing wave kernels for certain second-order operators was suggested in [9]. The method is based on transmutation formulas and on the Trotter product formula. Some kind of a "Feynman integral representation" is obtained, and the expression for a wave kernel involves differentiating to a high order a very high-dimensional integral. In [9] expressions were obtained for the wave kernel $\cos(tL^{1/2})$ where $-L/2$ is the harmonic oscillator and when $-L/2$ is the Heisenberg Laplacian. A direct proof of the identity of the expressions from [9] with the expression obtained here appears to be nontrivial.

In the remainder of this section we collect some preliminary material concerning sub-Riemannian geometry, complex action, and separation of variables.

### *1.1. Sub-Riemannian (Carnot-Caratheodory) metrics*

Recall the definition of sub-Riemannian (Carnot-Caratheodory, also known as C-C) metrics (see [4, pp. 4 – 7]): Let $X_1, \ldots, X_m$ be smooth vector fields on a manifold $M$. For $x \in M$ and $v \in T_x M$,

$$\|v\|_x^2 = \inf\big\{u_1^2 + \cdots + u_m^2 \text{ s.t. } u_1 X_1(x) + \cdots + u_m X_m(x) = v\big\}. \tag{1.7}$$

In particular, $\|v\|_x^2 = \infty$ if $v$ is not contained in $\mathrm{sp}(X_1, \ldots, X_m)$.

The length $l(c)$ of an absolutely continuous curve $c(t)$ $(a \le t \le b)$ contained in $M$ (*absolute continuity* is well defined in terms of local charts) is given by the integral $\int_a^b \|\dot{c}(t)\|_{c(t)}\, dt$, and the energy of $c$ is equal to $\int_a^b \|\dot{c}(t)\|_{c(t)}^2\, dt$.

The distance between two points is defined by $d(p, q) = \inf l(c)$, where the infimum is taken over all absolutely continuous curves joining $p$ and $q$.

In this paper we always assume that the vector fields $X_1, \ldots, X_m$ and their brackets $[X_i, X_j]$, $1 \leq i, j \leq m$, span the tangent space $T_x M$ at every point of $M$, and $M$ is connected. By Chow's theorem (see [4, p. 15]), any two points in $M$ can be joined by an absolutely continuous curve with finite length. Hence $d(p, q) < \infty$ for any points $p, q \in M$. (Note that we consider here only the so-called step-two case.)

We can define the Hamiltonian associated with the sub-Riemannian metric by

$$H(x, \xi) = \frac{1}{2} \sum_{i=1}^{m} \langle X_i, \xi \rangle^2. \tag{1.8}$$

Note that we do not have a finite metric defined on the tangent bundle; we are forced to study the cotangent bundle. It is well known that any two points $p, q$ may be joined by a curve whose length equals $d(p, q)$. Thus the distance between $p$ and $q$ is attained as the length of a minimizing geodesic joining $p$ and $q$. Moreover, the geodesic curves are projections onto $M$ of bicharacteristics of Hamiltonian $H$ (see [14], [1]). Observe that if we normalized the "time" to be equal to 1, then $d^2(p, q)$ is equal to the energy of the minimizing geodesic joining $p$ with $q$ and is also equal to twice the action $S$ computed along the corresponding bicharacteristic.

Perhaps the simplest example of a sub-Riemannian metric is the metric associated with the Grušin operator. In the Grušin plane, $\mathbb{R}^2$, the sub-Riemannian metric is given by the vectors

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

The vector fields span the tangent plane everywhere, except along the line $x = 0$. But since

$$[X_1, X_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Chow's conditions are satisfied and it follows that the sub-Riemannian distance between any two points is finite (see [4, p. 24]).

In the complement of the line $x = 0$, the sub-Riemannian metric is Riemannian, $G = (\mathbb{R}^2, ds)$, where $ds^2 = dx^2 + dy^2/x^2$. The Hamiltonian is given by

$$H(x, y, \eta, \xi) = \frac{1}{2}(\xi^2 + x^2\eta^2) \tag{1.9}$$

and is equal to the symbol of the Grušin operator.

The distance between two points is

$$d(P, Q) = \inf_{c(t) \in C^1([0,1], G),\ c(0)=P,\ c(1)=Q} \int_0^1 \|\dot{c}(t)\|_{c(t)}\, dt. \tag{1.10}$$

A simple computation yields the Euler-Lagrange equations

$$\frac{\dot{y}}{x^2} = b,$$
$$\ddot{x} + \frac{\dot{y}^2}{x^3} = 0, \tag{1.11}$$

where $b$ is a constant. All geodesics may be computed explicitly from (1.11). In particular, the geodesics starting at the origin are given by

$$x(t) = \frac{c}{b}\sin(bt),$$
$$y(t) = \frac{c^2}{b}\Big(\frac{t}{2} - \frac{\sin(2bt)}{4b}\Big), \tag{1.12}$$

where $b$ and $c$ are arbitrary real parameters. It is easy to see that these geodesics are projections of certain bicharacteristics—the solutions of the system

$$\dot{x} = \frac{\partial H(x, y, \eta, \xi)}{\partial \xi} = \xi,$$
$$\dot{\xi} = -\frac{\partial H(x, y, \eta, \xi)}{\partial x} = -x\eta^2,$$
$$\dot{y} = \frac{\partial H(x, y, \eta, \xi)}{\partial \xi} = x^2\eta,$$
$$\dot{\eta} = -\frac{\partial H(x, y, \eta, \xi)}{\partial y} = 0,$$
$$\tag{1.13}$$

with the initial conditions $x(0) = y(0) = 0$, $\xi(0) = c$, $\eta(0) = b$. A similar system was studied in [6] and in [13] for the Heisenberg group.

Observe that

$$\frac{y}{x^2} = \frac{b}{2}\Big(\frac{t}{\sin^2(bt)} - \frac{\cos(bt)}{b\sin(bt)}\Big) = \frac{1}{2}\mu(\theta), \tag{1.14}$$

where

$$\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot(\theta), \tag{1.15}$$

and $\theta = bt$.

It follows that if $x \neq 0$, then for every solution of the equation

$$2\frac{y}{x^2} = \mu(\theta) \tag{1.16}$$

there corresponds a geodesic joining the origin with the point $(x, y)$. The graph of the function $\mu$ is portrayed in Figure 1. The Hamiltonian is constant along any bichar-

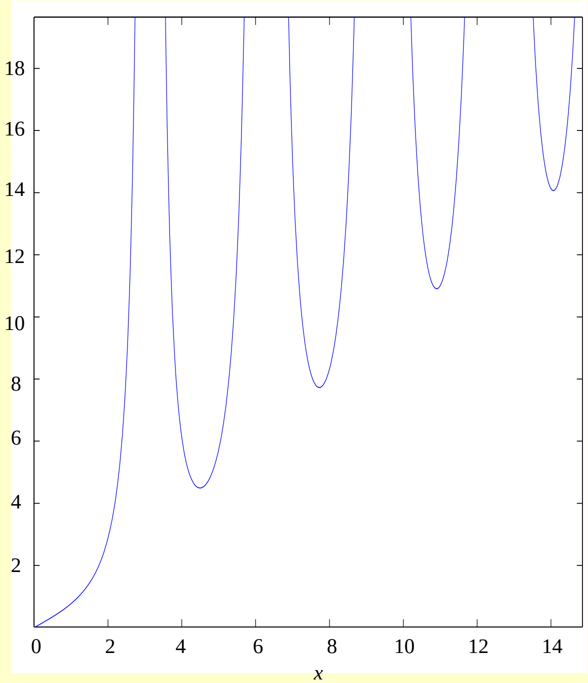


Figure 1. $\phi \to \phi/\sin^2\phi - \cot\phi$

acteristic. For a geodesic starting at the origin, $H$ is equal to $(1/2)\xi(0)^2 = (1/2)c^2$. Hence the energy is equal to

$$2S(x, y) = \int_0^t 2H(t)\,dt = \int_0^t c^2\,dt = c^2 t = \frac{c^2\theta}{b}, \tag{1.17}$$

and a similar computation shows that the length of the geodesic is equal to $c\theta/b$. (Note that once $\theta$ is found from (1.16), $c/b$ is obtained from the first equation in (1.12).) If $x \neq 0$, then the number of geodesics joining $(x, y)$ with the origin is finite and grows from 1 to $\infty$ as $y/x^2$ varies from 0 to $\infty$. If $x = 0$, then $\mu(\theta) = \infty$ so that $\theta = k\pi$ for any integer $k > 0$. Correspondingly, there exist infinitely many geodesics joining the origin to $(0, y)$ with lengths satisfying $d^2 = 2\pi k|y|$. (Here $c/b$ is calculated from the second equation of (1.12).)

Another example is the Heisenberg group $H_n$. The Carnot-Caratheodory metric associated with the left-invariant vector fields has been discussed in detail (see [13], [1], [4]). In particular, the Hamiltonian is the symbol of the Heisenberg Laplacian. If $x = (x_1, \ldots, x_{2n}) \neq 0$, then there exist (in the isotropic case) finitely many geodesics joining the origin with $(x_1, \ldots, x_{2n}, x_0)$ parametrized by the solutions of the equation $\mu(\theta) = 2x_0/r^2$, where $r^2 = |x|^2 = \sum_{i=1}^{2n} x_i^2$. If $x = 0$, then there exist infinitely many geodesics joining the origin with $(0, x_0)$ parametrized by the Cartesian product of $S^{2n-1}$ with the set of nonzero integers. Otherwise, the computations are similar to the case of the Grušin plane.

Degenerate elliptic operators of the kind studied in Section 5.2 form a generalization of the Grušin operator. A subclass consists of operators of the form

$$L = -\left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + |x_1|^2 \frac{\partial^2}{\partial y^2}\right), \tag{1.18}$$

where $x_i \in V_i$ $(i = 1, 2)$, $V_1$, $V_2$ are real Euclidean vector spaces, and $\partial/\partial x_i$ denotes the gradient in $V_i$ $(i = 1, 2)$. In order to cover both the Heisenberg Laplacian and the degenerate elliptic operators, let us replace $y$ by $x_0/a$, where $a$ is a positive constant. In the case of operators described by (1.18), we set $r^2 = |x_1|^2$. The Hamiltonian is the symbol of the operator. Once again we treat here only geodesics starting at the origin. The following propositions hold in all cases.

PROPOSITION 1

*There is a finite number of geodesics joining the origin with $(x, x_0)$ if and only if $r \neq 0$. They are parametrized by the solutions $\theta$ of equation (1.16). Their lengths increase with $\theta$. With $\phi_1$ denoting the first critical value of $\mu$, there is only one geodesic if and only if $2|x_0|/r^2 < \mu(\phi_1)$. The number of geodesics increases to $\infty$ with $|x_0|/r^2$.*

*The C-C distance $d_c(x, x_0)$ between the origin and the point $(x, x_0)$ is given by the length of the shortest geodesic joining these points and $d_c^2(x, x_0) = 2S(x, |x_0|)$, where $S$ is the action along the shortest geodesic.*

PROPOSITION 2

*There is an infinite number of geodesics that join the origin to the point $(0, x_0)$ of length*

$$d_k^2 = 2\pi k \frac{|x_0|}{a}, \quad k = 1, 2, \ldots. \tag{1.19}$$

*The Carnot-Caratheodory distance from the origin to $(0, x_0)$ is given by $d_c(0, x_0) = \sqrt{2\pi(|x_0|/a)}$.*

These propositions are proved in [1] for the Heisenberg case; the degenerate elliptic case is very similar.

*1.2. A curve in the complex plane and the complex action*

We continue to use the notation introduced in Section 1.1. Thus let $x$ denote either the vector $(x_1, \ldots, x_{2n})$ (the Heisenberg group case) or the vector $(x_1, x_2) \in V_1, V_2$ (the generalized Grušin case), and let $x_0 \in \mathbb{R}^1$, $r = |x|$ (Heisenberg) or $r = |x_1|$ (Grušin),

$z \in \mathbb{C}$, and $a \in \mathbb{R}_+$. Consider the function

$$f(x, x_0, z; a) = \frac{a}{2} r^2 z \coth(az) - i x_0 z. \tag{1.20}$$

Then

$$f'(x, x_0, z; a) = \frac{a}{2} r^2 \Big( \coth(az) - \frac{az}{\sinh^2(az)} \Big) - i x_0 \tag{1.21}$$

and

$$f''(x, x_0, z; a) = \frac{a^2 r^2}{\sinh^2(az)} \Big( -1 + \frac{az}{\tanh(az)} \Big), \tag{1.22}$$

where $'$ denotes differentiation with respect to $z$. The function $f(x, x_0, z; a)$, which appears in the heat kernel of the Heisenberg Laplacian (where $a = 2\alpha$) and other degenerate operators (for the Grušin operator $a = 1$), has been studied in, for example, [15] and [1], and it may be regarded as a complex action, associated with complex Hamiltonian mechanics and extending the C-C metric to the complex plane. In fact, $f$ satisfies the following analog of the Hamilton-Jacobi equation (cf., e.g., [1]):

$$H(x, x_0, \nabla f) + z \frac{\partial f}{\partial z} = f. \tag{1.23}$$

In the sequel we sometimes suppress the parameter $a$. Let $\Gamma_{0,x,x_0}$ denote the set (besides the imaginary axis) in the complex plane where $f$ is real; that is, let

$$\Gamma_{0,x,x_0} = \Big\{ z \in \mathbb{C},\ \operatorname{Im} f(x, x_0, z) = 0,\ \operatorname{Re} z \neq 0 \Big\}. \tag{1.24}$$

We recall the main properties of the curve $\Gamma_{0,x,x_0}$ (see [1], [15]). Here we assume, without loss of generality, that $x_0 \geq 0$ and that geodesics are understood with respect to a Carnot-Caratheodory metric associated with $H$ (see Fig. 2).

PROPOSITION 3

(1) *If $f(x, x_0, z)$ is real and $\partial f(x, x_0, z)/\partial z = 0$, then $z$ is purely imaginary.*

(2) *If $r > 0$, then the number $N$ of purely imaginary solutions of $f'(x, x_0, z) = 0$ is finite depending on $\lambda = 2x_0/r^2$. More precisely, the set $Z_{f'}$ of purely imaginary zeros of $f'$ is given by*

$$Z_{f'} = \Big\{ i\frac{\theta}{a}, \theta \in \mathbb{R} - \pi\mathbb{Z} \text{ s.t. } \frac{\theta}{\sin^2\theta} - \cot\theta = \lambda \Big\} \tag{1.25}$$

*(see Fig. 1). Let $p_k$ denote the $k$ th positive root of the equation $\tan\theta = \theta$; the elements $(i\theta_j)_{j=1,\dots,N}$ of $Z_{f'}$ are such that $\theta_1 < \pi/a < \theta_2 \leq p_1/a < \theta_3 < 2\pi/a < \cdots < \theta_{2K} \leq p_K/a \leq \theta_{2K+1} < (K+1)\pi/a$, where $K = [N/2]$. (It may happen that $\theta_{2K} = \theta_{2K+1}$. Then the three curves intersecting at $i\theta_N$ form angles of $\pi/3$ radians with each other, one of them being the imaginary axis; see Fig. 3 and paragraph (5) below.)*

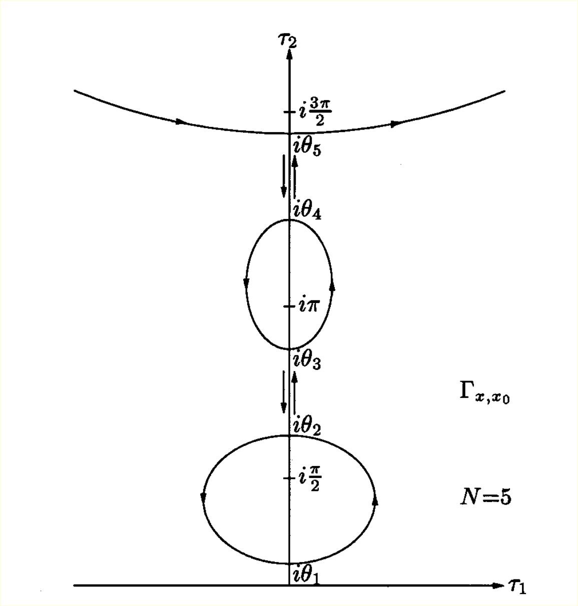


Figure 2. The curve $\Gamma$ in the complex plane for $a = 2$

(3) *$\Gamma_{0,x,x_0}$ is symmetric with respect to the imaginary axis. If $x_0 = 0$, then $\Gamma_{0,x,x_0}$ coincides with the real axis. If $r > 0$, then the curve branches off to $\infty$ in both directions from $i\theta_N$ and the branches are asymptotic to the lines $\lambda|\operatorname{Re} z| = \operatorname{Im} z$. Between $i\theta_{2k-1}$ and $i\theta_{2k}$, $\Gamma_{0,x,x_0}$ encircles the pole $ik(\pi/a)$ of $f$.*

(4) *Let $\Gamma_{x,x_0}$ denote the union of $\Gamma_{0,x,x_0}$ and $[0, i\theta_1] \cup [i\theta_2, i\theta_3] \cup \cdots \cup [i\theta_{2K}, i\theta_{2K+1}]$. The real function $f$ is strictly increasing along the path $\Gamma_{x,x_0}$ assuming all values between $ar^2/2$ and $\infty$.*

(5) *Let $i\theta$ be a zero of $f'(x, x_0, z)$. It is also a zero of $f''(x, x_0, z)$ if and only if $\theta$ is equal to one of the real numbers $p_k$, and in this case, $f^{(3)}(x, x_0, ip_k) \neq 0$ and $f(i\theta_k) = (a/2)r^2(\theta_k^2/(\sin\theta_k)^2)$.*

(6) *If $r > 0$, then there exist $N$ geodesic curves joining $(x, x_0)$ with the origin. The length of the $j$th geodesic is given by*

$$d_j(x, x_0) = \sqrt{2f(x, x_0, i\theta_j)}, \quad 1 \leq j \leq N;$$

*and $d_1(x, x_0) \leq d_2(x, x_0) \leq \cdots \leq d_N(x, x_0)$.*

Observe that the equation defining $Z_{f'}$, (1.25), is equivalent to (1.16).

The case $r = 0$ is degenerate. In that case, the function $f(0, x_0, z)$ is real if and only if $z$ is purely imaginary. $\Gamma$ coincides with the ray $\operatorname{Im}(\tau) > 0$, traversed twice in

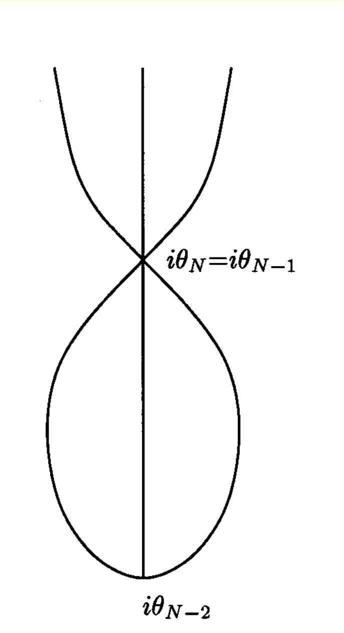


Figure 3. $\Gamma$ near a double root of $f$

opposite directions. The numbers $\theta_k$ tend to $\pi k/a$ as $r \to 0$ ($\lambda \to \infty$) (cf. also Taylor [15, pp. 82–83]).

We thus see from Proposition 3 that $\sqrt{2f}$ can be interpreted as a "distance" along $\Gamma_{x,x_0}$.

In several applications (nonisotropic Heisenberg Laplacian, certain degenerate elliptic operators), we have to use a more general form of the function $f$. Let $a_1, \dots, a_m$ be positive numbers, and let $x \in \mathbb{R}^m$. Set

$$f(x, x_0, z) = \sum_{j=1}^{m} \frac{a_j x_j^2}{2} z \coth(a_j z) - i x_0 z. \tag{1.26}$$

Assume, without loss of generality, that $a_1 \le a_2 \le \cdots \le a_p < a_{p+1} = \cdots = a_m$. Set $x'' = (x_{p+1}, \dots, x_m)$, $r = |x''|$. It is well known (see [1]) that Proposition 3 is valid, mutatis mutandis, in this case as well.

*1.3. Lorentz-Carnot-Caratheodory metric*

In analogy to the standard case, we introduce a Lorentz-Carnot-Caratheodory Hamiltonian on $M \times \mathbb{R}$ defined at a point $(m, \xi, t, \tau) \in T^*(M \times \mathbb{R})$ by $Q(m, \xi, \tau) = \tau^2/2 - H(m, \xi)$, where $H(m, \xi)$ is the Hamiltonian defined in formula (1.8). ($Q$ is independent of $t$.) For the model cases discussed in Sections 1.1 and 1.2, $m = (x, x_0)$. In these cases, consider the functions

$$F(x, x_0, t, z, a) = i x_0 z + \frac{t^2}{2} - \frac{a}{2} r^2 z \coth(az) \tag{1.27}$$

and

$$\phi(x, x_0, t, z) = -\frac{a}{2} r^2 \coth(az) + i x_0 + \frac{t^2}{2z}. \tag{1.28}$$

We need these functions to analyze the wave kernel for all operators discussed in the sequel. In our applications, the function $\phi$ satisfies the equation

$$\frac{\partial \phi(x, x_0, t, z)}{\partial z} + Q\Big(\nabla_m \phi, \frac{\partial \phi}{\partial t}\Big) = 0. \tag{1.29}$$

As an example, note that in the particular case of the Grušin plane we have

$$Q_{x,y,t}(\xi, \eta, \tau) = \frac{\tau^2 - \xi^2 - x^2 \eta^2}{2}, \tag{1.30}$$

and we can check by computation that $\phi$ is a solution of

$$\frac{\partial \phi(x, y, t, z)}{\partial z} + Q\Big(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial t}\Big) = 0. \tag{1.31}$$

In the general case, it follows from Proposition 3 that $f(x, x_0, z) = t^2/2 - F(x, x_0, t, z) = z\phi(x, x_0, t, z) - t^2/2$ is a complexification of the action computed along the bicharacteristics of $H$, and for $z = i\theta_1$, $2f(x, x_0, i\theta_1)$ is exactly the square of Carnot-Caratheodory distance from zero to the point $(x, x_0)$. Hence $2F(x, x_0, t, i\theta_1)$ is the square of the associated Lorentz indefinite metric. Note that equation (1.29) may have other solutions, not of the type (1.28).

### *1.4. Separation of variables*

The general solution of the wave equation for the Grušin operator may be found using separation of variables. Writing $u(x, t) = e^{ikt} h(x) g(y)$, we obtain two families of solutions:

$$u_{a,n}(x, y, t) = e^{iat\sqrt{2n+1}} \cos(a^2 y) H_n(ax) e^{-a^2 x^2/2}$$

and

$$v_{a,n}(x, y, t) = e^{iat\sqrt{2n+1}} \sin(a^2 y) H_n(ax) e^{-a^2 x^2/2}, \tag{1.32}$$

where $a$ is a real parameter, $n$ is a nonnegative integer, and $H_n$ is the $n$th Hermite polynomial,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \tag{1.33}$$

so that $z(x) = H_n(x)e^{-x^2/2}$ is a solution of the ordinary differential equation (ODE)

$$z'' + (2n + 1 - x^2)z = 0. \tag{1.34}$$

The Hermite polynomials $H_n$ are orthogonal, and

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}\,dx = \delta_{nm}\pi^{1/2}2^n n!. \tag{1.35}$$

Recall the Mehler formula (see W. Magnus, F. Oberhettinger, and R. Soni [11, p. 252]), according to which for all real $x$, $y$ and complex $z$ ($|z| < 1$),

$$\sum_{n=0}^{+\infty} \frac{H_n(x)H_n(y)z^n}{2^n n!} = \frac{1}{\sqrt{1-z^2}} e^{y^2-(y-zx)^2/(1-z^2)}. \tag{1.36}$$

We want to express the Dirac distribution $\delta((x, y); (0, 0))$ using the family $u_{a,n}$. Recall that in the distribution sense

$$\delta_y = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iwy}\,dw = \frac{2}{\pi}\int_0^{+\infty} a\cos(a^2 y)\,da; \tag{1.37}$$

the last equality in (1.37) follows from a change of variable $w = a^2$. Using the base induced by the Hermite polynomials, we have in the distribution sense

$$\delta(x_1 - x_2) = \sum_0^{\infty} \frac{H_n(x_1)e^{-x_1^2/2}H_n(x_2)e^{-x_2^2/2}}{\|H_n\|^2}, \tag{1.38}$$

where $\|H_n\|^2 = \sqrt{\pi}n!2^n$. Replacing $x_1, x_2$ by $ax_1, ax_2$, we get

$$\delta(x_1 - x_2) = \sum_0^{\infty} \frac{H_n(ax_1)e^{-a^2x_1^2/2}H_n(ax_2)e^{-a^2x_2^2/2}}{\|H_n\|_a^2}, \tag{1.39}$$

where $\|H_n\|_a^2 = \sqrt{\pi}n!2^n/a$. Hence in two-dimensional space, the Dirac distribution at (0, 0) has the form

$$\delta\big((x, y); (0, 0)\big) = \sum_0^{\infty} \frac{2}{\pi}\int_0^{+\infty} \frac{a^2}{\sqrt{\pi}n!2^n}\cos(a^2 y)H_n(ax)e^{-a^2x^2/2}H_n(0)\,da. \tag{1.40}$$

Applying (1.40) to the function $\phi(x, y) = f(x)g(y) \in D(\mathbb{R}^2)$, we get

$$g(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(w)e^{-iw0}\,dw = \frac{2}{\pi}\int_0^{+\infty} \hat{g}(a^2)a\,da \tag{1.41}$$

and

$$f(0) = \sum_0^{\infty} \frac{H_n(0)(f, h_{n,a})}{\|H_n\|_a}, \tag{1.42}$$

where $h_{n,a}(x) = H_n(ax)e^{-(ax)^2/2}$ and $(f, h_{n,a}) = \int_{\mathbb{R}} f(x)H_m(ax)e^{-a^2x^2/2}\,dx$.

In performing the computations connected with the separation of variables, we also use the following formula (see [11, p. 83]) for the Bessel function $J_\nu$:

$$2i\pi\, J_\nu(\alpha z) = z^\nu \int_{-\infty}^{0+} e^{(\alpha/2)(t-z^2/t)} t^{-\nu-1}\,dt, \tag{1.43}$$

where $\operatorname{Re}(\alpha) > 0$ and $|\arg(t)| \le \pi$, and the integral is extended over a contour starting at $\infty$, going clockwise around 0, going back to $\infty$, and never cutting the semiaxis $x < 0$. This contour can be deformed so that it becomes parallel to the $x$-axis and

$$2i\pi\, J_\nu(\alpha z) = z^\nu \int_{c-i\infty}^{c+i\infty} e^{(\alpha/2)(t-z^2/t)} t^{-\nu-1}\,dt, \tag{1.44}$$

where $c, \alpha > 0$ and $\operatorname{Re}\nu > 0$. Also, it is well known that

$$J_{1/2}(z) = \frac{\sin z}{\sqrt{\pi z/2}}. \tag{1.45}$$

## 2. The wave kernel for the Grušin operator

In this chapter, we study the properties of the fundamental solution of the wave equation associated to the Grušin operator $L/2 = -(1/2)(\partial^2/\partial x^2 + x^2(\partial^2/\partial x_0^2))$ using separation of variables. We are interested in computing the fundamental solution that satisfies the initial condition $u(x, x_0, 0) = 0$ and $u_t(x, x_0, 0) = \delta(0, 0))$, where $\delta(0, 0))$ denotes the Dirac distribution at $(0, 0)$ for the variable $(x, x_0)$. The kernel can be expressed as

$$K_w(x, x_0, t) = \frac{\sin(tL^{1/2})}{L^{1/2}}\delta(0, 0). \tag{2.1}$$

Applying formula (1.40), we obtain

$$K_w(x, x_0, t) = \sum_0^\infty \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\sqrt{2n+1}at)}{a\sqrt{2n+1}} \frac{a^2}{\sqrt{\pi}n!2^n} \times \cos(a^2x_0)H_n(ax)e^{-a^2x^2/2}H_n(0)\,da. \tag{2.2}$$

We wish to sum the series (2.2) so as to obtain a more manageable form for the kernel. The situation is summed up in the following theorem.

THEOREM 1

*The wave kernel* $K_w(x, x_0, t)$ *defined by* $K_w(x, x_0, t) = (\sin(tL^{1/2})/L^{1/2})\delta(0, 0)$ *is given by*

$$K_w(x, x_0, t) = \frac{K(x, x_0, t) + K(x, -x_0, t)}{2}. \tag{2.3}$$

*Here*

$$K(x,x_0,t) = iK_0' \int_C \sqrt{\frac{1}{u\sinh(u)}}\frac{1}{\Phi(x,x_0,t,u)}\,du, \tag{2.4}$$

*where* $K_0' = 1/(2(\pi)^2)$ *is a constant,* $\Phi(x,x_0,t,z)$ *is the phase given by the expression*

$$\Phi(x,x_0,t,z) = ix_0 + \frac{t^2}{2z} - \frac{x^2\coth(z)}{2}, \tag{2.5}$$

*and* $C$ *(the precise description is given in the following subsections) is a closed contour of integration lying outside the set where* $\operatorname{Re}\Phi > 0$ *and avoiding the zeros of* $\sinh(z)$.

*The phase* $\Phi$ *satisfies the Hamilton-Jacobi equation*

$$2\frac{\partial\phi}{\partial z} = \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - x^2\left(\frac{\partial\phi}{\partial x_0}\right)^2. \tag{2.6}$$

*The wave kernel satisfies the finite speed property: it vanishes identically for* $t^2 < d_c^2(x,x_0)$. *Equivalently, the kernel is zero before the first geodesic of the C-C metric arrives at the point* $(x,x_0)$. *Moreover, the kernel vanishes when the time satisfies the conditions*

$$2f(i\theta_{2k}) < t^2 < 2f(i\theta_{2k+1}), \tag{2.7}$$

*where the points* $\theta_k$ *are introduced in Proposition 3.*

*Remark.* The singularities of the wave kernel are computed using the zeros of the function $\Phi(x,x_0,t,z)$ defined by (1.28) with $a = 1$ (or by (2.5)). When time increases to the value $t = \sqrt{2f(i\theta_{2k+1})}$, this means that a new geodesic hits the point $(x,x_0)$ and then the kernel becomes singular.

*Proof*

All the computations in this paragraph are to be understood in the distribution sense. From the identity $J_{1/2}(z) = \sin z/\sqrt{\pi z/2}$ and from formula (1.44), we obtain for arbitrary $c' > 0$,

$$\frac{\sin\alpha z}{\alpha z} = \sqrt{\frac{\pi}{2}}\frac{1}{2i\pi}\int_{c'-i\infty}^{c'+i\infty} e^{(\alpha/2)(u-z^2/u)}u^{-3/2}\,du. \tag{2.8}$$

Let us choose $\alpha = 1$ and $z = at\sqrt{2n+1}$; then $c' = a^2t^2c$, where $c > 0$. Then

$$\frac{\sin at\sqrt{2n+1}}{at\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}\frac{1}{2i\pi}\int_{a^2t^2c-i\infty}^{a^2t^2c+i\infty} e^{(1/2)(u-(2n+1)(at)^2/u)}u^{-3/2}\,du, \tag{2.9}$$

and using the change of variable $u = (at)^2 v$ for $at > 0$,

$$\frac{\sin at\sqrt{2n+1}}{a\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}\frac{1}{2i\pi a}\int_{c-i\infty}^{c+i\infty} e^{(1/2)(a^2t^2v-(2n+1)/v)}v^{-3/2}\,dv, \tag{2.10}$$

we see that the kernel takes the form

$$K_w(x,x_0,t) = -iK_0\int_0^{\infty}\sum_0^{\infty}\int_{c-i\infty}^{c+i\infty} e^{(1/2)(a^2t^2v-(2n+1)/v)}v^{-3/2}\,dv\frac{a}{n!2^n}\cos(a^2x_0) \cdot H_n(ax)e^{-a^2x^2/2}H_n(0)\,da \tag{2.11}$$

with

$$K_0 = \sqrt{\frac{1}{2}}\frac{1}{2\pi}\frac{2}{\pi} = \frac{1}{\pi^2\sqrt{2}}. \tag{2.12}$$

To sum the series, we use the Mehler formula (1.36) (see [11, p. 252]). Set $z = e^{-1/v}$ so that $|z| < 1$ since $\mathrm{Re}(-1/v) < 0$ to get

$$e^{-1/2v}\sum_{n=0}^{+\infty}\frac{H_n(ax)H_n(0)e^{-n/v}}{2^n n!} = \frac{e^{-1/2v}}{\sqrt{1-e^{2/v}}}e^{-(ax)^2e^{-2/v}/(1-e^{-2/v})}. \tag{2.13}$$

Using $e^{-1/2v}/\sqrt{1-e^{2/v}} = 1/\sqrt{2\sinh(1/v)}$ and $\cos(a^2x_0) = (1/2)(e^{ia^2x_0}+e^{-ia^2x_0})$, the kernel can be expressed as $K_w(x,x_0,t) = (1/2)(K(x,x_0,t)+K(x,-x_0,t))$, where

$$K(x,x_0,t) = -iK_0\int_0^{+\infty}\int_{c-i\infty}^{c+i\infty} v^{-3/2}\frac{a}{\sqrt{2\sinh(1/v)}} \cdot e^{-(ax)^2e^{-2/v}/(1-e^{-2/v})-a^2x^2/2+ia^2x_0+(1/2)(a^2t^2v)}\,dv\,da.$$

Changing to a new variable $z = 1/v$, $dv/v = -dz/z$, and $\sqrt{v} = 1/\sqrt{z}$, the contour $\mathrm{Re}\,v = c$ is transformed to the circle $C(1/2c, 1/2c)$, centered at $(1/2c, 0)$ with radius $1/2c$. The previous integral becomes

$$K(x,x_0,t) = iK_0\int_C\int_0^{+\infty}\frac{az^{-1}\sqrt{z}}{\sqrt{2\sinh(z)}} \cdot e^{-(ax)^2e^{-2z}/(1-e^{-2z})-a^2x^2/2+ia^2x_0+(1/2)(a^2t^2/z)}\,dz\,da, \tag{2.14}$$

and the term in the exponential can be rewritten as

$$\frac{(ax)^2e^{-2z}}{1-e^{-2z}} - \frac{a^2x^2}{2} + ia^2x_0 + \frac{1}{2}\Big(\frac{a^2t^2}{z}\Big) = a^2\Big(-\frac{x^2\coth(z)}{2} + ix_0 + \frac{t^2}{2z}\Big).$$

Recall that the phase is given by

$$\Phi(x,x_0,t,z) = -\frac{x^2\coth(z)}{2} + ix_0 + \frac{t^2}{2z}. \tag{2.15}$$

An elementary computation proves that Hamilton-Jacobi equation (2.6) is satisfied.

Now for $\operatorname{Re}\Phi = \operatorname{Re}(-x^2\coth(z)/2 + t^2/(2z)) < 0$,

$$\int_0^\infty a e^{a^2\Phi}\,da = \frac{-1}{2\Phi}, \tag{2.16}$$

so that the kernel is

$$K(x, x_0, t) = \frac{iK_0}{2}\int_C dz\sqrt{\frac{1}{2z\sinh(z)}}\frac{1}{\Phi(x, x_0, t, z)}, \tag{2.17}$$

which is equivalent to

$$K(x, x_0, t) = \frac{iK_0}{2}\int_C dz\sqrt{\frac{z}{2\sinh(z)}}\frac{1}{F(x, x_0, t, z)}, \tag{2.18}$$

where we recall

$$F(x, x_0, t, z) = ix_0 z + \frac{t^2}{2} - \frac{x^2}{2}z\coth(z).$$

It follows from here that if $t < |x|$, then the kernel is zero (unless $x = x_0 = 0$). Indeed, $\operatorname{Re}\Phi < 0$ if $\operatorname{Re} z > 0$. Now since $\Phi$ is a continuous function of the complex variable $z$, $\Phi(x, x_0, t, 0) \neq 0$, there exists a neighborhood $V$ of zero such that $\Phi(x, x_0, t, z)$ does not vanish for $z \in V$. Hence the constant $c$ can be chosen large enough so that the circle $C(1/2c, 1/2c)$ is small enough and contained in $V$. Finally, a simple application of the residue theorem implies that $K(x, x_0, t) = 0$.

We complete the proof in the following subsections, where we also describe the contour of integration.

### *2.1. The boundary of the forbidden set*

In this paragraph we describe the contours of integration which are used in the proof of Theorem 1. From the discussion leading to (2.17) and (2.18), we see that the contours have to be contained in the region $\operatorname{Re}(\Phi) < 0$.

By analytic continuation, we may enlarge the region to $\operatorname{Re}(\Phi) \leq 0$.

Let us denote $\lambda = x^2/t^2$. Then the set of points $\Theta_{x,t}$, where $\operatorname{Re}(\Phi) = 0$, is defined by

$$\Theta_{x,t} = \left\{ z = u + iv \in \mathbb{C},\ \frac{u}{u^2 + v^2} = \lambda\frac{\sinh u\cosh u}{\sin^2 v + \sinh^2 u} \right\}. \tag{2.19}$$

The curve $\Theta_{x,t}$ depends on the time and can be described as follows. It is symmetric with respect to $v \to -v$. The $v$-axis is always contained in $\Theta_{x,t}$ since $u = 0$ is a solution.

There are the following two cases.

(1) If $\lambda \geq 1$, the curve has no intersection with the axis $v = 0$ except at the origin. In this case, there exists no bifurcation point on the $v$-axis.

(2) If $\lambda < 1$, then there are several branches that intersect the imaginary axis (see Fig. 4). First by continuity, there exists a curve starting at the point $(u_\lambda, 0)$, where $\tanh(u_\lambda)/u_\lambda = \lambda$ with a vertical tangent, connecting to a point $(0, v_\lambda)$, where $v_\lambda$ is a solution of the following equation:

$$\left(\frac{\sin v}{v}\right)^2 = \lambda < 1. \tag{2.20}$$

Equation (2.20) has a finite number of solutions depending on the relative sizes of $(\sin p_k/p_k)^2$ and $\lambda$, where $p_k$ is the $k$-root of $\tan x = x$. Note first that for $t, x$ fixed the non-purely-imaginary part of the curve $\Theta_{x,t}$ is bounded. This follows from the fact that $u$ cannot go to $\infty$ in the expression of $\Theta_{x,t}$.

Now $\operatorname{Re}\phi > 0$ on the real interval $[0, u_\lambda]$, and for $v$ large enough $\operatorname{Re}(\Phi) < 0$ uniformly in $u \in [0, u_\lambda]$. By continuity, we deduce that there exists a continuous curve that joins the point $(0, v_\lambda)$ to the point $(u_\lambda, 0)$. By the inverse function theorem, we have a unique tangent at a neighborhood of the point $(u_\lambda, 0)$. Hence the curve starting at this point is unique.

More precisely, consider first the case where

$$\frac{1}{1 + p_1^2} < \lambda < 1, \tag{2.21}$$

so that equation (2.20) has only one solution. Hence there exists one curve connecting $(u_\lambda, 0)$ to $(0, v_\lambda)$, and the tangent at $(0, v_\lambda)$ is parallel to the real axis.

In general, at the point $(0, v_\lambda)$, the curve $\Theta_{x,t}$ has a horizontal tangent except when $v_\lambda = (p_k)$ for a certain $k$. At these points, a bifurcation appears. This results from a simple perturbation analysis: writing $v_0 = v_\lambda + \delta$ near the point $(0, v_\lambda)$ in (2.19) gives for $u$ small

$$2v_\lambda\left(1 - \frac{v_\lambda}{\tan(v_\lambda)}\right)\delta = \left(\frac{1}{\lambda} - 1\right)u^2. \tag{2.22}$$

Now, except at points $v_\lambda = p_k$, the perturbation $\delta$ is of second order and only one horizontal branch can start from this point. At the bifurcation points $p_k$, we obtain, after some computations,

$$\left(1 - \frac{\cos(2v_\lambda)}{\lambda}\right)\delta^2 = \left(\frac{1}{\lambda} - 1\right)u^2. \tag{2.23}$$

Thus $\delta$ is linear in $u$, and the curve has two tangents that are not horizontal.

For $1/(1 + p_{n+1}^2) < \lambda < 1/(1 + p_n^2)$, we obtain $2n + 1$ solutions $v_1(t), \ldots, v_{2n+1}(t)$, and when $\lambda = 1/(1 + p_n^2)$, a double solution appears.

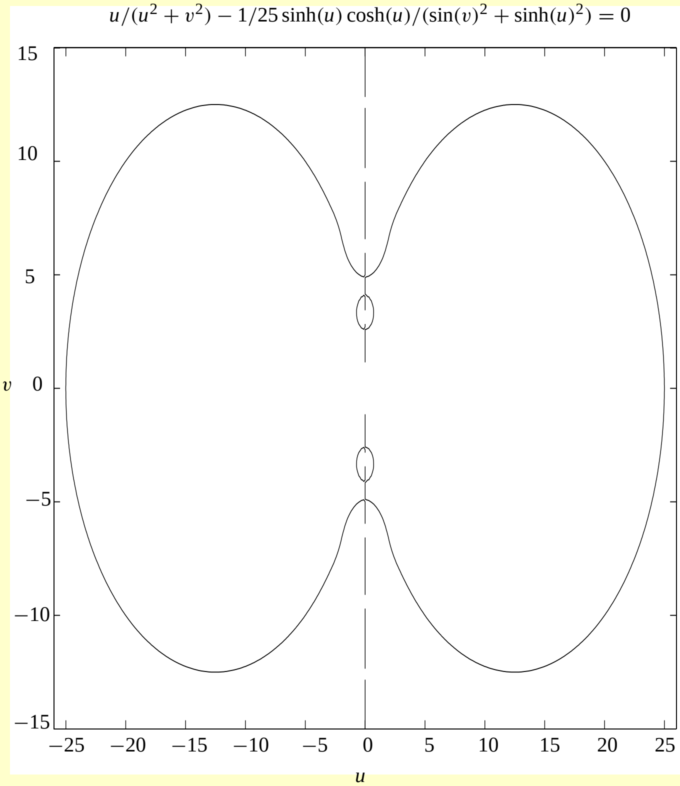


Figure 4. Curve $\Theta_{x,t}$ for $\lambda = 1/25$

At the first time when $\lambda = 1/(1+p_1^2)$, the curve $\Theta_{x,t}$ jumps to reach the upper point $(0, v_2)$, where $\pi < v_2 < 2\pi$. Indeed, otherwise the branch of the curve starting at $(0, v_2)$ would return to itself and could not be connected to the rest of the curve.

Thus $(u_\lambda, 0)$ is connected to $(0, v_2)$, and then an arc joins $(0, v_2(t))$ to $(0, v_1(t))$ in the region $u > 0$. After some time all tangents of the curve near the imaginary axis become horizontal until we reach the second bifurcation point $1/(1+p_2^2) = \lambda$. As time goes on, the part of the curve near the imaginary axis turns around the point $(0, k\pi)$, and the two points $v_{2p+1}(t) < (p+1)\pi < v_{2p+2}(t)$ converge to $(p+1)\pi$ for $p \in \mathbb{N}$ when $t$ converges to $\infty$.

### *2.2. The zero of the phase and the integration path*

We see in this subsection that the relevant pole of the function $1/\Phi$ is located at the intersection of the curves $\Gamma$ and $\Theta$.

It has been proved in Section 2.1 that the curve $\Theta$ has a unique branch starting on the positive $u$-axis and intersecting the positive $v$-axis. This curve cuts the curve $\Gamma$

only once. This intersection point is exactly the pole.

Recall that by Proposition 3, the function $f$ is strictly increasing along the curve $\Gamma$. Moreover,

$$\Phi(x, x_0, t, z) = -\frac{x^2}{2}\coth(z) + ix_0 + \frac{t^2}{2z} = 0$$

is equivalent to

$$t^2 = \frac{zx^2}{\tanh(z)} - 2ix_0 z = 2f(x, x_0, z), \tag{2.24}$$

so that if $z$ is a zero of $\Phi$, then $\operatorname{Im} f(z) = 0$. If $t^2 > x^2$, then $t^2$ is bigger than the minimum of $2f$ along $\Gamma$. This proves that there is precisely one solution $z$ on $\Gamma$ of equation (2.24). This point is at the intersection of the two curves $\Gamma$ ($\operatorname{Im} f = 0$) and $\Theta$ ($\operatorname{Re}\Phi = 0$).

The converse holds as well. If $t^2/z - x^2/\tanh(z)$ is purely imaginary (equals $i\alpha$, say), then multiplying by $z$, we get $2\Phi = i(\alpha + 2x_0)z$, so that the equation $\operatorname{Im}\Phi = 0$ implies that if $\operatorname{Im} z > 0$, then $\alpha + 2x_0 = 0$ and $\Phi = 0$. If $z$ is purely imaginary, the statement remains correct due to the strictly increasing property of $f$ along $\Gamma$. Indeed, no jump occurs as $\operatorname{Im}(z)$ tends to zero due to the continuity of the zero with respect to the arguments. The other purely imaginary zeros of $\Phi$ do not contribute to the integral.

Hence we may deform the path of integration $C$ and choose it starting in the region where $\operatorname{Re}\Phi < 0$ (taking into account the singularity), going along the imaginary axis, and avoiding the poles $ik\pi$ for $k \in \mathbb{N}$. This completes the statement of Theorem 1 and the proof of formula (2.4).

We finish the proof of the theorem in the next subsection, after obtaining an explicit formula.

### *2.3. Explicit formula for the integral*

We obtain an explicit expression for the wave kernel when we perform the integration along the contour described in the previous paragraph and apply the residue theorem.

PROPOSITION 4

*The wave kernel $K_w$ for the Grušin operator is given at a point $(x, x_0)$, where $x_0 \neq 0$,* *by the expression*

$$
\begin{aligned}
&K_w(x, x_0, t)\\
&= \frac{1}{2\pi}\left(\sqrt{\frac{z(x,x_0,t)}{2\sinh(z(x,x_0,t))}}\frac{1}{-t^2/(z(x,x_0,t)) + x^2 z(x,x_0,t)/\sinh^2(z(x,x_0,t))}\right.\\
&\quad\left. + \sqrt{\frac{\bar z(x,x_0,t)}{2\sinh(\bar z(x,x_0,t))}}\frac{1}{-t^2/(\bar z(x,x_0,t)) + x^2 \bar z(x,x_0,t)/\sinh^2(\bar z(x,x_0,t))}\right),
\end{aligned}
\tag{2.25}
$$

*where* $z(x, x_0, t)$ *is the unique solution on* $\Gamma$ *of* $F(x, x_0, t, z) = 0$*, and* $F'(x, x_0, t, z) \neq 0$*; that is,*

$$
t^2 = x^2 \frac{z(x,x_0,t)}{\tanh z(x,x_0,t)} - 2i x_0 z(x,x_0,t), \tag{2.26}
$$

*and the denominator in formula (2.25) does not vanish. The kernel is analytic except at points where* $F'(x, x_0, t, z(x, x_0, t)) = 0$*. Put differently,*

$$
\begin{aligned}
K_w(x,x_0,t) = -\frac{1}{4\pi t}&\left(\sqrt{\frac{z(x,x_0,t)}{2\sinh(z(x,x_0,t))}}\frac{\partial z(x,x_0,t)}{\partial t}\right.\\
&\left. + \sqrt{\frac{\bar z(x,x_0,t)}{2\sinh(\bar z(x,x_0,t))}}\frac{\partial \bar z(x,x_0,t)}{\partial t}\right),
\end{aligned}
$$

*where an elementary computation yields that*

$$
\frac{\partial z(x,x_0,t)}{\partial t} = \frac{2t}{t^2/(z(x,x_0,t)) - x^2 z(x,x_0,t)/\sinh^2(z(x,x_0,t))}. \tag{2.27}
$$

*When* $x = 0$ *and* $x_0 \neq 0$*, the wave kernel is given for* $t > 0$ *by*

$$
K_w(0, x_0, t) = \frac{(-1)^j}{2\pi|x_0|}\frac{t}{\sqrt{4x_0\sin(-t^2/(2|x_0|))}}\chi_{\{\sin(t^2/(2|x_0|))<0\}}, \tag{2.28}
$$

*where* $\chi_I$ *is the characteristic function of the interval* $I$ *and* $j \in \mathbb{N}$ *is such that* $(2j-1)\pi < t^2/(2x_0) < 2j\pi$*. Moreover, the kernel is singular for* $t^2 = 2k|x_0|\pi$*, where* $k \in \mathbb{Z}$*.*

*Proof*

Since the path avoids the zeros of $\sinh(z) = 0$ and stays in the right half of the complex plane $u \geq 0$, we can apply the residue theorem. Our path of integration $C$ contains only singularities generated by the zeros of the function $\Phi(x, x_0, t, z)$. Let us recall the expression of the kernel $K$,

$$
K(x, x_0, t) = iK_0/2\int_C \frac{1}{\Phi(x,x_0,t,z)} w(z)\,dz, \tag{2.29}
$$

where

$$w(z) = \sqrt{\frac{z}{2\sinh(z)}}, \tag{2.30}$$

and $C$ has been described earlier. We have seen that the relevant solution of $\Phi(x, x_0, t, z) = 0$ is unique and is located exactly at the intersection of the curves $\Gamma_{x,x_0}$ and $\Theta_{x,t}$.

In the case where $x_0 \neq 0$, we compute the residue at points where $\Phi'(x, x_0, t, z) \neq 0$, where $z$ satisfies the equation

$$t^2 = x^2 \frac{z(x, x_0, t)}{\tanh z(x, x_0, t)} - 2ix_0 z(x, x_0, t). \tag{2.31}$$

In this case, the function $(x, x_0, t) \to z(x, x_0, t)$ is an analytic function of $(x, x_0, t)$. We discuss below the set of points $(x, x_0, t)$ in $\mathbb{R}^3$ where the derivative is zero (see also A. Nachman [13]).

Applying the residue theorem, we get

$$\int_C \frac{1}{\Phi(x, x_0, t, z)} w(z)\, dz = 2i\pi \sqrt{\frac{z(x, x_0, t)}{2\sinh z(x, x_0, t)}} \frac{1}{\Phi'(x, x_0, t, z(x, x_0, t))}. \tag{2.32}$$

Hence

$$K(x, x_0, t) = 2\pi K_0 \frac{w(z(x, x_0, t))}{\Phi'(x, x_0, t, z(x, x_0, t))} \tag{2.33}$$

and

$$\Phi'\big(x, x_0, t, z(x, x_0, t)\big) = -\frac{t^2}{2z} + \frac{x^2 z}{2\sinh^2(z)}, \tag{2.34}$$

so that finally

$$K(x, x_0, t) = \frac{1}{2\pi} \cdot \left(\sqrt{\frac{z(x, x_0, t)}{2\sinh(z(x, x_0, t))}} \frac{2}{-t^2/(z(x, x_0, t)) + x^2 z(x, x_0, t)/\sinh^2(z(x, x_0, t))}\right). \tag{2.35}$$

Now to obtain $K_w$, we need to compute $K(x, -x_0, t)$. But by symmetry, the solution $\eta \in \mathbb{C}$ of $\Phi(x, -x_0, t, \eta) = 0$ is exactly $\bar{z}$, where $z$ is the solution of $\Phi(x, x_0, t, z) = 0$. The pole in the integrand occurring in the expression for $K(x, -x_0, t)$ is exactly $\bar{z}$, and

$$K(x, -x_0, t) = 2\pi K_0 \frac{w(\bar{z}(x, x_0, t))}{\Phi'(x, x_0, t, \bar{z}(x, x_0, t))}. \tag{2.36}$$

We conclude that the wave kernel is given by

$$\begin{aligned} &K_w(x, x_0, t)\\ &= \frac{1}{2\pi}\Bigg(\sqrt{\frac{z(x, x_0, t)}{2\sinh(z(x, x_0, t))}}\frac{1}{-t^2/(z(x, x_0, t)) + x^2 z(x, x_0, t)/\sinh^2(z(x, x_0, t))}\\ &\quad + \sqrt{\frac{\bar{z}(x, x_0, t)}{2\sinh(\bar{z}(x, x_0, t))}}\frac{1}{-t^2/(\bar{z}(x, x_0, t)) + x^2 \bar{z}(x, x_0, t)/\sinh^2(\bar{z}(x, x_0, t))}\Bigg). \end{aligned} \tag{2.37}$$

$K_w(x, x_0, t)$ is thus analytic except at the critical points of the phase $\Phi$.

To finish the proof, consider the case where $x = 0$. Since the expression of the kernel is analytic in its arguments, when $x = 0$ the expression is still valid and the pole is now located at $z_0 = it^2/(2x_0)$. Using expression (2.37) for $z = z_0$, we obtain

$$K_w(0, x_0, t) = \frac{1}{\pi}\operatorname{Re}\left(\sqrt{\frac{z_0}{2\sinh(z_0)}}\frac{1}{-t^2/z_0}\right). \tag{2.38}$$

If $\sin(t^2/(2x_0)) \geq 0$ is positive, we obtain a purely imaginary number and $K_w(0, x_0, t)$ vanishes. This proves that cancellation phenomena exist for an infinite number of time intervals. Now suppose that $\sin(t^2/(2x_0)) < 0$. Then by evaluating (2.38), we obtain for $x_0 > 0$,

$$K_w(0, x_0, t) = \frac{t}{2x_0\pi}\sqrt{\frac{1}{-4x_0\sin(t^2/(2x_0))}}. \tag{2.39}$$

If $x_0 < 0$, then

$$K_w(0, x_0, t) = -\frac{t}{2x_0\pi}\sqrt{\frac{1}{-4x_0\sin(t^2/(2x_0))}}. \tag{2.40}$$

In formulas (2.39) and (2.40), the sign of the square root alternates.

It follows from this expression that the singularities are located at the points $t^2 = 2k|x_0|\pi$. □

*Remark.* We could have derived the formula (2.28) of Proposition 4 from formula (2.4) of Theorem 1. For in this case the integration contour is contained in the imaginary axis where the pole $z_0 = it^2/(2x_0)$ is located. More precisely, the contour $\Gamma$ starts at the origin, continues along the imaginary axis encircling the pole $z_0$, and returns to the origin, oriented clockwise. When $x$ tends to zero, the curve $\Gamma_{x,x_0}$ converges pointwise to the imaginary axis and $F(x, x_0, z, t)$ converges to $F(0, x_0, t, z)$, except at the points $i\pi k$ for $k \in \mathbb{Z}$. Since $K_w(x, x_0, t) = (1/2)(K(x, x_0, t) + K(x, -x_0, t))$, we need to evaluate the integral

$$K(0, x_0, t) = \frac{i}{\pi^2} \int_\Gamma \sqrt{\frac{u}{\sinh(u)}} \frac{1}{2uix_0 + t^2}\, du, \tag{2.41}$$

where $\Gamma$ is the part of the contour contained in the imaginary axis. The points $i\pi k$ do not really contribute a singularity to the integral. In fact, the computation shows that the integral along the integration contour vanishes and only the singularity at the point $z_0$ is relevant.

We may now prove the statements of Theorem 1 about the finite speed of propagation, using the expression of the wave kernel obtained above. Note that it is possible to deform the path of integration below the $u$-axis. Using this remark, it is possible to prove that the kernel is vanishing before it reaches $d_c^2(x, x_0)$.

Indeed, for $x^2 < t^2 < d_c^2(x, x_0)$, the zero of the function $\Phi(x, x_0, t, z)$ is exactly situated on the imaginary axis between 0 and $i\theta_1$ with $\theta_1 < \pi$. Deforming the contour of integration in order to include the pole of $1/\Phi$, we obtain a certain residue. The pole where the residue is computed is of the form $i\alpha$, where $\alpha > 0$. Using exactly the same argument, we see that the residue for the kernel $K(x, -x_0, t)$ is computed at the pole $-i\alpha$. Since the kernel $K_w$ is an odd function of $z$, the two residues cancel.

This result is valid each time that $f(x, x_0, t, z) = 0$ has a solution $i\alpha$ in the imaginary axis such that $\alpha \in [2k\pi, (2k+1)\pi]$, $k \in \mathbb{N}$. Applying the description of the curve $\Gamma_{x,x_0}$, given in Proposition 3, this situation appears exactly when $2f(i\theta_{2k}) < t^2 < 2f(i\theta_{2k+1})$, where $\theta_k$ are defined in the same proposition and, as above, the terms $K(x, x_0, t)$ and $K(x, -x_0, t)$ cancel. □

*2.4. Analysis of the wave kernel near the singularities*

We describe here the wave kernel near the singularities. We have found in the last paragraph an explicit expression for the wave kernel. We are going to focus on the set

$$Z(x, x_0, t) = \left\{z \in \mathbb{C} \text{ s.t. } F(x, x_0, t, z) = 0 \text{ and } F'(x, x_0, t, z) = 0\right\}. \tag{2.42}$$

The characteristic set may be parametrized by a real parameter $\theta$. After some elementary computations, we find that

$$S = \left\{(x, x_0, t) \in \mathbb{R}^3 \text{ s.t. } \frac{x^2}{t^2} = \frac{\sin^2\theta}{\theta^2} \text{ and } \frac{2x_0}{x^2} = \frac{\theta - \cos\theta\sin\theta}{\sin^2\theta}, \theta \in \mathbb{R} - \pi\mathbb{Z}\right\},$$
$$S = S_- \cup S_+, \tag{2.43}$$

where

$$S_\pm = \Big\{(x, x_0, t) \in \mathbb{R}^3 \text{ s.t. } x = \pm t \Big|\frac{\sin\theta}{\theta}\Big| \\ \text{and } \frac{2x_0}{x^2} = \frac{\theta - \cos\theta\sin\theta}{\sin^2\theta}, \theta \in \mathbb{R} - \pi\mathbb{Z}\Big\}.$$

The main result of this subsection is the following.

PROPOSITION 5

*The singular set of the wave kernel is $S$. This set is the disjoint union of the sets $S_+$ and $S_-$. Moreover, near the singular set, the main singularity of the kernel is given by the following expression:*

$$K_w(x, x_0, t) \sim \frac{1}{2\pi}\Big(\sqrt{\frac{\theta_k}{2\sin(\theta_k)}}\Big)\frac{1}{(x^2/\sin^2\theta_k)(-1 + \theta_k/\tan\theta_k)} \\ \cdot\Big(\sqrt{\frac{H(x, x_0, t, i\theta_k)}{t^2 - d_k^2(x, x_0)}} + \sqrt{\frac{H(x, -x_0, t, -i\theta_k)}{t^2 - d_k^2(x, -x_0)}}\Big),$$

*where $d_k$ is the length of the $k$th geodesic joining $(x, x_0)$ with the origin, $k = 1, \ldots, N$, $H$ is an analytic function, nonzero on $S$ and $\theta_k$ is a solution of the equation $F'(x, x_0, t, i\theta) = 0$ with $F$ real, and $\theta_k$ is not a root of $\tan\theta = \theta$. When $\theta_k$ is one of the points $p_k$, $K_w(x, x_0, t)$ grows like $1/(t^2 - d_k^2(x, x_0))^{3/2}$.*

*Proof*

The singular sets $Z$ and $S$ are essentially the same as the singular sets analyzed and shown graphically in [13]. Here we study the behavior of $K_w$ near the set $S$ and determine the singularity there.

In the set $Z(x, x_0, t)$, the points $z = ip_k = i\tan p_k$ are isolated and $F''(x, x_0, t, ip_k) = 0$, but the third derivative is not zero. At points of $S$ corresponding to the latter, the singularity of the wave kernel is of higher order.

For $z$ in a neighborhood of $Z(x, x_0, t) - \{ip_k, k \in \mathbb{N}\}$, $F(x, x_0, t, z)$ has the following expansion:

$$\begin{aligned} F(x, x_0, t, z) &= F(x, x_0, t, i\theta_k) + F'(x, x_0, t, i\theta_k)(z - i\theta_k) \\ &\quad + (z - i\theta_k)^2 H(x, x_0, t, z) \\ &= (z - i\theta_k)^2 H(x, x_0, t, z), \end{aligned} \tag{2.44}$$

where $H(x, x_0, t, z)$ does not vanish in a neighborhood of $i\theta_k$, and by Taylor expansion we have

$$-\frac{t^2}{z(x, x_0, t)} + \frac{x^2 z(x, x_0, t)}{\sinh^2(z(x, x_0, t))} = \frac{x^2}{\sin\theta_k}\Big(-1 + \frac{\theta_k}{\tan\theta_k}\Big)(z - i\theta_k) + o(z - i\theta_k)$$

and

$$K_w(x, x_0, t) \sim \frac{1}{2\pi}\left(\sqrt{\frac{\theta_k}{2\sin(\theta_k)}}\right)\frac{1}{(x^2/\sin^2\theta_k)(-1+\theta_k/\tan\theta_k)}\left(\frac{1}{(z-i\theta_k)}+\frac{1}{(\bar{z}+i\theta_k)}\right). \tag{2.45}$$

In order to express (2.45) in terms of the distance to the set $S$, note that

$$F(x, x_0, t, z) = \frac{t^2}{2} - f(x, x_0, z) = (z-i\theta_k)^2 H(x, x_0, t, z), \tag{2.46}$$

and using the length of the $k$ th geodesic, $f(x, x_0, z) \sim d_k^2(x, x_0)/2$, we see that

$$K_w(x, x_0, t) \sim \frac{1}{2\pi}\left(\sqrt{\frac{\theta_k}{2\sin(\theta_k)}}\right)\frac{1}{(x^2/\sin^2\theta_k)(-1+\theta_k/\tan\theta_k)} \cdot\left(\sqrt{\frac{H(x, x_0, t, z)}{t^2-d_k^2(x, x_0)}}+\sqrt{\frac{H(x, -x_0, t, \bar{z})}{t^2-d_k^2(x, -x_0)}}\right),$$

where $H$ does not vanish. When $\theta_k$ is one of the points $p_k$, the same type of analysis shows that $K$ grows like $1/(t^2-d_k^2(x, x_0))^{3/2}$. □

*2.5. The operator satisfies the wave equation*

Using integration by parts, we prove that the wave kernel $K_w$ satisfies the wave equation. Starting with the fact that $K_w(x, x_0, t) = (K(x, x_0, t) + K(x, -x_0, t))/2$, we only need to show that $K$ satisfies the wave equation. Recall that

$$K(x, x_0, t) = K_0 \int_C \frac{1}{\Phi(x, x_0, z, t)} w(z)\, dz, \tag{2.47}$$

where $w(z) = \sqrt{z}/(2\sinh(z))$ and $\Phi(x, x_0, z, t) = ix_0 z + t^2/2 - x^2 z\coth(z)/2$, $K_0$ is constant, and $C$ is a closed contour that may enclose singularities but not pass through them. We have already proved that $K_w$ is zero for $t$ small enough. Note that $w$ satisfies

$$w'(z) = \frac{w(z)}{2}\left(\frac{1}{z} - \frac{1}{\tanh(z)}\right), \tag{2.48}$$

and recall the formula for the derivatives of $\Phi$:

$$\Phi'(z) = ix_0 - \frac{x^2}{2}\left(\frac{1}{\tanh z} - \frac{z}{\sinh^2(z)}\right). \tag{2.49}$$

By direct computations,

$$\frac{\partial^2 K(x,x_0,t)}{\partial t^2} = K_0 \int_C \Big( -\frac{1}{\Phi^2(x,x_0,z,t)} + \frac{2t^2}{\Phi^3(x,x_0,z,t)}\Big) w(z)\,dz, \quad (2.50)$$

$$\frac{\partial^2 K(x,x_0,t)}{\partial x^2} = K_0 \int_C \Big(\frac{z}{\tanh(z)\Phi^2(x,x_0,z,t)} + \frac{2x^2z^2}{\tanh^2(z)\Phi^3(x,x_0,z,t)}\Big) w(z)\,dz, \quad (2.51)$$

$$x^2\frac{\partial^2 K(x,x_0,t)}{\partial y^2} = -2K_0 \int_C \Big(\frac{z^2x^2}{\Phi^3(x,x_0,z,t)}\Big) w(z)\,dz, \quad (2.52)$$

and

$$\frac{\partial^2 K(x,x_0,t)}{\partial x^2} + x^2\frac{\partial^2 K(x,x_0,t)}{\partial y^2} = K_0 \int_C \Big(\frac{z}{\tanh(z)\Phi^2(x,x_0,z,t)} + \frac{2x^2z^2}{\sinh^2(z)\Phi^3(x,x_0,z,t)}\Big) w(z)\,dz.$$

Consider the first term of the right-hand side in the last expression. Using the properties of the function $w$ and integrating by parts, we obtain

$$\begin{aligned}
\int_C \frac{z}{\tanh(z)\Phi^2(x,x_0,z,t)} w(z)\,dz &= \int_C \frac{w(z)-2zw'(z)}{\Phi^2(x,x_0,z,t)}\,dz \\
&= \int_C \Big(\frac{w(z)}{\Phi^2(x,x_0,z,t)} - 2\Big[\frac{wz}{\Phi^2(x,x_0,z,t)}\Big]\Big)\,dz \\
&\quad + 2\int_C \Big(\frac{z}{\Phi^2(x,x_0,z,t)}\Big)' w(z)\,dz \\
&= \int_C \frac{w(z)}{\Phi^2(x,x_0,z,t)}\,dz + 2\int_C \frac{w(z)}{\Phi^2(x,x_0,z,t)}\,dz \\
&\quad - 4\int_C \frac{zw(z)\Phi'(x,x_0,z,t)}{\Phi^3(x,x_0,z,t)}\,dz \\
&= 3\int_C \frac{w(z)}{\Phi^2(x,x_0,z,t)}\,dz \\
&\quad - 4\int_C \Big(ix_0z - \frac{x^2}{2}\Big(\frac{z}{\tanh(z)} - \frac{z^2}{\sinh^2(z)}\Big)\Big)\frac{w(z)}{\Phi^3(x,x_0,z,t)}\,dz. \quad (2.53)
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\partial^2 K(x,x_0,t)}{\partial x^2} &+ x^2 \frac{\partial^2 K(x,x_0,t)}{\partial y^2} \\
&= 3\int_C \frac{w(z)}{\Phi^2(x,x_0,z,t)}\,dz - 4\int_C \frac{ix_0\,zw(z)}{\Phi^3(x,x_0,z,t)}\,dz \\
&\quad + 2\int_C \frac{x^2 w(z) z}{\tanh(z)\Phi^3(x,x_0,z,t)}\,dz \\
&= -\int_C \frac{w(z)}{\Phi^2(x,x_0,z,t)}\,dz + 4\int_C \frac{\Phi(x,x_0,z,t) - ix_0 z}{\Phi^3(x,x_0,z,t)} w(z)\,dz \\
&\quad + 2\int_C \frac{x^2 w(z) z}{\tanh(z)\Phi^3(x,x_0,z,t)}\,dz \\
&= -\int_C \frac{w(z)}{\Phi^2(x,x_0,z,t)}\,dz + 2\int_C \frac{t^2}{\Phi^3(x,x_0,z,t)} w(z)\,dz \\
&= \frac{\partial^2 K(x,x_0,t)}{\partial t^2}.
\end{aligned} \tag{2.54}$$

Observe that $w$ plays the role of the transport term in wave theory.

### 2.6. Source not at the origin

In this subsection we make several remarks concerning the case where the Dirac distribution is given at a point $(y, y_0) \neq (0, 0)$ and the observer is fixed at the point $(x, x_0)$. An analog of Theorem 1 is valid in this case as well. The following is true.

PROPOSITION 6

*The wave kernel $K_w(x, x_0, t)$ defined by $K(x, x_0, t) = (\sin(tL^{1/2})/L^{1/2})\delta(y, y_0)$, where $-L/2$ is the Grušin operator, is given by*

$$K_w(y, y_0, x, x_0, t, z) = \frac{K(y, y_0, x, x_0, t, z) + K(y, y_0, x, x_0, t, z)}{2}, \tag{2.55}$$

*where*

$$K(x, x_0, t) = K_0' \int_\Gamma \sqrt{\frac{1}{u\sinh(u)}}\frac{1}{\Phi(x, x_0, t, u)}\,du, \tag{2.56}$$

*and $K_0' = 1/(2\pi)^2$, $\Gamma$ is an appropriate contour, and $\Phi(y, y_0, x, x_0, t, z)$ is the phase given by the expression*

$$\Phi(y, y_0, x, x_0, t, z) = -\frac{x^2 + y^2}{2\tanh z} + \frac{xy}{\sinh z} + i(y_0 - x_0) + \frac{t^2}{2z}. \tag{2.57}$$

*Set $f(y, y_0, x, x_0, t, z) = z\Phi(y, y_0, x, x_0, t, z)$. The singularities of the integrand are located at the zeros of $F(y, y_0, x, x_0, t, z) = 0$. More precisely, the singularity is lo-*

*cated at the intersection of* $\operatorname{Re}\Phi(y, y_0, x, x_0, t, z) = 0$ *and* $\operatorname{Im} f(y, y_0, x, x_0, t, z) = 0$.

*Proof*
The proof is similar to the proof of Theorem 1, and using the same arguments, we obtain the following expression of the phase:

$$\Phi(y, y_0, x, x_0, t, z) = -\frac{x^2 + y^2}{2\tanh z} + \frac{xy}{\sinh z} + i(y_0 - x_0) + \frac{t^2}{2z}. \tag{2.58}$$

The contour starts in the region where $\operatorname{Re}\Phi(y, y_0, x, x_0, t, z) < 0$, but it turns out to be more complex. □

The analog of Proposition 4 is more complex and will be discussed in a future work.

## 3. Wave kernels in one dimension

In this section we study the wave kernel for two operators. The first operator is the harmonic oscillator $\partial_{xx} - x^2$, where we are able to give an integral representation formula, and the second is the Klein Gordon operator $\partial_{xx} - a^2$ ($a$ is a constant), where we recover well-known results.

### *3.1. Wave kernel for the harmonic oscillator*

The purpose of this subsection is to discuss the wave kernel for the equation and initial conditions

$$\begin{aligned} \partial_{tt} u &= \partial_{xx} u - x^2 u, \\ \partial_t u(x, 0) &= \delta_x, \\ u(x, 0) &= 0. \end{aligned}$$

We obtain the following result.

PROPOSITION 7
*The wave kernel for the operator* $\partial_{xx} - x^2$ *can be expressed in the form*

$$K(x, t) = \frac{i}{4\pi} \int_C \sqrt{\frac{1}{2z\sinh(z)}} e^{t^2/2z - x^2\coth(z)/2}\, dz, \tag{3.1}$$

*where* $C$ *is a contour symmetric with respect to the* $x$*-axis going through the origin, obtained by a smooth deformation of the circle* $C(1/2c, 1/2c)$. *The singularity at* $z = 0$ *is essential. For* $t < |x|$ *the kernel vanishes. The phase* $\phi(x, t, z) = t^2/(2z) - x^2\coth(z)/2$ *satisfies the Hamilton-Jacobi equation*

$$2\frac{\partial\phi}{\partial z} = -\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 - x^2. \tag{3.2}$$

*Proof*
Using expression (1.38) for the Dirac operator and separation of variables, we can express the kernel in the form

$$K(x,t) = \sum_{n=0}^{+\infty} \frac{1}{n!2^n\sqrt{\pi}} \frac{\sin(\sqrt{2n+1}t)}{\sqrt{2n+1}} H_n(0)H_n(x)e^{-x^2/2}. \tag{3.3}$$

We now use integral formula (2.10) to express the time dependence:

$$\frac{\sin t\sqrt{2n+1}}{\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(1/2)(t^2v-(2n+1)/v)}v^{-3/2}\,dv. \tag{3.4}$$

Then the kernel assumes the form

$$K_w(x,x_0,t) = \frac{1}{2i\pi\sqrt{2}}\sum_{0}^{\infty}\int_{c-i\infty}^{c+i\infty} e^{(1/2)(t^2v-(2n+1)/v)}v^{-3/2}\,dv\frac{1}{n!2^n}H_n(x)e^{-x^2/2}H_n(0).$$

By Mehler formula (1.36) for $\operatorname{Re} v > 0$, we have

$$e^{-1/2v}\sum_{n=0}^{+\infty}\frac{H_n(x)H_n(0)e^{-n/v}}{2^n n!} = \frac{e^{-1/2v}}{\sqrt{1-e^{2/v}}}e^{-x^2e^{-2/v}/(1-e^{-2/v})}. \tag{3.5}$$

We can simplify as in (2.14), and we find that in the new variable $z = 1/v$, $dv/v = -dz/z$, and $\sqrt{v} = 1/\sqrt{z}$, the contour $x = c$ is transformed to a circle $C = C(1/2c, 1/2c)$, centered at $(1/2c, 0)$ of radius $1/2c$, obtaining finally

$$K(x,t) = \frac{i}{4\pi}\int_C \sqrt{\frac{1}{2z\sinh(z)}}e^{t^2/(2z)-x^2\coth(z)/2}\,dz. \tag{3.6}$$

The details of the computation are given in Section 2 for the Grušin operator.

We can deform $C$ to a closed contour symmetric with respect to the $u$-axis ($u = \operatorname{Re}(z)$) as follows. The contour starts outside the region defined by $\operatorname{Re}(t^2/(2z) - x^2\coth(z)/2) > 0$ and $\operatorname{Re} z \geq 0$ (see Fig. 4 for the curve $\Theta_{x,t}$). At the point where $\Theta_{x,t}$ meets the imaginary axis for the first time, we continue along this axis until the next point $v_\lambda$ (where $\lambda = x^2/t^2$), defined in Section 2.1. Then the contour is continued by being allowed to come back to $\operatorname{Re} z > 0$, avoiding the singularity at a multiple of $i\pi$. This construction is continued each time a singularity has to be avoided. The contour reaches the origin along the imaginary axis. Then the path is symmetrized with respect to the $u$-axis. Thus the contour avoids the singularities $\sinh(z) = 0$ for $z \neq 0$ and stays in the region $u \geq 0$.

To prove that the operator satisfies finite speed propagation and vanishes when $t < x$, we may use the residue theorem, and since the singularity at $z = 0$ is removable,

$$K(x, t) = 0 \quad \text{when } t < x. \tag{3.7}$$

We may verify by an elementary computation that the phase $\phi$ satisfies Hamilton-Jacobi equation (3.2). □

*Remark.* When $t > x$, the point zero is an essential singularity and yields the main contribution. Indeed, the integral on the imaginary axis can be expressed as

$$K_a(x, t) = \frac{i}{4\pi} \int_{-a}^{a} \sqrt{\frac{1}{2y \sin(y)}} \left( \cos\left( \frac{t^2}{2y} - \frac{x^2 \coth(y)}{2} \right) - i \sin\left( \frac{t^2}{2y} - \frac{x^2 \coth(y)}{2} \right) \right) dy,$$

which reduces by symmetry of the first term (understood as principal value) to

$$K_a(x, t) = \frac{1}{4\pi} \int_{-a}^{a} \sqrt{\frac{1}{2y \sin(y)}} \sin\left( \frac{t^2}{2y} - \frac{x^2 \coth(y)}{2} \right) dy, \tag{3.8}$$

so that

$$K_a(x, t) = \frac{1}{4\pi} \int_{-a}^{a} \sqrt{\frac{1}{2y \sin(y)}} \sin\left( \frac{t^2 - x^2}{2y} \right) dy + O(1). \tag{3.9}$$

$K_a$ depends (to a first approximation) only on $t^2 - x^2$, and we have for $a$ close to zero,

$$K_a(x, t) \sim \frac{1}{2\sqrt{2}\pi} \int_{0}^{a} \frac{1}{y} \sin\left( \frac{t^2 - x^2}{2y} \right) dy + O(1). \tag{3.10}$$

The integral in (3.10) is convergent, as can be seen by using the following change of variable $z = 1/y$.

### *3.2. Wave kernel for the Klein-Gordon operator*

In this paragraph we show how it is possible to recover the well-known result concerning the wave kernel for the translation-invariant Klein-Gordon operator. The result is given in term of the Bessel function $J_0$. We prove the following.

PROPOSITION 8

*The wave kernel for the equation*

$$\begin{aligned} \frac{\partial^2}{\partial t^2} w &= \partial_{xx} w - a^2 w, \\ w(x, 0) &= 0, \\ w_t(x, 0) &= \delta_0(x), \end{aligned} \tag{3.11}$$

*is given by*

$$K(x,t) = \frac{1}{2} J_0(a\sqrt{t^2 - x^2}) H(t^2 - x^2), \tag{3.12}$$

*where $H$ is the Heaviside function and $J_0$ is the Bessel function.*

*Proof*
The general solution of (3.11) is given by the family of functions $e^{ikx}\sin(\sqrt{k^2+a^2}t)$ and $e^{ikx}\cos(\sqrt{k^2+a^2}t)$. Integrating the family (recall (1.9)), we obtain

$$K(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \cos(kx)\frac{\sin(\sqrt{k^2+a^2}t)}{\sqrt{k^2+a^2}}\,dk. \tag{3.13}$$

Using the relation

$$\frac{\sin(\sqrt{k^2+a^2}t)}{\sqrt{k^2+a^2}} = \frac{\sqrt{\pi/2}}{2i\pi}\int_{-\infty}^{0+} e^{(1/2)(u-(k^2+a^2)t^2/u)} u^{-3/2}\,du \tag{3.14}$$

(see (1.43)), we get

$$K(x,t) = \frac{1}{2\pi}\frac{\sqrt{\pi/2}}{2i\pi}\int_{\mathbb{R}}\int_{-\infty}^{0+} e^{(1/2)(u-(k^2+a^2)t^2/u)} u^{-3/2}\,du\cos(kx)\,dk. \tag{3.15}$$

But

$$\int_{\mathbb{R}} e^{-k^2t^2/2u}\cos(kx)\,dk = \frac{\sqrt{2\pi u}}{t} e^{-x^2u/(2t^2)} \tag{3.16}$$

for $u \in \mathbb{C} - \mathbb{R}^-$ (usual cut). Hence

$$K(x,t) = \frac{1}{2}\frac{1}{2i\pi}\int_{-\infty}^{0+} e^{((1-x^2/t^2)/2)(u-t^2a^2/(1-x^2/t^2))} u^{-1}\,du, \tag{3.17}$$

and applying (1.43) again, we see that for $t^2 - x^2 > 0$,

$$K(x,t) = \frac{1}{2} J_0(a\sqrt{t^2-x^2}). \tag{3.18}$$

It is well known that $K$ is zero for $t^2 < x^2$, and if $H$ denotes the Heaviside function, we obtain (3.12). □

## 4. The Heisenberg wave kernel via the heat kernel

In this section we construct a representation formula for the Heisenberg wave kernel (the kernel of $\cos(\sqrt{-2\Delta_H}\,t)$) by inverting the so-called transmutation formulas. We also have to deform an integration path in the complex plane. The results are expressed in Theorems 2 and 3 of this section. In the corollaries, analyticity results are given, and the behavior of the kernel near its singular support is described.

PROPOSITION 9
*Let L be a nonnegative self-adjoint operator, $u > 0$. Then*

$$e^{-Lu} = \frac{1}{\sqrt{4\pi u}} \int_{-\infty}^{\infty} e^{-t^2/(4u)} \cos(\sqrt{L}\, t)\, dt \ = \frac{1}{\sqrt{\pi u}} \int_{0}^{\infty} e^{-t^2/(4u)} \cos(\sqrt{L} t)\, dt. \tag{4.1}$$

Proposition 9 is a well-known transmutation formula. A proof may be found in [8].

PROPOSITION 10
*For every $a \neq 0$, $u > 0$, and nonnegative integer $n$,*

$$\frac{e^{-a^2/(2u)}}{u^{n+1}} = \frac{4^{n+1/2}}{\sqrt{u}} \int_0^{\infty} e^{-t^2/(4u)} \frac{\partial}{\partial t} \delta^{(n-1/2)}(t^2 - 2a^2)\, dt. \tag{4.2}$$

*Proof*
We have

$$\int_0^{\infty} e^{-t^2/(4u)} \frac{\partial}{\partial t} \delta^{(n-1/2)}(t^2 - 2a^2)\, dt = 1/(4u) \int_0^{\infty} e^{-t^2/(4u)} \delta^{(n-1/2)}(t^2 - 2a^2) 2t\, dt.$$

Setting $t^2 - 2a^2 = y$, we see (recalling the formula for fractional differentiation, cf. [5, pp. 739–740]) that the right-hand side is, by definition, equal to

$$\begin{aligned} \frac{1}{4u} \int_0^{\infty} e^{-(2a^2+y)/(4u)} \delta^{(n-1/2)}(y)\, dy &= \frac{e^{-a^2/(2u)}}{4u} \int_0^{\infty} \frac{e^{-y/(4u)}}{\Gamma(1/2)} \frac{\partial^n}{\partial y^n} \Big(\frac{1}{\sqrt{y}}\Big)\, dy \\ &= \frac{e^{-a^2/(2u)}}{\Gamma\big(1/2\big)(4u)^{n+1} \int_0^{\infty} e^{-y/(4u)} / \sqrt{y}\, dy} \\ &= \frac{\sqrt{\pi}}{\sqrt{\pi}\, (4u)^{n+1/2}} e^{-a^2/(2u)} = \frac{e^{-a^2/u}}{4u^{n+1/2}}, \end{aligned}$$

and the proposition follows. □

Note that one may derive (4.2) from (4.1) and the well-known expressions for $\cos(\sqrt{-\Delta}\, t)$.

We set

$$P(x, x_0; u) = \frac{1}{(2\pi u)^{n+1}} \int_{-\infty}^{\infty} e^{-f(x, x_0; \tau)/u} V(\tau)\, d\tau, \tag{4.3}$$

where

$$f(x, x_0; \tau) = \alpha\tau \coth(2\alpha\tau)|x|^2 - i x_0 \tau = f(x, x_0, \tau; \alpha) \tag{4.4}$$

and

$$V(\tau) = \left(\frac{2\alpha\tau}{\sinh(2\alpha\tau)}\right)^n. \tag{4.5}$$

Here $\alpha$ is a positive constant and $f(x, x_0, \tau; \alpha)$ is the function introduced in Section 1.2.

PROPOSITION 11

*Let*

$$\Delta_H = \frac{1}{2}\Big[\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j-1}} + 2\alpha x_{2j}\frac{\partial}{\partial x_0}\Big)^2 + \frac{1}{2}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j}} - 2\alpha x_{2j-1}\frac{\partial}{\partial x_0}\Big)^2\Big]. \tag{4.6}$$

*Then*

$$P(x, x_0; t) = e^{t\Delta_H}(x, x_0; 0, 0). \tag{4.7}$$

A proof may be found in [1]. Observe that the kernel $P(x, x_0; t)$ is the fundamental solution of the heat equation

$$\frac{\partial P}{\partial t} = \Delta_H P \tag{4.8}$$

with the singularity at the origin.

*4.1. Deforming the path of integration*

We deform the path of integration used in formula (4.3) from the real axis to the curve $\Gamma$ discussed in Section 1.2. This path has been introduced for studying singularities of the Heisenberg wave kernel by Taylor [15, pp. 80–86]. We consider the case where $x_0 \geq 0$; the other case is similar.

PROPOSITION 12

*If* $x \neq 0$, $x_0 \neq 0$, *then*

$$P(x, x_0; u) = \frac{1}{(2\pi u)^{n+1}}\int_{\Gamma_{x,x_0}} e^{-f(x,x_0,\tau)/u}V(\tau)\,d\tau. \tag{4.9}$$

*Proof*

Let $N = N(x_0/|x|^2)$ denote the number of purely imaginary zeros of $\partial f/\partial\tau$, and for $R \to \infty$ consider the closed contour $\Gamma_R$ (see Fig. 5) formed by the interval $\{-R \leq \tau_1 \leq R, \tau_2 = 0\}$, the vertical segments $\tau_1 = \pm R$ joining $(-R, 0)$ and $(R, 0)$ with the unbounded branches of $\Gamma_{x,x_0}$, and the portion of $\Gamma_{x,x_0}$ between those intersection points.

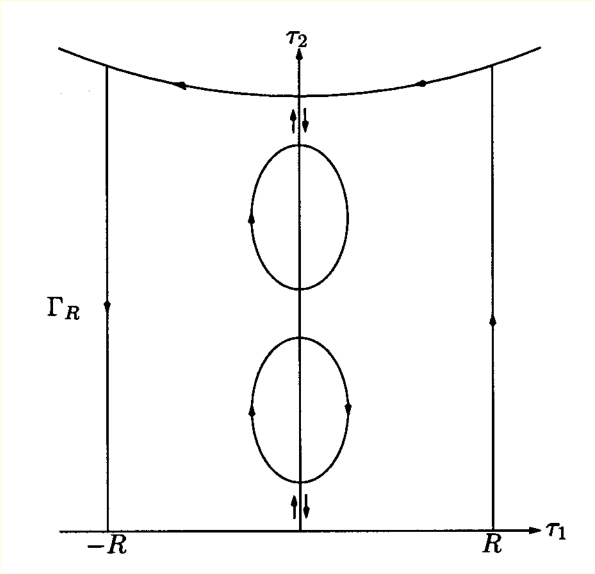


Figure 5. The integration path $\Gamma_R$

By Cauchy's integral theorem,

$$\int_{\Gamma_R} e^{-f(x,x_0;\tau)/u} V(\tau)\, d\tau = 0.$$

Recall that $V(R+i\tau_2) \sim 2(R+i\tau_2)^n$ as $R \to \infty$. Moreover,

$$f(x, x_0; R+i\tau_2) = (R+i\tau_2)\coth(R+i\tau_2) - iRx_0 + x_0\tau_2,$$

so that for every $\varepsilon > 0$,

$$|e^{-f(x,x_0;R+i\tau)/u}| \leq \exp\left[-\frac{(|R|+\tau_2)(1-\varepsilon)+x_0\tau_2}{u}\right]$$

if $R$ is large enough. The length of the vertical interval is proportional (asymptotically) to $R$ (see Prop. 3). Hence the integral on the vertical line $\tau_1 = R$, and similarly on $\tau_1 = -R$, tends to zero as $R \to \infty$. The proposition follows from (4.3). □

*Remark.* Recall that $\Gamma_{0,x,x_0}$ denotes the closure of the set of non-purely-imaginary points of $\Gamma_{x,x_0}$. The analyticity of $V(\tau)$ and of $f(x, x_0; \tau)$ on $(i\theta_{2k}, i\theta_{2k+1})$ implies that the two integrals over $(i\theta_{2k}, i\theta_{2k+1})$, performed in opposite directions, cancel out, and we may rewrite (4.9) as

$$P(x, x_0; u) = \frac{1}{(2\pi u)^{n+1}} \int_{\Gamma_{0,x,x_0}} e^{-f(x,x_0,\tau)/u} V(\tau)\, d\tau. \tag{4.10}$$

*4.2. Computation of the wave kernel*

We analyze the wave kernel for the isotropic Heisenberg Laplacian, that is, the solution $w(x, x_0; t)$ of the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} = \Big[\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j-1}} + 2\alpha x_{2j}\frac{\partial}{\partial x_0}\Big)^2 + \sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j}} - 2\alpha x_{2j-1}\frac{\partial}{\partial x_0}\Big)^2\Big] w(x, x_0, t) \tag{4.11}$$

with the initial conditions $w(x, x_0; 0) = \delta(x, x_0; 0, 0)$ and $(\partial w/\partial t) w(x, x_0; 0) = 0$. This is the kernel of $\cos(\sqrt{-2\Delta_H}\, t)$, and we calculate it by combining Propositions 9, 10, and 12 and by setting $L = -2\Delta_H$.

PROPOSITION 13

*If* $x \neq 0$, $x_0 \neq 0$, *then*

$$P(x, x_0; 2u) = \frac{1}{2\pi^{n+1}\sqrt{u}} \int_{\Gamma_{x,x_0}} V(\tau) \int_0^\infty e^{-t^2/(4u)} \times \frac{\partial}{\partial t}\delta^{(n-1/2)}\big(t^2 - 2f(x, x_0; \tau)\big)\, dt\, d\tau. \tag{4.12}$$

*Proof*

Recall that $f(x, x_0; \tau)$ is positive on $\Gamma_{x,x_0}$. By (4.2) (and by setting $f = a^2$),

$$\frac{1}{(4\pi u)^{n+1}} e^{-f(x,x_0,\tau)/(2u)} = \frac{1}{2\pi^{n+1}}\frac{1}{\sqrt{u}} \int_0^\infty e^{-t^2/(4u)} \frac{\partial}{\partial t}\delta^{(n-1/2)}\big(t^2 - 2f(x, x_0; \tau)\big)\, dt.$$

Substituting in (4.9), we get (4.12). □

Note also that

$$P(x, x_0; 2u) = \frac{1}{2\pi^{n+1}\sqrt{u}} \int_{\Gamma_{0,x,x_0}} V(\tau) \int_0^\infty e^{-t^2/(4u)} \times \frac{\partial}{\partial t}\delta^{(n-1/2)}\Big(t^2 - 2f(x, x_0; \tau)\Big)\, dt\, d\tau. \tag{4.13}$$

THEOREM 2
*If* $x \neq 0$, $x_0 \neq 0$, *then*

$$w(x, x_0; t) = \cos(\sqrt{-2\Delta_H}\, t)(x, x_0) = \frac{1}{\pi^{n+1/2}} \int_{\Gamma_{x,x_0}} \frac{\partial}{\partial t} \delta^{(n-1/2)}\big(t^2 - 2f(x, x_0; \tau)\big) V(\tau)\, d\tau. \qquad (4.14)$$

*Proof*
Set $L = -2\Delta_H$, and apply Proposition 9 to (4.12), getting the relation

$$\frac{1}{\sqrt{4\pi u}} \int_{-\infty}^{\infty} e^{-t^2/(4u)} w(x, x_0; t)\, dt = \frac{1}{2\pi^{n+1}\sqrt{u}} \int_{\Gamma_{x,x_0}} V(\tau) \int_0^{\infty} e^{-t^2/(4u)} \frac{\partial}{\partial t} \delta^{(n-1/2)}\big(t^2 - 2f(x, x_0; \tau)\big)\, ds\, d\tau. \qquad (4.15)$$

Recall that on the unbounded branch of $\Gamma_{x,x_0}$, $\coth(2\tau) \to 1$ as $\tau \to \infty$, so that

$$f(x, x_0; \tau) = \operatorname{Re} f(x, x_0; \tau) \sim |x|^2 \tau_1 + x_0 \tau_2 \geq \delta|\tau|$$

for a certain $\delta = \delta(|x|, x_0) > 0$. Proceeding as in the proof of Proposition 10 and setting $t^2 - 2f(x, x_0; \tau) = y$, we see that the inner integral on the right-hand side of (4.15) is equal to $\big(e^{-f(x,x_0;\tau)/(2u)}/(\sqrt{\pi}\, u^{n+1})\big) \int_0^{\infty} e^{-y/(4u)}/\sqrt{y}\, dy$, so that for fixed $u$, the right-hand side of (4.15) is proportional to

$$\int_{\Gamma_{x,x_0}} V(\tau) e^{-f(x,x_0;\tau)/(2u)} \int_0^{\infty} \frac{e^{-y/(4u)}}{\sqrt{y}} dy\, d\tau \leq C \int_{\Gamma_{x,x_0}} V(\tau) e^{-\delta|\tau|}\, d\tau \int_0^{\infty} \frac{e^{-y/(4u)}}{\sqrt{y}}\, dy\, dc.$$

Hence we may interchange the order of integration in (4.15) to obtain the equation

$$\int_{-\infty}^{\infty} e^{-t^2/(4u)} w(x, x_0; t)\, dt = 2 \int_0^{\infty} e^{-t^2/(4u)} w(x, x_0; t)\, dt = \frac{1}{\pi^{n+1/2}} \int_0^{\infty} e^{-t^2/(4u)} \Big[ \int_{\Gamma_{x,x_0}} V(\tau) \frac{\partial}{\partial t} \delta^{(n-1/2)} (t^2 - 2f(x, x_0; \tau))\, d\tau \Big] ds. \qquad (4.16)$$

Relation (4.16) holds for all $u > 0$; hence for all $1/u > 0$ and by uniqueness for the Laplace transform, we get (4.14). □

*Note.* Formula (4.14) remains valid if the integration is extended over $\Gamma_{0,x,x_0}$.

COROLLARY 1
*If* $x \neq 0$*,* $x_0 \neq 0$ *and* $t \neq d_j$ *for all* $i \leq j \leq N$*, then* $w(x, x_0; t)$ *is a real analytic function of all its arguments.*

*Note.* Recall the formula $d_j(x, x_0) = \sqrt{2f(x, x_0; i\theta_j)}$ for $i \leq j \leq N$. Note also that $w$ is a solution of the Heisenberg wave equation.

*Proof*
Let $t_0$ be different from $d_j$ for all $1 \leq j \leq N$. Note that $w(x, x_0; t) \equiv 0$ for $t < d_1$ (finite speed; see [13] or [15]). We may assume that $t_0 \in (d_{2k+1}, d_{2k+2})$ for a certain $k$ (or $d_N < t_0$). The only possible singularities of $w$ may arise from the contributions of the parts of $\Gamma_{x,x_0}$ between $i\theta_{2k+1}$ and $i\theta_{2k+2}$ (or $i\theta_N$ and $\infty$). There exists a positive $\varepsilon$ such that the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ contains no $d_j$. Recall that $\partial f/\partial\tau \neq 0$ between $i\theta_{2k-1}$ and $i\theta_{2k+2}$ (or between $i\theta_N$ and $\infty$). The reality of $f$ on $\Gamma_{x,x_0}$ along with the Cauchy-Riemann equations imply that there exists an analytic function $h(f)$ (the inverse function of $f$) such that $h(d_{2k+1}^2/2) = i\theta_{2k+1}$, $h(d_{2k+2}^2/2) = i\theta_{2k+2}$. Let $\psi = C_0^\infty(t_0 - \varepsilon, t_0 + \varepsilon)$ be such that $\psi(t) \equiv 1$ for $t$ in a neighborhood of $t_0$. To prove Corollary 1, it suffices to prove the regularity of the distribution $v(x, x_0; t)$ given by

$$v(x, x_0; t) = \int_{\Gamma_{x,x_0}} \delta^{(n-1/2)}\big(t^2 - 2f(x, x_0; \tau)\big)V(\tau)\psi\Big(\sqrt{2f(x, x_0; \tau)}\Big)\,d\tau.$$

Introduce a new real variable $\sigma = 2f(x, x_0, \tau)$. Then the function $h(\sigma/2)$ is well defined in $(\sqrt{t_0 - \varepsilon}, \sqrt{t_0 + \varepsilon}) \supset \operatorname{supp}\psi$. Hence

$$v(x, x_0; t) = \int \delta^{(n-1/2)}(t^2 - \sigma)V\left(h\left(\frac{\sigma}{2}\right)\right)\psi(\sigma)\frac{h'(\sigma/2)}{2}\,d\sigma.$$

Set $V(h(\sigma/2))\psi(\sigma)(h'/2)(\sigma/2) = g(\sigma)$. Thus $g(\sigma)$ is real analytic near $t_0^2$, and so is

$$v(x, x_0; t) = \int g(t^2 - \sigma)\delta^{(n-1/2)}(\sigma)\,d\sigma$$

near $t_0$. □

It is possible to use the same change of variables on all of $\Gamma_{0,x,x_0}$ in order to obtain a representation of $w(x, x_0; t)$. Recall that $f(x, x_0; \tau)$ maps the branch of $\Gamma_{x,x_0}$ which joins $i\theta_{2k-1}$ and $i\theta_{2k}$ and for which $\tau > 0$ in a differentially invertible manner onto $(d_{2k-1}^2/2, d_k^2/2)$. Let $h_k(\sigma)$ denote the inverse function of $f(x, x_0; \tau)$ defined on $(d_{2k-1}^2/2, d_k^2/2)$. Set $w_k(\sigma) = V(h_k(\sigma)) + V(\overline{-h_k(\sigma)})$. Similarly, let $h_N(\sigma)$ denote the inverse of $f(x, x_0; \tau)$, defined on $(d_N^2/2, \infty)$ and parametrizing the right branch of $\Gamma_N$ joining $i\theta_N$ and $\infty$. The note after the proof of Theorem 2 yields the following.

THEOREM 3

*If* $x \neq 0$, $x_0 \neq 0$, *then*

$$w(x,x_0;t) = \cos(\sqrt{-2\Delta_H}\,t)(x,x_0) = \frac{1}{\pi^{n+1}}\Big\{\sum_{k=1}^{[N/2]}\int_{d_{2k-1}^2}^{d_k^2}\frac{\partial}{\partial t}\delta^{(n-1/2)}(t^2-\sigma)W_k(\sigma)\frac{h_k'(\sigma/2)}{2}\,d\sigma + \int_{d_N^2}^{\infty}\frac{\partial}{\partial t}\delta^{(n-1/2)}(t^2-\sigma)W_k(\sigma)\frac{h_k'}{2}\Big(\frac{\sigma}{2}\Big)\,d\sigma\Big\}. \tag{4.17}$$

One may read off (4.17) the nature of the singularities of $w(x,x_0;t)$ when $t$ is near $d_j$. An alternative determination of the singularities may be found in Nachman [13, p. 713]. We restrict ourselves to the leading singularity.

COROLLARY 2

*Let* $x \neq 0$, $x_0 \neq 0$, *and let all the* $\theta_j$ *be distinct. Then there exist* $C_j(x,x_0)$ *such that*

$$\begin{aligned} &w(x,x_0;t) \sim C_j(x,x_0)\delta^{(n)}(t-d_j) \quad \text{if } j \text{ is odd,}\\ &w(x,x_0,t) \sim C_j(x,x_0)(t-d_j)^{-n-1} \quad \text{if } j \text{ is even,} \end{aligned} \tag{4.18}$$

*for* $t$ *near* $d_j(x,x_0)$.

*Proof*

Consider first the case where $j$ is odd, $j = 2k-1$. By Theorem 3, the only contribution to the singularities of $w(x,x_0;t)$ for $t$ near $d_{2k-1}$ or $d_N$ arises from integration near the local minimum of $f$ at $i\theta_{2k-1}$ (or $\sigma = d_{2k-1}^2$). By assumption, $(\partial^2 f/\partial\tau^2)(x,x_0;i\theta_{2k-1}) \neq 0$, so that $h_k'(\sigma/2) \sim C_j/(\sigma - d_{2k-1}^2)^{1/2}$. (We denote different constants by $C_j$.) Hence the main singularity of $w(x,x_0;t)$ is given by

$$\begin{aligned} C_j\int_{d_{2k-1}^2}\frac{\partial}{\partial t}\delta^{(n-1/2)}(t^2-\sigma)\frac{d\sigma}{(\sigma-d_{2k-1}^2)^{1/2}} &= C_j\int_0\frac{\partial}{\partial t}\frac{\delta^{(n-1/2)}(t^2-d_{2k-1}^2-\sigma)}{\sqrt{\sigma}}\,d\sigma\\ &= C_j\int_0\frac{\partial}{\partial t}\Big(\frac{1}{2}\frac{\partial}{\partial t}\Big)^{(n)}\delta^{(-1/2)}(t^2-d_{2k-1}^2-\sigma)\frac{d\sigma}{\sqrt{\sigma}}\\ &= C_j\int_0\frac{\partial}{\partial t}\Big(\frac{1}{2}\frac{\partial}{\partial t}\Big)^{(n)}\frac{1}{\sqrt{t^2-d_{2k-1}^2-\sigma}}\frac{d\sigma}{\sqrt{\sigma}}, \end{aligned}$$

where the integrations and the differentiations are to be understood in the distribution sense, and only the left endpoint of the $\sigma$-interval is indicated. Applying this distribution to a test function $\varphi(t)$, we see that the leading part of $w(x,x_0;t)$ applied to $\varphi(t)$

is

$$C_j \int \int_0 \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(n)} \frac{1}{\sqrt{t^2 - d_{2k-1}^2 - \sigma}} \frac{d\sigma}{\sqrt{\sigma}} \varphi(t)\, dt$$

$$= C_j \int \int_0^{t^2 - d_{2k-1}^2} \frac{1}{\sqrt{t^2 - d_{2k-1}^2 - \sigma}} \frac{d\sigma}{\sqrt{\sigma}} \Big[\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^n \frac{\partial}{\partial t}\varphi\Big](t)\, dt$$

$$= C_j \int_{d_{k-1}^2}^{\infty} \Big[\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^n \frac{\partial}{\partial t}\varphi\Big](t)\, dt = C_j \varphi^{(n)}(2d_{2k-1}^2) + \cdots$$

(recall that $\int_0^a d\sigma/\sqrt{(a-\sigma)\,\sigma} = \pi$ if $a > 0$), yielding the first relation in (4.18).

Similarly, if $j = 2k$ is even, then $f$ has a local maximum at $i\theta_{2k}$ ($\sigma = d_k^2$). Then the main singularity of $w(x, x_0; t)$ is given by

$$C_j \int^{d_{2k}^2} \frac{\partial}{\partial t} \delta^{(n-1/2)}(t^2 - \sigma) \frac{d\sigma}{(d_{2k}^2 - \sigma)^{1/2}}$$

$$= C_j \int_0 \frac{\partial}{\partial t} \delta^{(n-1/2)}(t^2 - d_{2k}^2 + \sigma) \frac{d\sigma}{\sqrt{\sigma}}$$

$$= C_j \int_0 \frac{\partial}{\partial t}\Big(\frac{1}{2}\frac{\partial}{\partial t}\Big)^{(n)} \frac{1}{\sqrt{t^2 - d_k^2 + \sigma}} \frac{d\sigma}{\sqrt{\sigma}}.$$

Applying this distribution to a test function $\varphi(t)$, we get the leading term

$$C_j \int \int_0 \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(n)} \frac{1}{\sqrt{t^2 - d_{2k}^2 + \sigma}} \frac{d\sigma}{\sqrt{\sigma}} \varphi(t)\, dt$$

$$= C_j \int \Big(\int_0 \frac{d\sigma}{\sqrt{(t^2 - d_k^2 + \sigma)\sigma}}\Big) \Big[\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^n \frac{\partial}{\partial t}\varphi\Big](t)\, dt.$$

Note that $1/\sqrt{(a+\sigma)\sigma}$ makes sense for $a < 0$ as well if $\sigma > -a$ and $\int_{\max(-a,0)}^1 d\sigma/\sqrt{(a+\sigma)\sigma} \sim_{a\to 0} C \ln|a|$. Hence the leading term is equal to

$$C_j \int \ln|t^2 - d_k^2| \Big[\Big[\frac{1}{t}\frac{\partial}{\partial t}\Big]^n \frac{\partial}{\partial t}\varphi\Big](t)\, dt.$$

Set $t - d_k = \rho$. Then $\ln|t^2 - d_k^2| = \ln|\rho| + \ln(2d_k + \rho) \sim \ln|\rho|$ as $\rho \to 0$ and

$$C_j \int \ln|\rho| \Big[\Big[\frac{1}{\rho + d_k}\frac{\partial}{\partial \rho}\Big]^n \frac{\partial}{\partial \rho}\varphi\Big](d_k + \rho)\, d\rho \sim C_j \int \frac{\varphi(d_k + \rho)}{\rho^{n+1}}\, d\rho,$$

where the last integral is understood in the distribution sense. □

*Remark.* The preceding results continue to hold in the anisotropic case (described at the end of Sec. 1.2) if the condition $x \neq 0$ is replaced throughout by $x'' \neq 0$.

The wave kernel may be computed if $x = 0$. We discuss the isotropic case first. Recall that $d_k^2(0, x_0) = k\pi x_0/\alpha$.

THEOREM 4

*For every positive integer $n$ and $\alpha, x_0 > 0$, there exist constants $a_{j,k}, k = 1, 2, \ldots, 1 \leq j \leq n-1$, such that*

$$w(0, x_0, t) = \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} a_{j,k} \frac{\partial}{\partial t} \delta^{(n+j-1/2)}(t^2 - d_k^2). \tag{4.19}$$

*If $n = 1$, then*

$$w(0, x_0, t) = \frac{\sqrt{\pi}}{2\alpha} \sum_{k=1}^{\infty} (-1)^{k+1} k \frac{\partial}{\partial t} \delta^{(1/2)}(t^2 - d_k^2). \tag{4.20}$$

*Proof*

By (4.3),

$$P(0, x_0; 2u) = \frac{1}{(4\pi u)^{n+1}} \int_{-\infty}^{\infty} e^{i x_0 \tau/(2u)} V(\tau)\, d\tau.$$

There exists a positive $\epsilon$ such that for every $k, n$ there exists a function $W_{k,n}(\tau)$ holomorphic in $|\tau - \pi k i/(2\alpha)| < 2\epsilon$, $W_{k,n}(\pi k i/(2\alpha)) \neq 0$, and $V(\tau) = W_{k,n}(\tau)/(\tau - \pi k i/(2\alpha))^n$. Using simple estimates (cf. [1]), it follows that

$$P(0, x_0; 2u) = \frac{1}{(4\pi u)^{n+1}} \sum_{k=0}^{\infty} \int_{|\tau - \pi k i/(2\alpha)| = \epsilon} e^{i x_0 \tau/(2u)} \frac{W_{k,n}(\tau)}{(\tau - \pi k i/(2\alpha))^n}\, d\tau. \tag{4.21}$$

Applying the residue theorem, we see that

$$P(0, x_0; 2u) = \frac{1}{u^{n+1}} \sum_{k=1}^{\infty} e^{-\pi k x_0/(4\alpha u)} \sum_{j=0}^{n-1} \frac{b_{j,k}}{u^j} = \sum_{k=0}^{\infty} e^{-d_k^2/(4u)} \sum_{j=0}^{n-1} \frac{b_{j,k}}{u^{n+1+j}}. \tag{4.22}$$

Application of (4.2) with $a^2 = d_k^2/2$ yields

$$P(0, x_0; 2u) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi u}} \int_0^{\infty} \sum_{j=0}^{n-1} a_{j,k} e^{-t^2/(4u)} \frac{\partial}{\partial t} \delta^{(n+j-1/2)}(t^2 - d_k^2)\, dt, \tag{4.23}$$

and (4.19) follows from Proposition 9 ($P(0, x_0; 2u) = e^{-2u\Delta_H}(0, x_0)$).

If $n = 1$, then (see [1, p. 654])

$$P(0, x_0; 2u) = \frac{1}{16\alpha u^2} \sum_{k=1}^{\infty} (-1)^{k+1} k e^{-d_k^2/(4u)}.$$

Applying (4.2) once again with $a^2 = d_k^2/2$, we get (4.20). □

In the anisotropic case, the function $V(\tau)$ has poles at the points $\pi k i/(2\alpha_j)$, $k = 1, 2, \ldots, \ 1 \leq j \leq n$. We leave the formulation and proof of the anisotropic analog of (4.19) to the diligent reader.

## 5. Wave kernels via the continuation method

Recall that $L$ denotes a second-order positive semidefinite self-adjoint operator. Set

$$W_L(t) = \frac{\sin(L^{1/2} t)}{L^{1/2}}, \tag{5.1}$$

so that $W_L(t)$ is the (operator-valued) solution of the wave equation

$$\frac{\partial^2 W_L}{\partial t^2} = -L W_L \tag{5.2}$$

with the initial conditions

$$W_L(0) = 0, \qquad W_L'(0) = I. \tag{5.3}$$

We find an explicit representation for the kernel of $W_L(t)$ when $-L/2$ is the Heisenberg Laplacian and when $-L$ is a degenerate elliptic operator of the type studied in [2]. Our method involves analytic continuation of the Green function of $L - \partial^2/\partial y^2$, and it is applicable whenever $L$ is positive definite or zero is in the continuous spectrum of $L$ (so that $L^{-1/2}$ is well defined, at least as a closed operator with a dense domain). The Green function $(-L + \partial^2/\partial y^2)^{-1}$ is defined as that (operator-valued) solution $G(y)$ of the equation $(-L + \partial^2/\partial y^2) G(y) = I \cdot \delta(y)$ which tends to zero as $|y| \to \infty$. The main tool is the following proposition, essentially due to Taylor [15].

PROPOSITION 14

(i) *Let $L$ be a positive semidefinite operator. Then*

$$\frac{1}{2}\Big(-L + \frac{\partial^2}{\partial y^2}\Big)(L^{-1/2} e^{-|y| L^{1/2}}) = -I, \tag{5.4}$$

$$\frac{1}{2} L^{-1/2} e^{-|y| L^{-1/2}} = -\Big(-L + \frac{\partial^2}{\partial y^2}\Big)^{-1}, \tag{5.5}$$

$$W_L(t) = \lim_{\epsilon \to 0} \operatorname{Im}\left(2\Big(-L + \frac{\partial^2}{\partial y^2}\Big)^{-1}(\cdot, it + \epsilon)\right). \tag{5.6}$$

(ii) *Let $m$ be a positive integer. Then for every $a > 0$,*

$$\lim_{\epsilon \to 0^+} \operatorname{Im} \frac{1}{(a^2 + (it + \epsilon)^2)^m} = c'_m \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \frac{\delta(t-a)}{t}, \tag{5.7}$$

$$\lim_{\epsilon \to 0^+} \operatorname{Im} \frac{1}{(a^2 + (it + \epsilon)^2)^{m-1/2}} = c''_m \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \frac{H(t-a)}{\sqrt{t^2 - a^2}}, \tag{5.8}$$

*where the limits are understood in the distribution sense (i.e., in $D'_m(\mathbb{R}_+)$ of $t$), and $c'_m$, $c''_m$ are negative constants.*

*Proof*
Note that

$$\frac{\partial}{\partial y} e^{-|y|L^{1/2}} = -L^{1/2} \operatorname{sign}(y) e^{-|y|L^{1/2}}, \tag{5.9}$$

$$\frac{\partial^2}{\partial y^2} e^{-|y|L^{1/2}} = L \operatorname{sign}^2(y) e^{-|y|L^{1/2}} - 2L^{1/2} \delta(y) e^{-|y|L^{1/2}}, \tag{5.10}$$

implying (5.4). The limits of $L^{-(1/2)} e^{-|y|L^{1/2}}$ and of $(L - \partial^2/\partial y^2)^{-1}$ as $|y| \to +\infty$ vanish. The solution of the second-order ordinary differential equation (in $y$) $(-L + \partial^2/\partial y^2)u = -I$ is uniquely determined by the limits $\lim_{y=\pm\infty} u(y)$, and (5.5) follows. The operator-valued function $L^{-(1/2)} e^{-yL^{1/2}}$ is holomorphic in the half-plane $\operatorname{Re} y > 0$, its boundary values at $y = it$ satisfy the wave equation (5.2), its imaginary part is uniformly bounded in compact subsets of the half-plane $\operatorname{Re} y \geq 0$, and the function $L^{-(1/2)} \sin(L^{1/2} t) = -\operatorname{Im}(L^{-(1/2)} e^{-itL^{1/2}})$ also satisfies the initial conditions (5.3), proving (5.6). To prove (5.7), set $L = -\Delta_{2m+1}$. Then the kernel of $(-L + \partial^2/\partial y^2)^{-1}$ is given by

$$\left( \Delta_{2m+1} + \frac{\partial^2}{\partial y^2} \right)^{-1} (x, y; 0, 0) = \frac{c_m}{(|x|^2 + y^2)^m} \quad (c_m < 0) \tag{5.11}$$

(for $x \in \mathbb{R}^{2m+1}$). On the other hand, it is well known that for $t > 0$,

$$W_{-\Delta_{2m+1}}(t)(x, 0) = d_m \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \frac{\delta(t - |x|)}{t} \quad (d_m > 0). \tag{5.12}$$

Formula (5.7) follows from (5.11), (5.12), and (5.6). Similarly, to prove (5.8), set $L = -\Delta_{2m}$, and recall the formulas (for $x \in \mathbb{R}^{2m}$)

$$\left( \Delta_{2m} + \frac{\partial^2}{\partial y^2} \right) (x, y; 0, 0) = \frac{\tilde{c}_m}{(x^2 + y^2)^{(2m-1)/2}}, \tag{5.13}$$

and (for $t > 0$)

$$W_{-\Delta_m}(t)(x, 0) = \tilde{d}_m \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \frac{H(t - |x|)}{\sqrt{t^2 - x^2}}. \tag{5.14}$$

□

Recall (see [3], [2]) that the Green kernels of operators such as the Heisenberg Laplacian, the Heisenberg Laplacian $+\partial^2/\partial y^2$, and certain degenerate elliptic second-order operators are known to be of the form

$$\int_R \frac{V(z)}{f(z)^q}\,dz,$$

where $V$ is an analytic function of $z$ (only), whereas $f$ is a complex-valued action (of the type introduced and discussed in Sec. 1.2) associated with the symbol of $L$ (or $L - \partial^2/\partial y^2$). The integration path may be deformed to a contour on which $f$ is real and (5.7), (5.8) may be applied.

*5.1. The Heisenberg Laplacian*

Here we consider for simplicity only the isotropic case. Thus, given a positive integer $n$ and a positive constant $\alpha$, the Heisenberg Laplacian $\Delta_H$ is defined on $\mathbb{R}^{2n+1} = \{(x_1, \ldots, x_{2n}, t)\}$ by

$$\Delta_H = \frac{1}{2}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j-1}} + 2\alpha x_{2j}\frac{\partial}{\partial y}\Big)^2 + \frac{1}{2}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_{2j}} - 2\alpha x_{2j-1}\frac{\partial}{\partial x_0}\Big)^2. \quad (5.15)$$

R. Beals and P. Greiner [3] computed the Green kernel of $2\Delta_H + \partial^2/\partial y^2$ with a pole at the origin (of $\mathbb{R}^{2n+2}$):

$$\begin{aligned} -\Big(2\Delta_H + \frac{\partial^2}{\partial y^2}\Big)^{-1}&(x_1, \ldots, x_{2n}, x_0, y) \\ &= c_n \int_{-\infty}^{\infty}\Big(\frac{2\alpha\tau}{\sinh(2\alpha\tau)}\Big)^n \frac{d\tau}{\big(|x|^2\alpha\tau\coth(2\alpha\tau) + y^2/2 - ix_0\tau\big)^{n+1/2}}, \end{aligned} \quad (5.16)$$

where $|x|^2 = \sum_{j=1}^{2n} x_j^2$ and $c_n$ is a positive constant. We utilize this computation in order to derive formulas for the wave kernel $W_H(t)(x, x_0; 0, 0)$, where $H$ denotes the operator $-2\Delta_H$. Setting

$$V(\tau) = \Big(\frac{2\alpha\tau}{\sinh(2\alpha\tau)}\Big)^n, \qquad f(x, x_0; \tau) = |x|^2\alpha\tau\coth(2\alpha\tau) - ix_0\tau, \quad (5.17)$$

we may rewrite (5.16) with a different $c_n$ as

$$-\Big(2\Delta_H + \frac{\partial^2}{\partial y^2}\Big)^{-1}(x_1, \ldots, x_n, x_0, y) = c_n\int_{-\infty}^{\infty}\frac{V(\tau)}{[2f(\tau) + y^2]^{n+1/2}}. \quad (5.18)$$

Note that $f(\tau) = f(x, x_0; \tau) = f(|x|, x_0, \tau; 2\alpha)$. Hence the properties of the curve $\Gamma_{x,x_0}$ (the set where $f$ is real) are as described in Proposition 3.

*Remark.* If $x_0 = 0$, then $\Gamma_{x,x_0}$ coincides with the real axis.

It is well known that one may perform the integrations in (5.16) or (5.18) on the deformed contour $\Gamma_{x,x_0}$. (Note that $\operatorname{Re} f > 0$ between $\mathbb{R}$ and $\Gamma_{x,x_0}$, so that the fractional power is single valued.) The following is a consequence of Proposition 14.

THEOREM 5
*If $x \neq 0$, then for $t > 0$,*

$$W_H(t)(x, x_0; 0, 0) = c_n \int_{\Gamma_{x,x_0}} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^n \frac{H\big(t - \sqrt{2f(x, x_0; \tau)}\big)V(\tau)\,d\tau}{\sqrt{t^2 - 2f(x, x_0; \tau)}}. \tag{5.19}$$

*Proof*
Note that $V(\tau)$ and $\tau\coth(2\alpha\tau)$ are both even functions and are real if $\tau \in \mathbb{R}$. Hence $V(-\overline{\tau}) = \overline{V(\tau)}$ and $f(-\overline{\tau}) = \overline{f(\tau)}$ ($x_0$ is real!). Moreover, the map $\tau \to -\overline{\tau}$ maps $\Gamma_{x,x_0}$ onto itself, with the orientation reversed. Hence

$$\begin{aligned}
\operatorname{Im} &\int_\Gamma \frac{V(\tau)\,d\tau}{(2f + (it+\epsilon)^2)^{n+1/2}} \\
&= \frac{1}{2i}\Big(\int_\Gamma \frac{V(\tau)\,d\tau}{(2f(\tau) + (it+\epsilon)^2)^{n+1/2}} - \int_\Gamma \frac{\overline{V(\tau)}\,d\overline{\tau}}{(2f(\tau) + (-it+\epsilon)^2)^{n+1/2}}\Big) \\
&= \frac{1}{2i}\Big(\int_\Gamma \frac{V(\tau)\,d\tau}{(2f(\tau) + (it+\epsilon)^2)^{n+1/2}} + \int_\Gamma \frac{V(-\overline{\tau})d(-\overline{\tau})}{(2f(-\overline{\tau}) + (-it+\epsilon)^2)^{n+1/2}}\Big) \\
&\quad \text{(we use the fact that } f(\tau) = \overline{f(\tau)} \text{ on } \Gamma) \\
&= \frac{1}{2i}\Big(\int_\Gamma \frac{V(\tau)\,d\tau}{(2f(\tau) + (it+\epsilon)^2)^{n+1/2}} - \int_\Gamma \frac{V(\tau)\,d\tau}{(2f(\tau) + (-it+\epsilon)^2)^{n+1/2}}\Big) \\
&= \int_\Gamma V(\tau)\operatorname{Im}\Big[\frac{1}{(2f(\tau) + (it+\epsilon)^2)^{n+1/2}}\Big]\,d\tau.
\end{aligned} \tag{5.20}$$

Application of (5.8) (with $m = n + 1$) to (5.18) proves (5.19). □

*Remark.* As in Section 4, (5.19) may be written as involving only integrals of a real variable.

If $x = 0$, $x_0 \neq 0$, then we may assume $x_0 > 0$. The integrals in (5.16) and (5.18) may be computed using the residue theorem: the function $V(\tau)$ has a pole of order $n$ at $\tau = k\pi i/(2\alpha)$, $k = \pm 1, \pm 2, \ldots$, and we may deform the path of integration so that only the poles with $\operatorname{Im}(\tau) > 0$ matter. For $y \neq 0$ there is no singularity at $\tau = 0$. For every $\epsilon > 0$ sufficiently small, we have

$$-\Big(2\Delta_H + \frac{\partial^2}{\partial y^2}\Big)^{-1}(0, x_0, y) = c_n \sum_{k=1}^{\infty} \int_{|\tau - k\pi i/(2\alpha)| = \epsilon} \frac{V(\tau)\,d\tau}{(-4ix_0\tau + y^2)^{n+1/2}}. \tag{5.21}$$

It is well known that there exist infinitely many geodesics joining $(0, x_0)$ to the origin with lengths

$$d_k = \sqrt{\frac{k\pi x_0}{\alpha}}, \quad k = 1, 2, \ldots . \tag{5.22}$$

THEOREM 6

*If $x = 0$, $x_0 > 0$, then for $t > 0$,*

$$W_H(t)(0, x_0; 0, 0) = \sum_{k=1}^{\infty} \sum_{j=0}^{n-1} c_{n,k,j} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{n+j} \frac{H(t - \sqrt{k\pi x_0/\alpha})}{\sqrt{t^2 - k\pi x_0/\alpha}}, \tag{5.23}$$

*where $c_{n,k,j}$ are functions of $x_0$.*

(Note that for fixed $t$ the sum in (5.22) is finite.)

*Proof*

By (5.17), we have for each $k = 1, 2, \ldots$ that

$$V(\tau) = \frac{W_k(\tau)}{(\tau - k\pi i/(2\alpha))^n},$$

where $W_k(\tau)$ is regular near $\tau = k\pi i/(2\alpha)$. Hence

$$\begin{aligned}
&\int_{|\tau - k\pi i/(2\alpha)| = \epsilon} V(\tau) \frac{d\tau}{(-2ix_0\tau + y^2)^{n+1/2}} \\
&\quad = \frac{2\pi i}{(n-1)!} \frac{\partial^{n-1}}{\partial \tau^{n-1}} \left( \frac{(W_k(\tau)}{(-2ix_0\tau + y^2)^{n+1/2}} \right)_{\tau = k\pi i/2} \\
&\quad = \frac{2\pi i}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} W_k^{(n-1-j)} \\
&\qquad \cdot \left( \frac{k\pi i}{2\alpha} \right) \frac{(n+1/2) \cdots (n+1/2+j-1)(2ix_0)^j}{[k\pi/\alpha + y^2]^{n+1/2+j}}.
\end{aligned} \tag{5.24}$$

Inserting (5.24) in (5.21) and using (5.8), we get (5.23). □

*Remark.* Note that for every nonnegative integer $m$ and a positive number $c$,

$$\delta^{m-1}(t^2 - c^2) = \left( \frac{1}{2t} \frac{\partial}{\partial t} \right)^m \frac{H(t-c)}{\sqrt{t^2 - c^2}}. \tag{5.25}$$

Moreover, (5.1) implies that $(\partial/\partial t) W_H(t) = \cos(\sqrt{-2\Delta_H}\, t)$. Hence Theorems 5 and 6 are equivalent to Theorems 2 and 4, respectively.

*5.2. Degenerate elliptic operators*

In this subsection we compute $W_L$ when $L$ is a degenerate elliptic operator of the type considered in [2]. For simplicity we consider here a subclass, consisting of operators of the form

$$L = -\frac{1}{2}\left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + (Bx_1, Bx_1)\frac{\partial^2}{\partial x_0^2}\right), \tag{5.26}$$

where $x_i \in V_i$ $(i = 1, 2)$, $V_1$, $V_2$ are real Euclidean vector spaces, and $\partial/\partial x_i$ denotes the gradient in $V_i$ $(i = 1, 2)$, $(\ ,\ )$ denotes the inner product in $V_1$, and $B$ is a positive definite matrix on $V_1$. It was proved in [2] that $-L$ has a fundamental solution of the form

$$G(x_1, x_2; x_0, x_1^0, x_2^0, x_0^0) = -c\int_{-\infty}^{\infty} \frac{V(\tau)\, d\tau}{\tilde{f}(x_1, x_1^0, x_2 - x_2^0, x_0 - x_0^0, \tau)^q}, \tag{5.27}$$

where

$$V(\tau) = \det\left(\frac{B\tau}{\sinh(B\tau)}\right)^{1/2}, \tag{5.28}$$

$$\begin{aligned}\tilde{f}(x_1, x_1^0, x_2, x_0, \tau) = -ix_0\tau + \frac{\tau}{2}\big(B\coth(B\tau)(x_1 - x_1^0), x_1 - x_1^0\big)\\ + \tau\left(B\tanh\left(\frac{B\tau}{2}\right)x_1, x_1^0\right) + \frac{|x_2 - x_2^0|^2}{2},\end{aligned} \tag{5.29}$$

$$q = \frac{\dim V_1 + \dim V_2}{2}, \tag{5.30}$$

and $c$ is a positive constant. We consider here only the case where $x_1^0 = 0$ (without loss of generality, we may assume $x_2^0 = t^0 = 0$), leaving the case where $x_1^0 \neq 0$ to the future.

*Examples.* Special cases of (5.26) are the Grušin operator $\partial^2/\partial x^2 + x^2(\partial^2/\partial x_0^2)$ and the Baouendi-Goulaouic operator $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + (x_1(\partial/\partial x_0))^2$.

Setting $\tilde{V}_2 = V_2 \times R$, we see from (5.27) that a fundamental solution of $2L - \partial^2/\partial y^2$ with a pole at the origin is given by the following (special case) of (5.27):

$$G(x_1, x_2, x_0, y; 0, 0, 0, 0) = c\int_{-\infty}^{\infty} \frac{V(\tau)\, d\tau}{(\tilde{f}(x_1, 0, x_2, x_0, \tau) + y^2/2)^{q+1/2}}. \tag{5.31}$$

Set $n = \dim V_1$, let $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ denote the eigenvalues of $B$ (repeated according to their multiplicities), let $(x_1)_j$ denote the component of $x_1$ in the $j$th eigendirection, and let $x_1''$ denote the projection of $x$ onto the eigenspace belonging to the largest eigenvalue $a_n$. Then

$$\tilde{f}(x_1, 0, x_2, x_0, \tau) = -ix_0\tau + \frac{\tau}{2}\sum_{j=1}^{n} a_j(x_1)_j^2 \coth(a_j\tau) + \frac{|x_2|^2}{2}. \tag{5.32}$$

Thus $\tilde{f}(x_1, 0, x_2, x_0, \tau)$ is the sum of a positive number and a function of the form (1.26).

Note that $G$ is arbitrarily small if $|(x_1, x_2, x_0, y)|$ is sufficiently large. Hence the right-hand side of (5.31) represents the kernel of $(-L + \partial^2/\partial y^2)^{-1}$. Observe that Proposition 3 holds if $x_1'' \neq 0,\ x_0 \geq 0$ (and $L$ is the (degenerate) Laplace-Beltrami operator associated with the distance $d$ on $V_1 \times V_2$). Hence

$$-\Big(-L + \frac{\partial^2}{\partial y^2}\Big)^{-1}(x_1, x_2, x_0, y; 0, 0, 0, 0) = c\int_{\Gamma_{x_1,t}} \frac{V(\tau)\, d\tau}{(\tilde{f}(x_1, 0, x_2, x_0, \tau) + y^2/2)^{q+1/2}}. \tag{5.33}$$

The next theorem follows from (5.33) and Proposition 14 in the same manner as in the proof of Theorem 5. Note, however, the distinction between the case where $\dim V_1 + \dim V_2$ is even and where it is odd. For fixed $x_1$, $x_2$, and $x_0$, let $h_k(\sigma)$ denote the inverse of the function (of $\tau$) $\sqrt{2\tilde{f}(x_1, x_2, x_0, \tau)}$ restricted to that part of $\Gamma_{x_1,x_0}$ ($\Gamma$ does not depend on $x_2$) lying between $i\Theta_{2k-1}$ and $i\Theta_{2k}$ and in the right half-plane.

THEOREM 7

*Let $x_1'' \neq 0$.*

(i) *If* $\dim V_1 + \dim V_2 = 2p$, *where* $p$ *is a positive integer, then*

$$W_L(t)(x_1, x_2, x_0; 0, 0, 0) = c' \int_{\Gamma_{x,x_0}} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^p \frac{H\big(t - \sqrt{2\tilde{f}(x_1, x_2, 0, x_0, \tau)}\big)}{\sqrt{t^2 - 2\tilde{f}(x_1, x_2, 0, x_0, \tau)}} V(\tau)\, d\tau. \tag{5.34}$$

(ii) *If* $\dim V_1 + \dim V_2 = 2p + 1$, *where* $p$ *is a nonnegative integer, then*

$$W_L(t)(x_1, x_2, x_0; 0, 0, 0) = c''\frac{1}{t}\Big(-\frac{1}{t}\frac{\partial}{\partial t}\Big)^p \operatorname{Re} \frac{V(h_k(t))}{h_k'(t)}. \tag{5.35}$$

*(Here $d_{2k-1} < t \leq d_{2k}$ or $d_N < t$.)*

*Proof*

We apply Proposition 14 to

$$\Big(-L+\frac{\partial^2}{\partial y^2}\Big)^{-1}(x_1,x_2,x_0,y;0,0,0,0)$$
$$= -\tilde{c}\int_{\Gamma_{x,x_0}}\frac{V(\tau)\,d\tau}{(2\tilde{f}(x_1,0,x_2,x_0,\tau)+y^2)^{q+1/2}}\quad(\tilde{c}>0). \tag{5.36}$$

If $\dim V_1+\dim V_2=2p$, where $p$ is a positive integer, then $q=p$ and (5.8) is applicable, with $m=p+1$, and (5.34) follows as in the proof of Theorem 5. If $\dim V_1+\dim V_2=2p+1$, where $p$ is a nonnegative integer, then $q=p+1/2$, so that $q+1/2=p+1$ and (5.7) is applicable to the integrand, with $m=p+1$. Hence

$$W_L(t)(x_1,x_2,x_0;0,0,0)=\hat{c}\int_{\Gamma_{x,x_0}}V(\tau)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^p\frac{\delta\big(t-\sqrt{2\tilde{f}(\cdot,\tau)}\big)}{t}\,d\tau. \tag{5.37}$$

Now let $t\in(d_{2k-1},d_{2k})$ or $t>d_N$. Then $h_k'(\sigma)$ is nonzero in an open interval $(d_{2k-1},d_{2k})$ (or $d_N,\infty$), and we may change the variable of integration from $\tau$ to $\sigma$, integrating over an interval rather than over an arc of $\Gamma_{x,x_0}$. Recall that we also have to take into account the branch of $\Gamma_{x,x_0}$ lying in the left half-plane, where $\tau=-\overline{h_k(\sigma)}$ and the orientation is reversed. Hence

$$W_L(t)(x_1,x_2,x_0;0,0,0)=\hat{c}\int\frac{V(h_k(\sigma))}{h_k'(\sigma)}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^p\frac{\delta(t-\sigma)}{t}\,d\sigma$$
$$+\hat{c}\int\frac{\overline{V(h_k(\sigma))}}{\hat{h}_k'(\sigma)}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^p\frac{\delta(t-\sigma)}{t}\,d\sigma, \tag{5.38}$$

the integration is performed over an interval containing $t$ (if $t<d_1$, then $W_L(t)(x_1,x_2,x_0;0,0,0)=0$), and (5.35) follows. □

*Remark.* For both the Grušin operator and the Baouendi-Goulaouic operator, $n=1$ and $x_1''=x_1$. More generally, if $B$ is a scalar operator, then the same equality $x_1''=x_1$ holds. The analysis of the case where $x_1\neq 0$, $x_1''=0$ is very complicated (cf. [1]) and is not attempted here. The case where $x_1=0$, $x_0=0$ is relatively simple, as the following theorem shows.

THEOREM 8
*Let $x_1=0$, $x_0=0$. If $\dim V_1+\dim V_2=2p$, where $p$ is a positive integer, then*

$$W_L(t)(0,x_2,0;0,0,0)=c\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^p\frac{H(t-|x_2|)}{\sqrt{t^2-x_2^2}}. \tag{5.39}$$

*If* $\dim V_1 + \dim V_2 = 2p + 1$, *where* $p$ *is a nonnegative integer, then*

$$W_L(t)(0, x_2, 0; 0, 0, 0) = c\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^p \frac{\delta(t - |x_2|)}{t}. \tag{5.40}$$

*Proof*
It follows from (5.29) and (5.33) that

$$-\Big(-L+\frac{\partial^2}{\partial y^2}\Big)^{-1}(0,x_2,0,y;0,0,0,0) = c\int_{-\infty}^{\infty}\frac{V(\tau)\,d\tau}{((x_2^2+y^2)/2)^{q+1/2}} = \frac{c_1}{(x_2^2+y^2)^{q+1/2}}. \tag{5.41}$$

If $\dim V_1 + \dim V_2 = 2p$, where $p$ is a positive integer, then $q = p$ and we may apply (5.8) to (5.41) (with $m = p+1$), obtaining (5.39). If $\dim V_1 + \dim V_2 = 2p+1$, where $p$ is a nonnegative integer, then $q = p + 1/2$ and we may apply (5.7) to (5.41) (with $m = p+1$), obtaining (5.40). □

The case where $x_1 = 0$, $x_0 \neq 0$ is considerably more complicated. The results depend on the parity of the multiplicities of the eigenvalues of $B$. We treat here only the case where $B$ is scalar, and we distinguish between even and odd $n$. The general case, where each eigenvalue has either even or odd multiplicity, is essentially a combination of the scalar cases.

THEOREM 9
*Let $x_1 = 0, x_0 > 0$. Assume that $B = aI$ and $\dim V_1 = n = 2n'$, where $n'$ is a positive integer. If $\dim V_1 + \dim V_2 = 2p$, where $p$ is a positive integer, then*

$$W_L(t)(0,x_2,x_0;0,0,0) = \sum_{k=1}^{\infty}\sum_{j=0}^{n'-1} c_{k,j}\cdot\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{p+j}\frac{H\big(t-\sqrt{2k\pi x_0/a + x_2^2}\big)}{\sqrt{t^2-2k\pi x_0/a - x_2^2}}. \tag{5.42}$$

*If $\dim V_1 + \dim V_2 = 2p+1$, where $p$ is a nonnegative integer, then*

$$W_L(t)(0,x_2,x_0;0,0,0) = \sum_{k=1}^{\infty}\sum_{j=0}^{n'-1} c'_{k,j}\cdot\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{p+j}\frac{\delta\big(t-\sqrt{2k\pi x_0/a + x_2^2}\big)}{t}. \tag{5.43}$$

*The constants $c_{k,j}$ and $c'_{k,j}$ in (5.42) and (5.43) also depend on $a$ and on $n$.*

Note that as in Theorem 6, the sums in (5.42) and in (5.43) are finite for $t$ fixed.

*Proof*
Let $x_0 > 0$ (the case where $x_0 < 0$ is treated similarly). Then $\tilde f(0,0,x_2,x_0,\tau) = -ix_0\tau + x_2^2/2$ is real for $\tau$ in the imaginary axis and is positive if $\operatorname{Im}\tau > 0$. The

function $V(\tau)$ defined in (5.28) is given by

$$V(\tau) = \Big(\frac{a\tau}{\sinh(a\tau)}\Big)^{n'},$$

and it has poles of order $n'$ at $\tau = k\pi i/a$, $k = \pm 1, \pm 2, \ldots$. As in the discussion leading to (5.18), we get from (5.36) the representation (valid for $y \neq 0$ and $\epsilon > 0$ sufficiently small)

$$\begin{aligned} -\Big(-L + \frac{\partial^2}{\partial y^2}\Big)^{-1}&(0, x_2, x_0, y; 0, 0, 0, 0) \\ &= c\sum_{k=1}^{\infty}\int_{|\tau - k\pi i/a| = \epsilon} \frac{V(\tau)\,d\tau}{[-2ix_0\tau + x_2^2 + y^2]^{q+1/2}} \\ &= c\sum_{k=1}^{\infty}\int_{|\tau - k\pi i/a| = \epsilon} \frac{V_k(\tau)}{(\tau - k\pi i/a)^{n'}}\frac{d\tau}{[-2ix_0\tau + x_2^2 + y^2]^{q+1/2}}, \end{aligned} \tag{5.44}$$

where $V_k$ is regular near $\tau = k\pi i/a$. But

$$\begin{aligned} \int_{|\tau - k\pi i/a| = \epsilon} \frac{V_k(\tau)}{(\tau - k\pi i/a)^{n'}}\frac{d\tau}{[-2ix_0\tau + x_2^2 + y^2]^{q+1/2}} = \frac{2\pi i}{(n'-1)!}\sum_{j=1}^{n'-1}\binom{n'-1}{j} \\ \cdot V_k^{(n'-j-1)}\Big(\frac{k\pi i}{a}\Big)\frac{(q+1/2)(q+1/2+1)\cdots(q+1/2+j-1)(2ix_0)^j}{|2k\pi x_0/a + x_2^2 + y^2|^{q+1/2+j}}. \end{aligned} \tag{5.45}$$

If $\dim V_1 + \dim V_2 = 2p$, where $p$ is a positive integer, then $q+1/2+j = p+j+1/2$, and we may apply (5.8) ($m = p+j+1$) and substitute (5.45) in (5.44) to obtain (5.42). If $\dim V_1 + \dim V_2 = 2p+1$, where $p$ is a nonnegative integer, then $q+1/2+j = p+j+1$; we may apply (5.7) ($m = p+j+1$), and, substituting (5.45) in (5.44), we get (5.43). □

If $n$ is odd, then $V(\tau)$ is no longer single-valued. First we have to determine the boundary values of $V(\tau)$ on the imaginary axis.

PROPOSITION 15

*Let $n$ be an odd integer, $n = 2n'+1$, and let $a$ denote a positive number. Then there exist temperate (one-dimensional) distributions $E_+$, $E_-$ such that*

$$\begin{aligned} E_+(s) &= \lim_{\epsilon\to 0_+}\Big(\frac{a(is+\epsilon)}{\sinh(a(is+\epsilon))}\Big)^{n/2}, \\ E_-(s) &= \lim_{\epsilon\to 0_+}\Big(\frac{a(is-\epsilon)}{\sinh(a(is-\epsilon))}\Big)^{n/2}. \end{aligned} \tag{5.46}$$

*Moreover,* $\operatorname{singsupp}(E_+) = \operatorname{singsupp}(E_-) = \{k\pi/a,\ k = \pm1, \pm2, \ldots\}$. *Outside of the singular support, the functions* $E_+(s)$ *and* $E_-(s)$ *coincide with each other and are real if* $|s| < \pi/a$ *or* $2k(\pi/a) < |s| < (2k+1)(\pi/a)$, $k = \pm1, \pm2, \ldots,$ *and are purely imaginary with* $E_+(s) = -E_-(s)$ *if* $(2k-1)(\pi/a) < |s| < 2k(\pi/a),\ k = \pm1, \pm2, \ldots,$ *and in this case,*

$$E_+(s) = (-1)^{k+1} i \Big(\frac{|as|}{|\sin(as)|}\Big)^{n/2}. \tag{5.47}$$

*Both* $E_+$ *and* $E_-$ *are* $n'$*th derivatives of a locally integrable function whose singularities are of the form* $1/(s - k(\pi/a))^{1/2}$.

*Proof*
The function $\sinh(z)/z$ vanishes if and only if $z = k\pi i,\ k = \pm1, \pm2, \ldots$. Moreover, $\sinh(z)/z$ is positive if $z$ is real. Hence the function $(ai\tau/\sinh(ai\tau))^{n/2}$ is well defined in the half-planes $\operatorname{Im}(\tau) > 0$ and $\operatorname{Im}(\tau) < 0$ with $(ai\tau/\sinh(ai\tau))^{n/2}$ positive if $\tau$ is real (except at the poles). The zeros of $\sinh(iz)$ at $z = k\pi/a$ are simple, so that

$$\Big(\frac{aiz}{\sinh(aiz)}\Big)^{n/2} = O\Big(\frac{1}{|z - k\pi/a|}\Big)^{n/2}\Big) \tag{5.48}$$

for $\operatorname{Im}(z) \neq o$, $z$ near $k\pi/a$. Hence the limits in (5.46) exist as distributions (see [7, p. 63]). If $s$ is real, then the function $\sin(as)/(as)$ is positive for $|s| < \pi/a$ or $2k(\pi/a) < |s| < (2k+1)(\pi/a)$, $k = \pm1, \pm2, \ldots,$ and negative elsewhere. Hence the functions $(aiz/\sinh(aiz))^{1/2}$ and $(aiz/\sinh(aiz))^{n/2}$ have branch points at $z = k\pi/a,\ k = \pm1, \pm2, \ldots,$ and cuts in the intervals where $\sin(as)/(as)$ is negative. The positive choice of the square root near $z = 0$ (in particular, for $z = is$ with $|s| < \pi/a$) implies that $E_+(s)$ and $E_-(s)$ must be negative for $2\pi < s < 3\pi$ and then positive for $3\pi < s < 5\pi$, and so on. Hence the alternating signs for the imaginary case (5.47). Recall that

$$\frac{z}{\sin(z)} = (-1)^k \Big(\frac{k\pi}{z - k\pi}\Big) + h_k(z), \tag{5.49}$$

where $h_k(z)$ is holomorphic near $k\pi$. Hence the function (having a branch point at $k\pi$)

$$\Big(\frac{ai\tau}{\sinh(ai\tau)}\Big)^{n/2} = \Big(\frac{ai\tau}{\sinh(ai\tau)}\Big)^{n'} \cdot \Big(\frac{ai\tau}{\sinh(ai\tau)}\Big)^{1/2}$$

can be expanded in an algebraic Laurent series in powers of $((\tau - k\pi)/a)^{1/2}$, starting with $-n = -(2n' + 1)$. This expansion may be integrated $n'$ times. □

Set now $E = E_+ - E_-$. Then $E(s) = 0$ if $|s| < \pi/a$ or $2k(\pi/a) < |s| < (2k + 1)(\pi/a),\ k = \pm1, \pm2, \ldots,$ and $E(s)$ is purely imaginary if $(2k-1)(\pi/a) < |s| <$

$2k(\pi/a)$, $k = \pm 1, \pm 2, \dots$; then

$$E(s) = (-1)^{k+1} 2i \Big( \frac{|as|}{|\sin(as)|} \Big)^{n/2}. \tag{5.50}$$

THEOREM 10

*Let $x_1 = 0$, $x_0 > 0$. Assume that $B = aI$ and $\dim V_1 = n = 2n' + 1$, where $n'$ is a nonnegative integer. If $\dim V_1 + \dim V_2 = 2p$, where $p$ is a positive integer, then*

$$W_L(t)(0, x_2, x_0; 0, 0, 0) = c \int_0^\infty E(s) \Big( \frac{1}{t} \frac{\partial}{\partial t} \Big)^p \frac{H\big(t - \sqrt{2sx_0 + x_2^2}\big)}{\sqrt{t^2 - 2sx_0 + x_2^2}} \, ds. \tag{5.51}$$

*If $\dim V_1 + \dim V_2 = 2p + 1$, where $p$ is a nonnegative integer, then*

$$W_L(t)(0, x_2, x_0; 0, 0, 0) = c \int_0^\infty E(s) \Big( \frac{1}{t} \frac{\partial}{\partial t} \Big)^p \frac{\delta\big(t - \sqrt{2sx_0 + x_2^2}\big)}{t} \, ds, \tag{5.52}$$

*and there exist real constants $c_k$ such that the leading singularity of $W_L(t)(0, x_2, x_0; 0, 0, 0)$ is given by*

$$\sum_{k=0}^\infty \frac{c_{2k+1} H\big(t - \sqrt{(2k+1)\pi/(ax_0)} - \pi/(2ax_0)\big)}{\big(t - \sqrt{(2k+1)\pi/(ax_0)}\big)_+^{n/2+p}} + \sum_{k=1}^\infty \frac{c_{2k} H\big(t - \sqrt{(2k)\pi/(ax_0)} - \pi/(2ax_0)\big)}{\big(t - \sqrt{(2k)\pi/(ax_0)}\big)_-^{n/2+p}}. \tag{5.53}$$

(The distributions $1/s_+^{n/2+p}$ and $1/s_-^{n/2+p}$ are defined, e.g., in [7, pp. 68–71].) Note that the integrals in (5.51) and in (5.52) are actually performed over compact intervals, and note that the sum in (5.53) is finite if $t$ is fixed.

*Proof*

Let $y \neq 0$. Recall that by (5.33),

$$-\Big( - L + \frac{\partial^2}{\partial y^2} \Big)^{-1} (x_1, x_2, x_0, y; 0, 0, 0, 0) = c \int_{\Gamma_{x_1,t}} \frac{(a\tau/\sinh(a\tau))^{n/2} \, d\tau}{(-2ix_0\tau + x_2^2 + y^2)^{q+1/2}}, \tag{5.54}$$

where $c$ is a positive constant, $V(\tau) = (a\tau/\sinh(a\tau))^{n/2}$, and $q = p + (1/2)$. An estimate similar to the one used in the proof of Proposition 12 implies that for every $\epsilon > 0$, $R > 0$, the integral in (5.54) may be extended over the contour composed of $\{-\infty < \tau_1 \leq -\epsilon, \tau_2 = R\}$, $\{\tau_1 = -\epsilon, 0 \leq \tau_2 \leq R\}$, $\{-\epsilon \leq \tau \leq \epsilon, \tau_2 = 0\}$, $\{\tau_1 = \epsilon, 0 \leq \tau_2 \leq R\}$, and $\{\epsilon \leq \tau_1 < \infty, \tau_2 = R\}$. For every positive integer $j$, set

$R_j = (j+1/2)\pi$. Then $\int_\epsilon^\infty |V(iR_j+\sigma)|\,d\sigma$, $\int_{-\epsilon}^{-\infty} |V(iR_j+\sigma)|\,d\sigma$ are $O(R_j^{n/2})$ while $(-2ix_0\tau + x_2^2 + y^2)^{-q-1/2} = O(R_j^{-q-1/2})$ for $\operatorname{Im}\tau = R_j$, $j\to\infty$. Hence

$$\begin{aligned}
-(-L+\frac{\partial^2}{\partial y^2})^{-1}&(x_1,x_2,x_0,y;0,0,0,0)\\
&= c\lim_{j\to\infty}\Big(\int_{R_j}^0 \frac{V(-\epsilon+is)\,d(is)}{[-2ix_0(-\epsilon+is)+x_2^2+y^2]^{q+1/2}}\\
&\qquad + \int_{-\epsilon}^{\epsilon} \frac{V(-\epsilon+is)\,d(is)}{[-2ix_0(-\epsilon+is)+x_2^2+y^2]^{q+1/2}}\\
&\qquad + \int_0^{R_j} \frac{V(\epsilon+is)\,d(is)}{[-2ix_0(\epsilon+is)+x_2^2+y^2]^{q+1/2}}\\
&= c\Big(\int_\infty^0 \frac{V(-\epsilon+is)\,d(is)}{[-2ix_0(-\epsilon+is)+x_2^2+y^2]^{q+1/2}}\\
&\qquad + \int_{-\epsilon}^{\epsilon} \frac{V(-\epsilon+is)\,d(is)}{[-2ix_0(-\epsilon+is)+x_2^2+y^2]^{q+1/2}}\\
&\qquad + \int_0^{\infty} \frac{V(\epsilon+is)\,d(is)}{[-2ix_0(\epsilon+is)+x_2^2+y^2]^{q+1/2}}\Big)\Big).
\end{aligned}$$

(As a side remark, note that the integrals do not converge absolutely if $\dim V_2 \le 1$. The following argument is valid without absolute convergence; one could also introduce additional artificial $x_2$-variables and apply a method of descent.) The regularity of the integrands at $\tau = 0$ (note that $y \neq 0$) implies that we may let $\epsilon$ tend to zero and obtain the representation

$$-\Big(-L+\frac{\partial^2}{\partial y^2}\Big)^{-1}(x_1,x_2,x_0,y;0,0,0,0) = ci\int_0^\infty \frac{E(s)\,ds}{[y^2+2x_0s+x_2^2]^{q+1/2}}. \tag{5.55}$$

Recall that $E(s)$ is purely imaginary. Hence

$$\begin{aligned}
\operatorname{Im}\Big(-L+\frac{\partial^2}{\partial y^2}\Big)^{-1}&(x_1,x_2,x_0,y;0,0,0,0)\\
&= ci\int_0^\infty E(s)\operatorname{Im}\Big(\frac{1}{[y^2+2x_0s+x_2^2]^{q+1/2}}\Big)\,ds.
\end{aligned}\tag{5.56}$$

Putting $y = it+\epsilon$, where $t>0$, and letting $\epsilon$ tend to zero from the right, we get (at least formally) (5.51) and (5.52) from (5.56), (5.7), and (5.8). The intersection of the wave front set $WF(E(s))$ with the set $WF'(K)_{\mathbb{R}^1}$, where $K$ is one of the right-hand sides of (5.7) or (5.8), is empty. By [7, Th. 8.2.13], the “integrals” in (5.51) and (5.52) make sense and are the limits of the integrals in (5.56).

Formula (5.53) follows from the observation that near the branch points $k\pi/a$, the leading singularity of $E_+(E_-)$ is given by $(1/(s - k\pi/a - i0)^{n/2})$ $(1/(s - k\pi/a + i0)^{n/2})$. □

For the Grušin operator $n = 1$, $\dim V_2 = 0$, so that $p = 0$, and we get from (5.52) the simple expression

$$W_L(t)(0, x_0; 0, 0) = c' \sum_{k=1}^{\infty} (-1)^k \int_{(2k-1)\pi}^{2k\pi} \Big( \frac{|s|}{|\sin(s)|} \Big)^{1/2} \frac{\delta\big(t - \sqrt{2x_0 s}\big)}{t} \, ds$$
$$= (-1)^j c' \Big( \frac{t^2/(2x_0)}{-\sin(t^2/(2x_0))} \Big)^{1/2}, \tag{5.57}$$

where $j$ is a positive integer determined by $(2j - 1)\pi < t^2/(2x_0) < 2j\pi$, and $W_L(t) = 0$ if no such $j$ exists.

## 6. Directions for further studies

We suggest here a certain number of open problems connected to this paper. The first question is how to extend the methods to find the wave kernel on a Riemannian manifold $(M_n, g)$ for a degenerate operator $L = \sum_1^m Ł_i \, Ł_i^*$, where $Ł_i$ are $m$ vector fields such that their Lie brackets generate the tangent space at each point of the manifold. In particular, we are interested in the role played by the geometry.

Another possible extension is to consider the wave kernel for

$$\frac{\partial^2}{\partial t^2} w = \Delta w - V(x) w,$$
$$w(P, 0) = 0,$$
$$w_t(P, 0) = \delta_P, \tag{6.1}$$

where V is a double well potential. A related question in dimension 2, is to study

$$\frac{\partial^2}{\partial t^2} w = \partial_{xx} w + V(x) \partial_{x_0 x_0} w,$$
$$w(x, x_0, 0) = 0,$$
$$w_t(x, x_0, 0) = \delta_P. \tag{6.2}$$

We can choose, for example, $V(x) = x^2(1 - x)^2$. The point $P$ can be $(0, 0)$ or $(1, 0)$. What is the picture of interferences?

Finally, we suggest a problem in the direction of the nonlinear wave equation. In particular, we are interested in the global existence or possible blow up in finite time for the following wave equation:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} w &= \Delta_H w + |w|^{p-1} w, \\ w(P,0) &= g, \\ w_t(P,0) &= f, \end{aligned} \tag{6.3}$$

where $p \le p_c = 1 + 2/n$ is the critical exponent in the Stein-Sobolev inequality on the Heisenberg group and where $\Delta_H$ is the Heisenberg Laplacian. The initial data are smooth enough. We expected some anisotropic phenomena due to the interaction between the nonlinearities and the propagation along the bicharacteristics.

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